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Assessment of Equivalence of Small and Large Orifice Computational Models

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Abstract

The aim of this paper was to analyze theoretical aspects of calculating steady water flow through unsubmerged circular orifices. Theoretical analysis shows that the values of discharge obtained by using formulas intended for small orifices are greater than those calculated using formulas for large orifices. These differences attain a maximum value when the water level reaches the upper edge of the orifice, and decrease when water head increases. It has been proven that the volumetric flow rate for circular unsubmerged orifices can be calculated by formulas for small orifices when the water level above the center of gravity is at least four times as high as the diameter of the orifice.

Key words: orifice, velocity, flow, computational models

1. Introduction

Steady liquid flow through an orifice is one of few hydrodynamic cases for which an analytical formula of the flow exists (Kubrak E. & Kubrak J. 2010, Mitosek 2014). When an orifice discharges freely into the air, as shown in Fig. 1, it is known as unsubmerged. The local velocity of the liquid flow varies with height within the jet of liquid. There are two methods of analyzing the discharge through an orifice. When the orifice is small in comparison with the head above the orifice, it is known as a small orifice. In this case, variations in velocity with height within the jet of liquid can be ignored, and the velocity is assumed to be constant. The analysis for large orifices takes into account the variation of velocity with height within the jet of liquid issuing from the orifice. This classification of orifices does not include the geometric dimensions and shape of the orifice.

When the outlet side of the orifice is beneath the surface of liquid, it is known as a submerged orifice.

The velocity of an elementary jet of liquid increases when the liquid level above its center of gravity increases. However, the width of the jet of liquid increases when head increases, but starts to decrease under the center of gravity of the orifice.

As already mentioned, it is possible to derive analytical formulas to calculate discharge from large circular orifices. However, in practical calculations of flow through a circular orifice, simplified formulas are used. A simplified formula is derived by using different lengths of series created for the integral function of the flow. Sometimes, it is possible to use formulas for small orifices.

In engineering practice, it is allowed to calculate the volumetric rate of flow through an unsubmerged circular orifice by formulas designed for small orifices if the D/H ratio is smaller than 0.1 (Mitosek 2014) (D – orifice diameter, H – head; liquid level above the center of the orifice S).

This simplifies calculations of discharge, but also has its implications for the accuracy of the calculations.

Accurate calculation of the volumetric rate of flow through unsubmerged large circular orifices is now possible with the use of mathematical software, e.g. Mathematica.

The aim of this article is to analyze the extent to which the shortening of a power series used in integrating the formula for the discharge of a large circular orifice affects the calculated values of the volumetric flow rate and to determine what H/D ratio can be used to calculate discharge as for small orifices.

2. Steady Liquid Flow through a Small Circular Orifice

In steady liquid flow through a small circular orifice, velocity is assumed to be constant (Fig. 1).

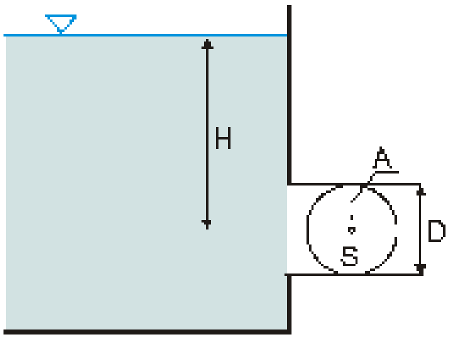

Fig. 1. Steady liquid flow through an unsubmerged circular orifice

The theoretical velocity of liquid is calculated by applying Bernoulli's theorem between the surface of the liquid and the center of the orifice S. The theoretical velocity of liquid passing through an orifice is given by

$$v = \varphi\sqrt{2gH}. \tag{1}$$

Factor φ is known as the Coefficient of Velocity (Kubrak E. & Kubrak J. 2010, Mitosek 2014).

The discharge Q is calculated by multiplying the theoretical velocity v by the cross sectional area at vena contracta A_s. Because of inertia forces, fluid streamlines cannot abruptly change direction. That is why the diameter of the jet of liquid discharged from the orifice is smaller than the orifice diameter (Fig. 2).

