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On the Description of Long Water Waves in Material Variables

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Abstract

Shallow water equations formulated in material variables are presented in this paper. In the model considered, a three-dimensional physical problem is substituted by a two-dimensional one describing a transformation of long waves in water of variable depth. The latter is obtained by means of the assumption that a vertical column of water particles remains vertical during the entire motion of the fluid. Under the assumption of small, continuous variation of the water depth, the equations for gravity waves are derived through Hamilton's principle formulated in terms of the material coordinates. This formulation ensures the conservation of mechanical energy. The approximation depends on the wave parameters as well as on the bed bathymetry. The latter may influence a solution of the model decisively; thus, one should be careful in applying the description to complicated geometries of fluid domains encountered in engineering practice.

Key words: long wave, shallow water, unsteady motion, sloping beach

1. Introduction

In description of water waves approaching a sloping beach, we frequently deal with a transformation of the waves induced by changes of the water depth. Usually, water depth diminishes towards a shoreline and an arriving wave changes its amplitude and celerity. Since the water depth is low compared to the length of the arriving wave, in theoretical description of the phenomenon we consider the problem as the transformation of the long wave in shallow water. In the theoretical analysis of the problem mentioned we introduce certain approximations which simplify the description. Commonly, we deal with quantities which are averaged over the water depth. In this way, the space dimension of the problem in question is reduced by one, i.e. the original three-dimensional problem is reduced to a two-dimensional one. There is considerable literature on this subject. Most formulations of the phenomenon make use of the space coordinates as independent variables. A certain drawback of such formulation is the difficulty of finding a solution to boundary conditions at moving boundaries of the fluid domain. On the other hand, formulation of the problem in terms of the material coordinates as independent variables simplifies the solution to boundary conditions but, at the same time, the governing equations describing the fluid motion are more complicated and thus, it is more difficult to find their solution. The classical theory of long waves propagating in water of variable depth has been developed by Stoker (1948). The analysis is confined to one-dimensional space problem of propagation of the waves in water with a constant bottom slope. The non-linear equations derived have been explicitly solved by means of the method of characteristics. Later on, Carrier and Greenspan (1958) developed a similar formulation allowing them to reduce non-linear shallow water equations to linear differential equations for the fluid velocity. With the equations obtained, they calculated a run-up height of non-breaking long waves on a sloping beach. The formulations presented in those papers were based on the fundamental assumption that the fluid pressure is given by a hydrostatic law. A more accurate theory of shallow water waves, which takes into account the vertical component of the fluid acceleration, has been formulated by Boussinesq (Abbott 1979). In the latter approach the magnitude of the vertical velocity is supposed to increase linearly from zero at the bed to a maximum value at the free surface. A detailed discussion of the long waves phenomenon, together with a wide bibliography of the subject may be found in the Dingemans' monograph (1997). The book covers most of the important problems associated with analysis of shallow water waves. Madsen and Schäffer (1999) have presented a detailed analysis of the Boussinesq-type equations for surface gravity waves. They discussed a velocity potential formulation in terms of an infinite series expansion. In the work, a number of various Boussinesq-type equations for water waves, existing in the literature of the subject, were analyzed. As far as the material description is concerned, Shuto (1967) has described the problem of run-up of long waves of small amplitude on a sloping beach. The first order approximation in the Lagrangian description of the phenomenon leads to momentum equations which do not contain vertical acceleration terms. A similar problem of long non-linear waves with a finite amplitude has been discussed by Goto (1979). The author derived a non-linear set of equations of the waves in the material description by means of a perturbation scheme with respect to finite displacements of water particles from their initial positions. The equations were solved numerically. Another formulation of the problem of long nonlinear gravity waves propagating over uneven bottoms has been given by Miles and Salmon (1985). Their formulation is based on the fundamental kinematical assumption that incompressible fluid motion is a 'columnar' motion for which the relevant Jacobian reduces to the form

$$J = \frac{\partial(x,y,z)}{\partial(x_0,y_0,z_0)} = \frac{\partial(x,y)}{\partial(x_0,y_0)}\frac{\partial z}{\partial z_0} = 1, \tag{A}$$

where (x, y, z) are the Cartesian coordinates of a fluid particle at an arbitrary moment of time and (x_0, y_0, z_0) are the coordinates of the same fluid particle in the equilibrium configuration at $t = 0$, which are thus chosen as the Lagrange coordinates of the particle.

The formula states that the horizontal displacements of fluid particles forming a vertical fluid column do not depend on the vertical z coordinate. With the formula (A), the Lagrangian density function in the Hamilton's principle is defined by a double integral over a horizontal plane. Recently, Wilde and Chybicki (2004) derived equations for the one-dimensional case of long water waves propagating in shallow water by means of a variational formulation in material description with the kinematical assumption that the vertical material lines of fluid particles remain vertical during entire motion of the fluid. Their kinematical assumption is similar to that given by Miles and Salmon (1985). A certain generalization of the formulation for cases of non-uniform water depth may be found in Chybicki (2006) and Szmidt (2006). The kinematical assumption allowed them to derive a partial differential equation describing a transformation of nonlinear long water waves. With respect to arbitrary bed changes of a sloping beach, two horizontal directions of a fluid domain should be taken into account in a description of the transformation of long waves approaching the beach. Thus, as compared to the one-dimensional cases, the problem becomes more complicated. Fundamental Boussinesq equations, formulated in spatial coordinates for the two-dimensional case, may be found in Abbott (1979). In order to simplify the analysis of long waves propagating in the two-dimensional water domain, Kapiński (2004) has applied a hybrid Lagrangian-Eulerian description in which both space and material variables are used. Final equations describing the problem have been derived under assumption of small continuous variation of the water bed. In order to derive momentum equations for the two-dimensional water domain with variable water depth Chybicki (2007) has applied a variational formulation with an approximate Lagrangian density function. The variation procedure has led to two non-linear partial differential momentum equations of very complicated structure. In most of the papers, final equations derived have been illustrated with examples corresponding to relatively regular fluid domains. The solutions obtained reflect desired characteristic features of wave propagation over uneven bottoms. It seems that with the material coordinates, chosen as independent variables, proper solutions to the problems considered are ensured. It may happen, however, that for more complicated bed bathymetries, the formulation fails to deliver proper results. Such cases are important from theoretical and practical points of view; therefore, the present paper aims to examine the main features of the description of the problem considered by means of the material variables. With respect to the need, we confine our attention to linear momentum equations obtained with the help of a rigorous variational formulation. In particular, it is shown that in a discrete solution of the momentum equations, a breaking of numerical computations occurs when a simply connected fluid domain becomes a multiply connected one. Such a case closely re-