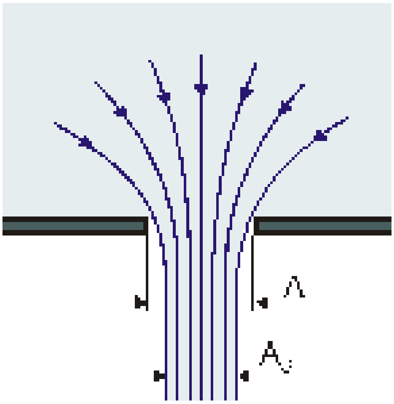

Fig. 2. Orifice flow contraction

This phenomenon is called orifice flow contraction. The ratio of the cross sectional area at vena contracta A_s to the cross sectional area of orifice is known as the Coefficient of Contraction β. The product of the Coefficient of Velocity φ and the Coefficient of Contraction β is called the Coefficient of Discharge μ. Discharge from a small unsubmerged circular orifice is given by

$$Q = vA_s = v\beta A = \varphi\beta A\sqrt{2gH} = \mu A\sqrt{2gH} = \mu\frac{\pi}{4}D^2\sqrt{2gH}. \tag{2}$$

The values of the Coefficient of Velocity φ and the Coefficient of Discharge φ may be determined through laboratory measurements (Kubrak E. & Kubrak J. 2010, Mitosek 2014). In order to compare the values of liquid flow calculated as for small and large orifices, in the following discussion it is assumed that for given H/D ratio the Coefficient of Discharge is constant and does not depend on the method of calculation. The local velocity of the perfect liquid flow ($\varphi = 1$) varies with height within the jet of liquid (Fig. 3)

3. Steady Liquid Flow through a Large Circular Orifice

When calculating discharge through a large orifice, it is necessary to take into account the variation of velocity with height within the jet of liquid issuing from the orifice (Fig. 4).

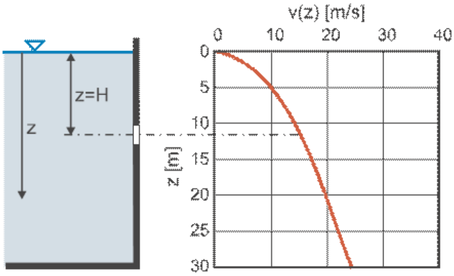

Fig. 3. Velocity of liquid flow as a function of the liquid level above the orifice

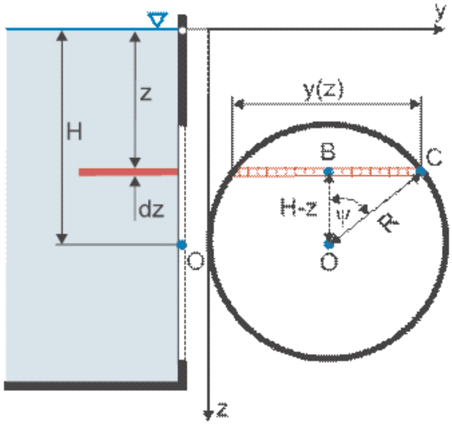

Fig. 4. Diagram for the analysis of liquid flow through a circular orifice

The total volumetric flow rate is equal to the sum of discharge from elementary jets of liquid with a width of $y(z)$ and a height of dz (Fig. 4). Discharge from an elementary jet of liquid that has a cross sectional area of $dA = y(z)dz$ is given by

$$dQ = \mu\sqrt{2gz}y(z)dz. \tag{3}$$

The width of the jet of liquid is given by

$$\left(\frac{y(z)}{2}\right)^2 = R^2 - (H - z)^2\,, \tag{4}$$

hence

$$y(z) = 2\sqrt{R^2 - (H - z)^2}. \tag{5}$$

By substituting $y(z)$ into (3), one obtains

$$dQ = \mu\sqrt{2gz} \cdot 2\sqrt{R^2 - (H - z)^2}dz. \tag{6}$$

By integrating the above formula in the range of $H - R$ to $H + R$, one obtains

$$Q = 2\mu\sqrt{2g}\int_{H-R}^{H+R}\sqrt{(R^2-(H-z)^2)z}\,dz = 2\mu\sqrt{2g}\int_{H-R}^{H+R}\sqrt{(R^2-vH^2)z+2Hz^2-z^3}\,dz. \quad (7)$$