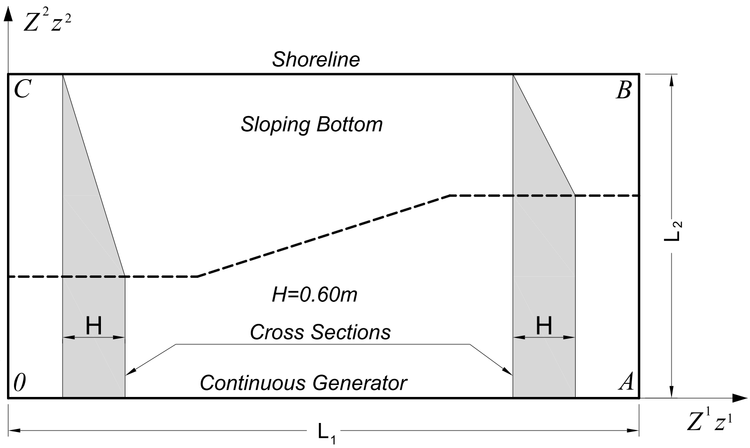

Fig. 1. Finite fluid domain of variable depth

lates to the physical situation where surface waves induce a water stream resulting from mixing of fluid particles.

2. Formulation of the Problem

Let us consider a rectangular fluid domain, as shown schematically in Fig. 1. The fluid area may be considered as a typical part of a sea shore with diminishing water depth towards the shore. A sea wave approaching the shore undergoes changes induced by the non-uniform water depth. It is assumed that the area shown in the figure repeats continuously and thus, the geometry and state of the fluid motion repeats along the beach. In order to describe the fluid motion, the Cartesian system of coordinates $z^r, r = 1, 2, 3$ is introduced, which is implemented to define an actual configuration of the fluid. A similar system of coordinates in the reference configuration is denoted by capital letters $Z^k, k = 1, 2, 3$. The latter coordinates define names of the fluid particles. The motion of incompressible fluid is described as the mapping of the names into actual positions of the material points.

$$
\begin{aligned}
z^1\left(Z^k, t\right) &= Z^1 + u\left(Z^1, Z^2, t\right), \\
z^2\left(Z^k, t\right) &= Z^2 + v\left(Z^1, Z^2, t\right), \\
z^3\left(Z^k, t\right) &= Z^3 + w\left(Z^1, Z^2, Z^3, t\right),
\end{aligned}
\tag{1}
$$

where: $u(Z^\alpha, t)$, $v(Z^\alpha, t)$ $(\alpha = 1, 2)$ and $w(Z^1, Z^2, Z^3, t)$ are horizontal alongshore and onshore components of the displacement vector, respectively.

The equations correspond directly to the ‘columnar’ assumption (A) that fluid particles forming a vertical material column at an initial moment of time will remain on the vertical column during entire motion of the fluid. For the incompressible fluid, the Jacobian of transformation (1) reads

$$J = \frac{\partial\left(z^1, z^2, z^3, t\right)}{\partial\left(Z^1, Z^2, Z^3, t\right)} = \left(1 + \frac{\partial w}{\partial Z^3}\right)\frac{\partial\left(z^1, z^2, t\right)}{\partial\left(Z^1, Z^2, t\right)} = \left(1 + \frac{\partial w}{\partial Z^3}\right) f\left(Z^1, Z^2, t\right) = 1. \quad (2)$$

In the relation $f\left(Z^\alpha, t\right)$ is two-dimensional Jacobian corresponding to the horizontal variables

$$f\left(Z^\lambda, t\right) = \left(1 + u_1\right)\left(1 + v_2\right) - u_2 v_1. \quad (3)$$

The subscripts 1 and 2 in the equation denote differentiation with respect to the coordinates Z^1 and Z^2, respectively. In the further discussion we will also need the time derivative of the function, i.e.

$$\dot{f} = \dot{u}_1\left(1 + v_2\right) + \left(1 + u_1\right)\dot{v}_2 - \left(\dot{u}_2 v_1 + u_2 \dot{v}_1\right), \quad (4)$$

where the dots denote differentiation with respect to time.