The analytical solution of the integral is not yet known (Kubrak E. & Kubrak J. 2010, Kubrak 2014). In practice, its value is calculated by expressing the integrated polynomial as a power series or by using a numerical integration method, e.g. Simpson's rule.

4. Calculation of the Volumetric Rate of Flow through a Large Circular Orifice by Using the Binomial Theorem

The formula for discharge from an orifice (6) can be expressed using the central angle ψ. Then

$$\frac{y(z)}{2} = \frac{D}{2}\sin\psi, \quad (8)$$

$$z = H - \frac{D}{2}\cos\psi, \quad (9)$$

$$dz = \frac{D}{2}\sin\psi d\psi, \quad (10)$$

where D – orifice diameter ($D = 2R$).
By substituting $y(z)$ into formula (3) and setting the range of the integration of 0 to π, one obtains

$$Q = \frac{1}{2}\mu D^2\sqrt{2gH}\int_0^{\pi}\sqrt{1-\frac{D}{2H}\cos\psi}\,\sin^2\psi d\psi. \quad (11)$$

In order to calculate the integral in equation (11), it is possible to use a property of a binomial. For any variable x in the interval $(-1; 1)$ it is possible to expand a binomial into a sum of the form (Wrona 1967):

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{n}x^n + \ldots . \quad (12)$$

For $m = 1/2$ it can be written as

$$\begin{aligned}\sqrt{1+x} &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \\ &+ \frac{m(m-1)(m-2)(m-3)}{4!}x^4 + \ldots = \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \ldots .\end{aligned} \quad (13)$$

Table 1. Solved integrals

Ordinal	Integral	Solved integral	Value of the integral within the range of 0 to π
I	$\int \sin^2 \psi d\psi$	$\frac{\psi}{2} - \frac{1}{4} \sin 2\psi + C$	$\frac{\pi}{2}$
II	$\int \cos \psi \sin^2 \psi d\psi$	$\frac{1}{3} \sin^3 \psi + C$	0
III	$\int \cos^2 \psi \sin^2 \psi d\psi$	$\frac{\psi}{8} - \frac{1}{32} \sin 4\psi + C$	$\frac{\pi}{8}$
IV	$\int \cos^3 \psi \sin^2 \psi d\psi$	$-\frac{1}{5} \sin \psi \cos^4 \psi + +\frac{1}{5} \left(\frac{1}{3} \cos^2 \psi \sin \psi + + \frac{2}{3} \sin \psi \right) + C$	0
V	$\int \cos^4 \psi \sin^2 \psi d\psi$	$-\frac{1}{6} \sin \psi \cos^5 \psi + +\frac{1}{6} \left(\frac{1}{4} \cos^3 \psi \sin \psi + + \frac{3}{8}\psi + \frac{3}{16} \sin 2\psi \right) + C$	$\frac{\pi}{16}$
VI	$\int \cos^5 \psi \ \sin^2 \psi d\psi$	$-\frac{1}{7} \sin \psi \cos^6 \psi + +\frac{1}{7} \left(\frac{1}{5} \cos^4 \psi \sin \psi + + \frac{4}{15} \cos^2 \psi \sin \psi + \frac{8}{15} \sin \psi \right) + C$	0
VII	$\int \cos^6 \psi \ \sin^2 \psi d\psi$	$-\frac{1}{8} \sin \psi \cos^7 \psi + +\frac{1}{8} \left(\frac{1}{6} \cos^5 \psi \sin \psi + \frac{5}{24} \cos^3 \psi \sin \psi + + \frac{5}{16}\psi + \frac{5}{32} \sin 2\psi \right) + C$	$\frac{5\pi}{128}$