Following the columnar assumption, the vertical component of the displacement field is written in the form

$$w\left(Z^k, t\right) = \Delta h + \frac{w^{\cdot}}{H - h^0}\left(Z^3 - h^0\right), \quad (5)$$

where Δh is the difference between the current and the reference bottom levels corresponding to rigid body translation of the water column, and $w^{\cdot}\left(Z^\alpha, t\right)$ is an additional displacement of the free surface material point resulting from stretching of the column. In the equation $H = \text{const}$, $h = h\left(Z^1, Z^2\right)$ and $h^0 = h\left(Z^1 = \text{const}, Z^2 = \text{const}\right)$ is the reference bottom level. The denominator of the second term in the equation describes the still water depth as the difference between the free surface level H and the reference bottom level h^0. The difference Δh in equation (5) is defined by the Taylor formula

$$\Delta h = h\left(Z^1 + u, Z^2 + v\right) - h\left(Z^1, Z^2\right) =$$

$$= \frac{\partial h}{\partial Z^1}u + \frac{\partial h}{\partial Z^2}v + \frac{1}{2}\left[\frac{\partial^2 h}{\partial\left(Z^1\right)^2} + 2\frac{\partial^2 h}{\partial Z^1 \partial Z^2} + \frac{\partial^2 h}{\partial\left(Z^2\right)^2}\right] + \cdots, \quad (6)$$

where all the derivatives are calculated at the reference point $\left(Z^1 = \text{const}, Z^2 = \text{const}\right)$.

In what follows we confine our attention to the linear part of the relation, namely

$$\Delta h \cong s_1 u + s_2 v, \tag{7}$$

where $s^1 = \partial h \big/ \partial Z^1$ and $s^2 = \partial h \big/ \partial Z^2$ are local reference bottom slopes.

From substitution of equations (2) and (5) into relation (3), the following is obtained

$$w\left(Z^{\alpha}, t\right) \cong s^1 u + s^2 v + \frac{1}{f}\left(1 - f\right)\left(Z^3 - h^0\right). \tag{8}$$

With respect to the geometry of the fluid domain shown in Fig. 1, the potential energy of the fluid is

$$E_{pot} = \rho g \int\limits_{Z^1} \int\limits_{Z^2} \int\limits_{h_0}^{H} \left[z^3\left(Z^k, t\right) - Z^3\right] J dZ^1 dZ^2 dZ^3, \tag{9}$$

where ρ is the fluid density and g is the gravitational acceleration.

Hereinafter the integrals $\int\limits_{Z^1}\int\limits_{Z^2}(\ )dZ^1 dZ^2$ denote integration with respect to the independent variables $\left(Z^1, Z^2\right)$ within the fluid domain, i.e. for $\left(0 \le Z^1 \le L_1\right) \times \left(0 \le Z^2 \le L_2\right)$ in our case. Substituting relations (1) and (8) into the integrand and performing integration with respect to the vertical coordinate gives

$$E_{pot} = \frac{1}{2}\rho g H \int\limits_{Z^1} \int\limits_{Z^2} \left[2\left(s^1 u + s^2 v\right)(1 - \alpha) + H \frac{1}{f}(1 - f)(1 - \alpha)^2\right] dZ^1 dZ^2, \tag{10}$$

where $\alpha = h\left(Z^1, Z^2\right) \big/ H$.

The kinetic energy of the fluid is defined by the formula

$$E_{kin} = \frac{1}{2}\rho \int\limits_{Z^1} \int\limits_{Z^2} \int\limits_{h}^{H} \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right] J dZ^1 dZ^2 dZ^3. \tag{11}$$

From substitution of equations (1) and (8) into the last formula one obtains

$$E_{kin} = \frac{1}{2}\rho H \int\limits_{Z^1}\int\limits_{Z^2} \Big\{ \Big[\left(1+s_1^2\right)\dot{u}^2 + \left(1+s_2^2\right)\dot{v}^2 + 2s_1 s_2 \dot{u}\dot{v} \Big] (1-\alpha) + \\ -\frac{\dot{f}}{f^2} H (s_1\dot{u} + s_2\dot{v})(1-\alpha)^2 + \frac{1}{3}H^2 \left(\frac{\dot{f}}{f^2}\right)^2 (1-\alpha)^3 \Big\} dZ^1 dZ^2. \tag{12}$$

With respect to the equations, the action integral reads

$$I = \int\limits_{t_1}^{t_2}\int\limits_{Z^1}\int\limits_{Z^2} \left(E_{kin} - E_{pot}\right) dZ^1 dZ^2 dt = \int\limits_{t_1}^{t_2}\int\limits_{Z^1}\int\limits_{Z^2} L dZ^1 dZ^2 dt, \tag{13}$$

where L is the Lagrangian density function.

The principle of least action (Hamilton's principle) gives

$$\delta I = \delta \int\limits_{t_1}^{t_2}\int\limits_{Z^1}\int\limits_{Z^2} L dZ^1 dZ^2 dt = 0. \tag{14}$$

Before calculating the variation of the action integral, it is convenient to establish the variation of the Jacobian function $f(Z^\lambda, t)$ and its partial derivative. From equations (3) and (4) it follows

$$\begin{aligned} \delta f &= \delta u_1 (1+v_2) + (1+u_1)\delta v_2 - (\delta u_2 v_1 + u_2 \delta v_1), \\ \delta \dot{f} &= \delta \dot{u}_1 (1+v_2) + \dot{u}_1 \delta v_2 + \delta u_1 \dot{v}_2 + (1+u_1)\delta \dot{v}_2 + \\ &\quad - (\delta \dot{u}_2 v_1 + \dot{u}_2 \delta v_1 + \delta u_2 \dot{v}_1 + u_2 \delta \dot{v}_1). \end{aligned} \tag{15}$$