For values of x close to zero, lower-order terms rapidly decrease to zero. Applying the binomial theorem to $\sqrt{1 - D/2H \cos \psi}$, one can write it as a sum of seven terms:

$$\sqrt{1 - \frac{D}{2H} \cos \psi} = 1 - \frac{1}{2} \frac{D}{2H} \cos \psi - \frac{1}{8} \left(\frac{D}{2H} \right)^2 \cos{}^2 \psi - \frac{1}{16} \left(\frac{D}{2H} \right)^3 \cos{}^3 \psi + \\ - \frac{5}{128} \left(\frac{D}{2H} \right)^4 \cos{}^4 \psi + - \frac{7}{256} \left(\frac{D}{2H} \right)^5 \cos^5 \psi - \frac{21}{1024} \left(\frac{D}{2H} \right)^6 \cos^6 \psi, \tag{14}$$

hence:

$$Q = \frac{1}{2} \mu D^2 \sqrt{2gH} \int_0^{\pi} \left[1 - \frac{1}{2} \frac{D}{2H} \cos \psi - \frac{1}{8} \left(\frac{D}{2H} \right)^2 \cos^2 \psi + \right. \\ - \frac{1}{16} \left(\frac{D}{2H} \right)^3 \cos^3 \psi \frac{5}{128} \left(\frac{D}{2H} \right)^4 \cos^4 \psi + \\ \left. - \frac{7}{256} \left(\frac{D}{2H} \right)^5 \cos^5 \psi - \frac{21}{1024} \left(\frac{D}{2H} \right)^6 \cos^6 \psi \right] \sin^2 \psi d\psi. \tag{15}$$

After the multiplication of the terms in brackets by $\sin^2 \psi d\psi$, seven integrals are obtained: $\int \sin^2 \psi d\psi$, $\int \cos \psi \sin^2 \psi d\psi$, $\int \cos^2 \psi \sin^2 \psi d\psi$, $\int \cos^3 \psi \sin^2 \psi d\psi$, $\int \cos^4 \psi \sin^2 \psi d\psi$, $\int \cos^5 \psi \sin^2 \psi d\psi$, $\int \cos^6 \psi \sin^2 \psi d\psi$. They were solved using integral tables (Piłat & Wasilewski 1985). The solved integrals are summarized in Table 1.

By substituting the solved integrals from Table 1 into formula (15), it can be written as

$$
\begin{aligned}
Q = \tfrac{1}{2}\mu D^2 \sqrt{2gH} \Bigg[& \underbrace{\frac{\psi}{2} - \frac{1}{4}\sin 2\psi}_{I} \underbrace{-\frac{1}{2}\frac{D}{2H}\frac{1}{3}\sin^3\psi}_{II} \underbrace{-\frac{1}{8}\left(\frac{D}{2H}\right)^2\left(\frac{\psi}{8} - \frac{1}{32}\sin 4\psi\right)}_{III} + \\
& \underbrace{-\frac{1}{16}\left(\frac{D}{2h}\right)^3\left(-\frac{1}{5}\sin\psi\cos^4\psi + \frac{1}{5}\left(\frac{1}{3}\cos^2\psi\sin\psi + \frac{2}{3}\sin\psi\right)\right)}_{IV} + \\
& \underbrace{-\frac{5}{128}\left(\frac{D}{2h}\right)^4\left(-\frac{1}{6}\sin\psi\cos^5\psi + \frac{1}{6}\left(\frac{1}{4}\cos^3\psi\sin\psi + \frac{3}{8}\psi + \frac{3}{16}\sin 2\psi\right)\right)}_{V} + \\
& \underbrace{-\frac{7}{256}\left(\frac{D}{2h}\right)^5\left(-\frac{1}{7}\sin\psi\cos^6\psi + \frac{1}{7}\left(\frac{1}{5}\cos^4\psi\sin\psi + \frac{4}{15}\cos^2\psi\sin\psi + \frac{8}{15}\sin\psi\right)\right)}_{VI} + \\
& \underbrace{-\frac{21}{1024}\left(\frac{D}{2h}\right)^6\left(-\frac{1}{8}\sin\psi\cos^7\psi + \frac{1}{8}\left(\frac{1}{6}\cos^5\psi\sin\psi + \frac{5}{24}\cos^3\psi\sin\psi + \frac{5}{16}\psi + \frac{5}{32}\sin 2\psi\right)\right)}_{VII} \Bigg]\Bigg|_0^{\pi}
\end{aligned}
\tag{16}
$$