Simple but tedious manipulations lead to the variation of the action integral

$$\delta I = \frac{1}{2}\rho H \int\limits_{t_1}^{t_2}\int\limits_{Z^1}\int\limits_{Z^2} \Big[R_1 \delta u + R_2 \delta \dot{u} + R_3 \delta u_1 + R_4 \delta u_2 + R_5 \delta \dot{u}_1 + R_6 \delta \dot{u}_2 + \\ + T_1 \delta v + T_2 \delta \dot{v} + T_3 \delta v_1 + T_4 \delta v_2 + T_5 \delta \dot{v}_1 + T_6 \delta \dot{v}_{22} \Big] dZ^1 dZ^2 dt, \tag{16}$$

where:

$$
\begin{aligned}
R_1 &= -2gs_1(1-\alpha)\,,\\
R_2 &= 2(1-\alpha)\left[\left(1+s_1^2\right)\dot{u}+s_1s_2\dot{v}\right]-\frac{s_1H}{f^2}(1-\alpha)^2\dot{f},\\
R_3 &= gH(1-\alpha)^2\frac{1}{f^2}(1+v_2)-\frac{H}{f^3}(1-\alpha)^2\left[f\dot{v}_2-2\dot{f}(1+v_2)\right](s_1\dot{u}+s_2\dot{v})+\\
&\quad+\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^5}\left[f\dot{v}_2-2\dot{f}(1+v_2)\right],\\
R_4 &= -gH(1-\alpha)^2\frac{1}{f^2}v_1+\frac{H}{f^3}(1-\alpha)^2\left(f\dot{v}_1-2\dot{f}v_1\right)(s_1\dot{u}+s_2\dot{v})+\\
&\quad-\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^5}\left(f\dot{v}_1-2\dot{f}v_1\right),\\
R_5 &= -\frac{H}{f^2}(1-\alpha)^2(1+v_2)(s_1\dot{u}+s_2\dot{v})+\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^4}(1+v_2)\,,\\
R_6 &= \frac{H}{f^2}(1-\alpha)^2v_1(s_1\dot{u}+s_2\dot{v})-\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^4}v_1,
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
T_1 &= -2gs_2(1-\alpha)\,,\\
T_2 &= 2(1-\alpha)\left[\left(1+s_2^2\right)\dot{v}+s_1s_2\dot{u}\right]-\frac{s_2H}{f^2}(1-\alpha)^2\dot{f},\\
T_3 &= -gH(1-\alpha)^2\frac{1}{f^2}u_2+\frac{H}{f^3}(1-\alpha)^2\left(f\dot{u}_2-2\dot{f}u_2\right)(s_1\dot{u}+s_2\dot{v})+\\
&\quad-\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^5}\left(f\dot{u}_2-2\dot{f}u_2\right),\\
T_4 &= gH(1-\alpha)^2\frac{1}{f^2}(1+u_1)-\frac{H}{f^3}(1-\alpha)^2\left[f\dot{u}_1-2\dot{f}(1+u_1)\right]\times\\
&\quad\times(s_1\dot{u}+s_2\;\dot{v})+\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^5}\left[f\dot{u}_1-2\dot{f}(1+u_1)\right],\\
T_5 &= \frac{H}{f^2}(1-\alpha)^2u_2(s_1\dot{u}+s_2\dot{v})-\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^4}u_2,\\
T_6 &= -\frac{H}{f^2}(1-\alpha)^2(1+u_1)(s_1\dot{u}+s_2\dot{v})+\frac{2}{3}H^2(1-\alpha)^3\frac{\dot{f}}{f^4}(1+u_1)\,.
\end{aligned}
\tag{18}
$$

For the linear operations considered, the terms in the integrand in Eq. (16) may be expressed in other forms. For example

$$R_2\delta\dot{u} = \frac{\partial}{\partial t}(R_2\delta u) - \dot{R}_2\delta u,$$
$$R_5\delta\dot{u}_1 = \frac{\partial}{\partial Z^1}\frac{\partial}{\partial t}(R_5\delta u) - \frac{\partial}{\partial Z^1}\left(\frac{\partial R_5}{\partial t}\delta u\right) - \frac{\partial}{\partial t}\left(\frac{\partial R_5}{\partial Z^1}\delta u\right) + \frac{\partial^2 R_5}{\partial Z^1 \partial t}\delta u. \tag{19}$$

Similar relations hold for the remaining terms. In view of formulae, the variation of the action integral (16) leads to the equation

$$\begin{aligned}
\delta I = &\int_{t_1}^{t_2}\int_{Z^1}\int_{Z^2}\left\{\left[R_1 - \frac{\partial R_2}{\partial t} - \frac{\partial R_3}{\partial Z^1} - \frac{\partial R_4}{\partial Z^2} + \frac{\partial^2 R_5}{\partial t \partial Z^1} + \frac{\partial^2 R_6}{\partial t \partial Z^2}\right]\delta u + \right.\\
&\left. + \left[T_1 - \frac{\partial T_2}{\partial t} - \frac{\partial T_3}{\partial Z^1} - \frac{\partial T_4}{\partial Z^2} + \frac{\partial^2 T_5}{\partial t \partial Z^1} + \frac{\partial^2 T_6}{\partial t \partial Z^2}\right]\delta v\right\} dZ^1 dZ^2 dt + \\
&+ \int_{Z^1}\int_{Z^2}\left[\left(R_2 - \frac{\partial R_5}{\partial Z^1} - \frac{\partial R_6}{\partial Z^2}\right)\delta u + \left(T_2 - \frac{\partial T_5}{\partial Z^1} - \frac{\partial T_6}{\partial Z^2}\right)\delta v\right]\Bigg|_{t_1}^{t_2} dZ^1 dZ^2 + \\
&+ \int_{t_1}^{t_2}\int_{Z^2}\left[\left(R_3 - \frac{\partial R_5}{\partial t}\right)\delta u + \left(T_3 - \frac{\partial T_5}{\partial t}\right)\delta v\right]\Bigg|_{Z^1=0}^{Z^1=L_1} dZ^2 dt + \\
&+ \int_{t_1}^{t_2}\int_{Z^1}\left[\left(R_4 - \frac{\partial R_6}{\partial t}\right)\delta u + \left(T_4 - \frac{\partial T_6}{\partial t}\right)\delta v\right]\Bigg|_{Z^2=0}^{Z^2=L_2} dZ^1 dt + \\
&+ \int_{Z^2}\left[R_5\delta u + T_5\delta v\right]\Bigg|_{t_1}^{t_2}\Bigg|_{Z^1=0}^{Z^1=L_1} dZ^2 + \int_{Z^1}\left[R_6\delta u + T_6\delta v\right]\Bigg|_{t_1}^{t_2}\Bigg|_{Z^2=0}^{Z^2=L_2} dZ^1 = 0,
\end{aligned} \tag{20}$$

where, for brevity, the constant multiplier $\rho H/2$ in equation (16) has been omitted.