After solving the integrals in the range of integration, it turns out that integrals II, IV, and VI are equal to zero. By substituting non-zero integrals into formula (16), one obtains

$$
Q = \frac{\pi}{2}\mu D^2 \sqrt{2gH}\left[\underbrace{\frac{1}{2}}_{I} \underbrace{-\left(\frac{D}{2H}\right)^2 \frac{1}{64}}_{III} \underbrace{-\left(\frac{D}{2H}\right)^4 \frac{5}{2048}}_{V} \underbrace{-\left(\frac{D}{2H}\right)^6 \frac{105}{131072}}_{VII}\right]. \tag{17}
$$

As is apparent from equation (17), the integrated value of the first term in formula (16) is constant, and it does not depend on the diameter of the orifice and total head. If one uses only the first term of the theorem in formula (17), the formula is converted into the one describing discharge from a small orifice (2). The values of terms III, V, and VII in the function of H/D are shown in Fig. 5.

Fig. 5 indicates that the inclusion of the subsequent terms of the theorem will decrease the calculated value of the volumetric flow rate. This is because all odd terms of the theorem are negative.

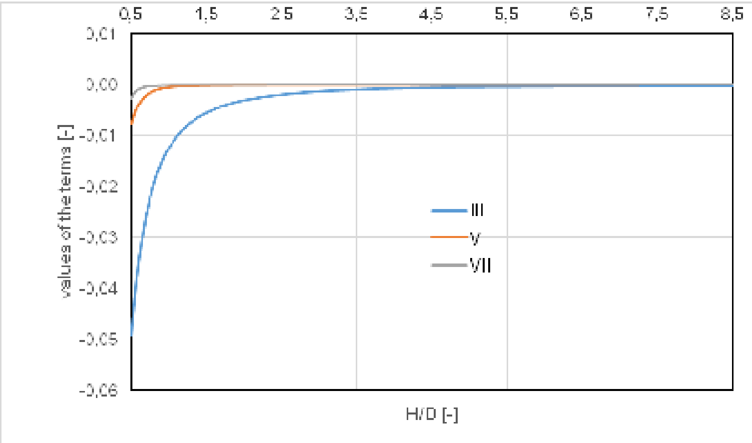

Fig. 5. Values of terms III, V, and VII in equation (17) in the function of H/D

An accurate value of integral (11) was also calculated by Mathematica. Thus obtained values at different H/D ratios were compared with values of discharge calculated by means of binomial theorem (13). Values of the volumetric flow rate for perfect liquid calculated as for a large orifice by Mathematica were labeled as Q_l (large orifice). Values of discharge calculated as for a small orifice were labeled as Q_s (small orifice).

In order to compare the values of discharge calculated as for a small orifice with those calculated as for a large orifice, percentage deviation in the function of the H/D ratio was calculated:

$$\frac{Q_s - Q_l}{Q_s} \cdot 100\%. \tag{18}$$

The results of calculations are shown in Fig. 6.

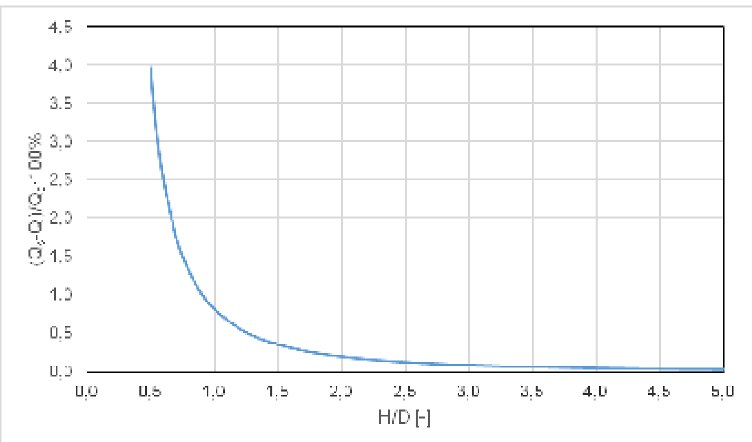

Fig. 6. alues of percentage deviation in the function of H/D

Fig. 6 shows, that the biggest difference between the values of discharge calculated by the formulas for small and large orifices amounts to 4% and occurs when $H/D = 0.5$. This occurs when the liquid level reaches the top edge of the orifice. When the liquid level above its center of gravity increases, percentage deviation (18) decreases to 0. For an H/D ratio equal to 4, the values of discharge calculated by the formulas for small and large orifices are practically the same.

In order to analyze the extent to which the shortening of a power series affects the calculated values of discharge, the following percentage deviation in the function of H/D was calculated:

$$\frac{Q_s}{Q_l\,(I + III)} \cdot 100\%, \tag{19}$$

$$\frac{Q_s}{Q_l\,(I + III + V)} \cdot 100\%, \tag{20}$$

$$\frac{Q_s}{Q_l\,(I + III + V + VII)} \cdot 100\%, \tag{21}$$

where: $Q_l\,(I + III)$ – discharge calculated as for a large orifice by formula (17), using terms I and III of the theorem; $Q_l\,(I + III + V)$ – discharge calculated as for a large orifice by formula (17), using terms I, III, and V of the theorem; $Q_l\,(I + III + V + VII)$ – discharge calculated as for a large orifice by formula (17), using terms I, III, V, and VII of the theorem.

The values of (19), (20), and (21) in the function of H/D are shown in Fig. 7.

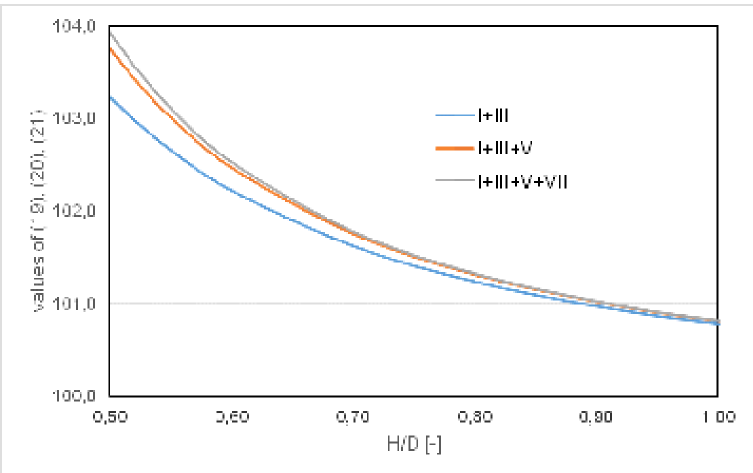

Fig. 7. Percentage deviation calculated by formulas (19), (20), and (21)

As is apparent from Fig. 7, the inclusion of the subsequent terms of the theorem when calculating discharge, reduces the value of the volumetric flow rate. Differences between discharge calculated as for large and small orifices decrease with increasing H/D.

5. Conclusion

The inclusion of only the first term of the theorem in formula (17) converts it into the formula describing discharge from a small orifice (2). The even terms of the theorem integrated within the range of 0 to π are equal to zero. The use of the subsequent terms of the theorem when calculating discharge, reduces the value of the volumetric flow rate. The analyses show that when $H \geq 4\ D$, the values of discharge calculated as for large and small orifices are practically the same. The biggest difference between the values of discharge calculated by formulas for small and large orifices amounts to 4% and occurs when $H/D = 0.5$.

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