In the further part we will discuss the fluid motion starting from rest, for which the variations δu and δv disappear at the end time points i.e. $\delta u|_{t_1} = \delta u|_{t_2} = 0$ and $\delta v|_{t_1} = \delta v|_{t_2} = 0$, respectively. Thus, the second and the last two integrals in the equations may be canceled out. At the same time, the third and the fourth integral define natural boundary conditions at the fluid boundaries. Following the variation procedure we require Eq. (20) to vanish for all δu and δv, which implies

$$\begin{aligned}
-R_1 + \frac{\partial R_2}{\partial t} + \frac{\partial R_3}{\partial Z^1} + \frac{\partial R_4}{\partial Z^2} - \frac{\partial^2 R_5}{\partial t \partial Z^1} - \frac{\partial^2 R_6}{\partial t \partial Z^2} = 0,\\
-T_1 + \frac{\partial T_2}{\partial t} + \frac{\partial T_3}{\partial Z^1} + \frac{\partial T_4}{\partial Z^2} - \frac{\partial^2 T_5}{\partial t \partial Z^1} - \frac{\partial^2 T_6}{\partial t \partial Z^2} = 0.
\end{aligned} \tag{21}$$

The equations define momentum equations for fluid motion in the domain of non-uniform water depth. In order to express the equations in terms of the dependent variables $u(Z^\lambda, t)$ and $v(Z^\lambda, t)$ $(\lambda = 1, 2)$, it is necessary to substitute the descriptions (17) and (18) into the equations and perform certain manipulations. For our needs it is sufficient to confine our attention to linear terms in the last equations, which describe key features of behavior for the model considered. Thus, from substitution of Eq. (17) and Eq. (18) into equations (21) the following relations are obtained

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \frac{1}{3}H^2(1-\alpha)^2\frac{\partial^4 u}{\partial (Z^1)^2\partial t^2} + Hs_1(1-\alpha)\frac{\partial^3 u}{\partial Z^1\partial t^2} + \\
&-\frac{1}{3}H^2(1-\alpha)^2\frac{\partial^4 v}{\partial Z^1\partial Z^2\partial t^2} + \frac{1}{2}H(1-\alpha)\left(s_1\frac{\partial^3 v}{\partial Z^2\partial t^2} + s_2\frac{\partial^3 v}{\partial Z^1\partial t^2}\right) + \\
&+g\left[s_1\left(2\frac{\partial u}{\partial Z^1} + \frac{\partial v}{\partial Z^2}\right) + s_2\frac{\partial v}{\partial Z^1} - H(1-\alpha)\left(\frac{\partial^2 u}{\partial (Z^1)^2} + \frac{\partial^2 v}{\partial Z^1\partial Z^2}\right)\right] = 0,
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} - \frac{1}{3}H^2(1-\alpha)^2\frac{\partial^4 v}{\partial (Z^2)^2\partial t^2} + Hs_2(1-\alpha)\frac{\partial^3 v}{\partial Z^2\partial t^2} + \\
&-\frac{1}{3}H^2(1-\alpha)^2\frac{\partial^4 u}{\partial Z^1\partial Z^2\partial t^2} + \frac{1}{2}H(1-\alpha)\left(s_2\frac{\partial^3 u}{\partial Z^1\partial t^2} + s_1\frac{\partial^3 u}{\partial Z^2\partial t^2}\right) + \\
&+g\left[s_1\frac{\partial u}{\partial Z^2} + s_2\left(\frac{\partial u}{\partial Z^1} + 2\frac{\partial v}{\partial Z^2}\right) - H(1-\alpha)\left(\frac{\partial^2 u}{\partial Z^1\partial Z^2} + \frac{\partial^2 v}{(\partial Z^2)^2}\right)\right] = 0.
\end{aligned} \tag{23}$$

In the equations, the slopes s_1 and s_2 are small quantities. In the case of a constant water depth $\alpha = s_1 = s_2 = 0$ and the equations reduce to simpler forms. The equations should be supplemented with initial and boundary conditions. Since the equations will be substituted by a system of ordinary differential equations, a description of the boundary conditions is left to the next section.

3. Approximations to the Momentum Equations

Although the momentum equations are linear, they have variable coefficients and therefore they are still difficult to be solved analytically. Therefore, in order to solve the equations we resort to approximate formulations by means of the finite difference method. With the method, the continuous displacement components are replaced by a discrete set of their values at chosen nodal points and, accordingly, their space derivatives are replaced by finite difference quotients at these points. In this way, instead of the partial differential equations we will consider a set of ordinary differential equations with respect to the time variable. In order to write the equations we assume the following approximations

$$
\begin{gathered}
\left.\frac{\partial u}{\partial Z^1}\right|_j \cong \frac{1}{2a}\left(u_{j+1}-u_{j-1}\right), \quad \left.\frac{\partial u}{\partial Z^2}\right|_k \cong \frac{1}{2b}\left(u_{k+1}-u_{k-1}\right), \\
\left.\frac{\partial^2 u}{\partial (Z^1)^2}\right|_j \cong \frac{1}{a^2}\left(u_{j-1}-2u_j+u_{j+1}\right), \quad \left.\frac{\partial^2 u}{\partial (Z^2)^2}\right|_k \cong \frac{1}{b^2}\left(u_{k-1}-2u_k+u_{k+1}\right), \\
\left.\frac{\partial^2 u}{\partial Z^1 \partial Z^2}\right|_{j,k} \cong \frac{1}{4ab}\left(u_{j+1,k+1}-u_{j-1,k+1}+u_{j-1,k-1}-u_{j+1,k-1}\right).
\end{gathered} \tag{24}
$$

where a, b are alongshore and onshore spacing of the nodal points, respectively.

Similar formulae hold for the second component of the displacement field. Thus, in the discrete in space approach, instead of the continuous fluid domain $(L_1 \times L_2)$ shown in Fig. 1 we consider an associated set of nodal points. From substitution of finite difference quotients (24) into equation (22) the following relation is obtained

$$
\begin{gathered}
\frac{\partial^2 u_{j,k}}{\partial t^2} - \frac{1}{3}H^2(1-\alpha_{j,k})^2\frac{\partial^2}{\partial t^2}\left[\frac{1}{a^2}\left(u_{j-1,k}-2u_{j,k}+u_{j+1,k}\right)\right]+ \\
+Hs^1_{j,k}(1-\alpha_{j,k})\frac{\partial^2}{\partial t^2}\left[\frac{1}{2a}\left(u_{j+1,k}-u_{j-1,k}\right)\right]+ \\
-\frac{1}{3}H^2(1-\alpha_{j,k})^2\frac{\partial^2}{\partial t^2}\left[\frac{1}{4ab}\left(v_{j+1,k+1}-v_{j-1,k+1}+v_{j-1,k-1}-v_{j+1,k-1}\right)\right]+ \\
+\frac{1}{2}H(1-\alpha_{j,k})\left\{s^1_{j,k}\frac{\partial^2}{\partial t^2}\left[\frac{1}{2b}\left(v_{j,k+1}-v_{j,k-1}\right)\right]\right\}+ \\
+\frac{1}{2}H(1-\alpha_{j,k})\left\{s^2_{j,k}\frac{\partial^2}{\partial t^2}\left[\frac{1}{2a}\left(v_{j+1,k}-v_{j-1,k}\right)\right]\right\}+ \\
+g\left\{s^1_{j,k}\left[\frac{1}{a}\left(u_{j+1,k}-u_{j-1,k}\right)+\frac{1}{2b}\left(v_{j,k+1}-v_{j,k-1}\right)\right]+\right. \\
+s^2_{j,k}\frac{1}{2a}\left(v_{j+1,k}-v_{j-1,k}\right)+ \\
-H(1-\alpha_{j,k})\left[\frac{1}{a^2}\left(u_{j-1,k}-2u_{j,k}+u_{j+1,k}\right)+\right. \\
\left.\left.+\frac{1}{4ab}\left(v_{j+1,k+1}-v_{j-1,k+1}+v_{j-1,k-1}-v_{j+1,k-1}\right)\right]\right\}=0.
\end{gathered} \tag{25}
$$

And, in a similar way,

$$
\begin{aligned}
&\frac{\partial^2 v_{j,k}}{\partial t^2} - \frac{1}{3}H^2(1-\alpha_{j,k})^2\frac{\partial^2}{\partial t^2}\left[\frac{1}{b^2}\left(v_{j,k-1} - 2v_{j,k} + v_{j,k+1}\right)\right] +\\
&\quad + Hs^2_{j,k}(1-\alpha_{j,k})\frac{\partial^2}{\partial t^2}\left[\frac{1}{2b}\left(v_{j,k+1} - v_{j,k-1}\right)\right] +\\
&-\frac{1}{3}H^2(1-\alpha_{j,k})^2\frac{\partial^2}{\partial t^2}\left[\frac{1}{4ab}\left(u_{j+1,k+1} - u_{j-1,k+1} + u_{j-1,k-1} - u_{j+1,k-1}\right)\right] +\\
&\quad + \frac{1}{2}H(1-\alpha_{j,k})\left\{s^2_{j,k}\frac{\partial^2}{\partial t^2}\left[\frac{1}{2a}\left(u_{j+1,k} - u_{j-1,k}\right)\right]\right\} +\\
&\quad \frac{1}{2}H(1-\alpha_{j,k})\left\{s^1_{j,k}\frac{\partial^2}{\partial t^2}\left[\frac{1}{2b}\left(u_{j,k+1} - u_{j,k-1}\right)\right]\right\} +\\
&+g\left\{s^1_{j,k}\frac{1}{2b}\left(u_{j,k+1} - u_{j,k-1}\right) + s^2_{j,k}\left[\frac{1}{2a}\left(u_{j+1,k} - u_{j-1,k}\right) + \frac{1}{b}\left(v_{j,k+1} - v_{j,k-1}\right)\right] +\right.\\
&\quad -H(1-\alpha_{j,k})\left[\frac{1}{4ab}\left(u_{j+1,k+1} - u_{j-1,k+1} + u_{j-1,k-1} - u_{j+1,k-1}\right)\right] +\\
&\quad \left. + \frac{1}{b^2}\left(v_{j,k-1} - 2v_{j,k} + v_{j,k+1}\right)\right\} = 0.
\end{aligned}
\tag{26}
$$

The equations are supplemented with relevant boundary conditions. As mentioned above, the fluid area shown in Fig. 1 and a state of the fluid are assumed to repeat continuously along a shoreline. Accordingly, for $Z^1 = 0$ and $Z^1 = L_1$ we have $u = 0$ and $\partial v \big/ \partial Z^1 = 0$. For $Z^2 = 0$ the onshore component of the displacement field equals the generator displacement. With respect to the discrete approach, the remainder derivatives of the onshore component of the displacement field with respect to Z^2 at the generator face and the shoreline are calculated with the help of a Gregory-Newton extrapolation formula (Chan and Street 1970). The formula allows to express a value of a function at an exterior point in the form of a weighted sum of its values at four interior points. Thus, all necessary boundary conditions are expressed in terms of the unknown variables at nodal points of the rectangular fluid domain considered. Knowing the boundary relations, one may write equations (25) and (26) for all the points. In this way, in place of the continuous functions $u(Z^\lambda, t)$ and $v(Z^\lambda, t)$, we will operate with the vectors $\mathbf{U}(t)$ and $\mathbf{V}(t)$ whose components describe the displacements at the nodal points. With the discrete description, the momentum equations are written in matrix form

$$
\begin{aligned}
\mathbf{AU}\cdot\ddot{\mathbf{U}} + \mathbf{BU}\cdot\mathbf{U} + \mathbf{CU}\cdot\ddot{\mathbf{V}} + \mathbf{DU}\cdot\mathbf{V} = \mathbf{P1},\\
\mathbf{CV}\cdot\ddot{\mathbf{U}} + \mathbf{DV}\cdot\mathbf{U} + \mathbf{AV}\cdot\ddot{\mathbf{V}} + \mathbf{BV}\cdot\mathbf{V} = \mathbf{P2},
\end{aligned}
\tag{27}
$$

where vectors **P1** and **P2** result from the assumed generation of the fluid motion.

All the matrices entering the equations results form the descriptions (25) and (26) and the relevant boundary conditions. In order to perform integration of the ordinary differential equations in the time domain, we introduce a countable set of the time steps $0, \Delta t, 2\Delta t, \ldots, t_{\max} = n\Delta t$ and make use of the Wilson θ method. In this method, the acceleration between subsequent time steps is approximated by a linear function of time. For a mechanical system the procedure is unconditionally stable for $\theta > 1.37$ (Bathe 1982). In order to make the discussion clear, some important relations of the method are summarized below. The velocity and acceleration are defined by means of the formulae

$$\begin{aligned} \dot{u}_3 &= \frac{3}{DT}(u_3 - u_1) - 2\dot{u}_1 - \frac{DT}{2}\ddot{u}_1, \\ \ddot{u}_3 &= \frac{6}{DT^2}(u_3 - u_1) - \frac{6}{DT}\dot{u}_1 - 2\ddot{u}_1, \end{aligned} \tag{28}$$

where $u_1 = u(t)$ is known value, $u_3 = u(t + DT)$ is unknown value and $DT = \theta\Delta t$ with $\theta = 1.47$ in our case. Similar formulae hold for the second component of the displacement field. For our further purposes it is convenient to write the second of the equations in the form

$$\frac{DT^2}{6}\ddot{u}_3 = u_3 - \left[u_1 + DT\dot{u}_1 + \frac{DT^2}{3}\ddot{u}_1\right] = u_3 - f_1. \tag{29}$$

And similarly

$$\frac{DT^2}{6}\ddot{v}_3 = v_3 - \left[v_1 + DT\dot{v}_1 + \frac{DT^2}{3}\ddot{v}_1\right] = v_3 - g_1. \tag{30}$$

The system of equations (27) holds for an arbitrary level of time. In particular, the system is written for the time $t_3 = t_1 + DT$. Substitution of Eqs. (29) and (30) into equations (27) gives

$$\begin{aligned} \mathbf{ABU} \cdot \mathbf{U}_3 + \mathbf{CDU} \cdot \mathbf{V}_3 &= \mathbf{AU} \cdot \mathbf{F}_1 + \mathbf{CU} \cdot \mathbf{G}_1, \\ \mathbf{CDV} \cdot \mathbf{U}_3 + \mathbf{ABV} \cdot \mathbf{V}_3 &= \mathbf{CV} \cdot \mathbf{F}_{eq1} + \mathbf{AV} \cdot \mathbf{G}_1, \end{aligned} \tag{31}$$

where:

$$\begin{aligned} \mathbf{ABU} &= \mathbf{AU} + \frac{DT^2}{6}\mathbf{BU}, \quad \mathbf{CDU} = \mathbf{CU} + \frac{DT^2}{6}\mathbf{DU}, \\ \mathbf{CDV} &= \mathbf{CV} + \frac{DT^2}{6}\mathbf{DV}, \quad \mathbf{ABV} = \mathbf{AV} + \frac{DT^2}{6}\mathbf{BV}. \end{aligned} \tag{32}$$

A remark here is necessary. The right-hand-side terms in equations (31) are known quantities. At the same time the vectors $\mathbf{U}_3$ and/or $\mathbf{V}_3$ contain prescribed values of the displacement field at some points of the fluid boundary. In such a case, the latter values should be added, as known quantities, to the right-hand-side terms of the equations. In the further section we will consider the motion of the fluid domain induced by known onshore component of fluid velocity assumed on the boundary $0 \leq Z^1 \leq L_1$, $Z^2 = 0$. This case corresponds directly to generation of the fluid motion by a continuous piston type generator placed at the boundary. Finally, the system of equations (31) is written in the form of the single matrix equation

$$\mathbf{AA} \cdot \mathbf{W} = \mathbf{P}, \tag{33}$$

where:

$$\mathbf{W}^{\mathbf{T}} = (\mathbf{U}, \mathbf{V})^{\mathbf{T}}, \tag{34}$$

and

$$\mathbf{AA} = \begin{bmatrix} \mathbf{ABU} & \mathbf{CDU} \\ \mathbf{CDV} & \mathbf{ABV} \end{bmatrix}. \tag{35}$$

The vector $\mathbf{P}$ in equation (33) results from the right hand side terms in equations (31), and depends on the form of generation of the fluid motion. The matrix $\mathbf{AA}$ in equations (33) does not depend on time. Having the solution at the time $t_3 = t_1 + DT$, one can calculate the solution at the subsequent moment of time i.e. at $t_2 = t_1 + \Delta t$

$$\begin{aligned} \ddot{u}(t + \Delta t) &= \ddot{u}_1 + \frac{\Delta t}{DT}(\ddot{u}_3 - \ddot{u}_1), \\ \dot{u}(t + \Delta t) &= \dot{u}_1 + \Delta t \ddot{u}_1 + \frac{(\Delta t)^2}{2DT}(\ddot{u}_3 - \ddot{u}_1), \\ u(t + \Delta t) &= u_1 + \Delta t \dot{u}_1 + \frac{(\Delta t)^2}{2}\ddot{u}_1 + \frac{(\Delta t)^3}{6DT}(\ddot{u}_3 - \ddot{u}_1). \end{aligned} \tag{36}$$

Similar equations hold for the onshore components of the displacement field.

4. Numerical Examples

In order to illustrate applicability of the model presented above and to learn more about the phenomenon, let us consider a particular case of fluid motion starting from rest. With respect to the fluid domain shown in Fig. 1, the fluid motion is induced by a piston type generator placed at the alongshore boundary $\left(0 \leq Z^1 \leq L_1, Z^2 = 0\right)$. The generator starts to move at an initial moment of time. In the model considered, the generator motion approaches the case of a simple harmonic motion within a few periods of time. Our main goal is to find a solution to the momentum equations within the fluid area shown in the figure. It should be stressed that only solutions in simply connected regions corresponding to deformations of the area $OABC$ in

Fig. 1 are proper solutions, and therefore, a possible transformation of the area to multiply connected regions terminates the solution validity. In particular, we will find evolutions of the shoreline in the time domain for an assumed set of parameters describing the generator motion. In the model considered, the shore boundary is formed by material particles; thus, the evolution of the boundary is defined by positions of the material points in the time domain. The generator motion is characterized by its amplitude and frequency. At the same time, the bottom elevation of the fluid domain is known in advance function of the material coordinates. In order to find a solution of the momentum equations, we need to integrate them in the time domain. The integration provides displacements of the chosen material points. As we have mentioned above, the fluid starts to move from rest. It is assumed that the motion beginning is very smooth, so not only the velocity, but also the acceleration disappears at the initial moment of time. Thus, the horizontal onshore motion of the piston-type generator is assumed in the following form (Wilde and Wilde 2001)

$$v_0(t) = d[A(\tau)\cos\omega t + D(\tau)\sin\omega t], \tag{37}$$

where d is a dimension unit, ω is the angular frequency, $\tau = \eta t$ is the non-dimensional time and

$$\begin{aligned} A(\tau) &= \frac{1}{3!}\tau^3\exp(-\tau), \\ D(\tau) &= 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right)\exp(-\tau). \end{aligned} \tag{38}$$

The parameter η in the relations is responsible for a growth in time of the generation amplitude. With increasing time elapse, measured from the starting point, the generation approaches the case of the steady harmonic generation with unit amplitude and the assumed frequency. With respect to the last formulae, it is a simple task to calculate the onshore components of the fluid velocity and the acceleration at the generator face. Having the forcing terms, and integrating equations (27) in the time domain, we obtain the displacement, velocity and acceleration components at nodal points of the fluid domain at successive moments of time. The discrete integrations have been carried out with the constant time step $\Delta t = 0.05$ s. Among others, the most interesting are displacement of nodal points at the shoreline. In order to illustrate the solution procedure we attach here some results of numerical computations. The calculations have been carried out for the rectangular fluid domain $L_1 \times L_2 = 30$ m $\times$ 9.6 m with equal spacing of the alongshore and onshore lines $a = b = 0.30$ m. The lines determine the set of $N_g = 3333$ nodal points with $N_u = 3267$ alongshore and $N_v = 3232$ onshore unknown components of the displacement field. The motion of the fluid has been induced by the piston type generator motion with the amplitude $A_g = 0.02$ m and chosen frequencies. The frequencies correspond to water waves of lengths $\lambda_1 = 6$ m, $\lambda_2 = 12$ m and

$\lambda_3 = 18$ m propagating in fluid of constant depth $H = 0.6$ m, respectively. Fig. 2a shows the distribution in time of the displacements components of chosen material points at the shoreline, i.e.: $P_1(Z^1 = 0.25L_1, Z^2 = L_2)$, $P_2(Z^1 = 0.5L_1, Z^2 = L_2)$ and $P_3(Z^1 = 0.75L_1, Z^2 = L_2)$ corresponding to the water wave of length $\lambda = 6H$. The plots in Fig. 2b shows the distribution of the shoreline in chosen moments of time. The similar cases for waves of lengths $\lambda = 12$ m and $\lambda = 18$ m are illustrated in the consecutive Figures 3 and 4, respectively. From the graphs in Figure 2 it may be seen, that with increasing time the vertical displacements of the chosen points increase. The evolution in time of the displacements is similar to displacements of a linear mass-spring system loaded with external harmonic force with frequency equal to, or close to, an eigenfrequency of the system mentioned. As a matter of fact, the equations derived above describe a finite continuous mechanical system without any damping mechanism, for which a set of its own eigenfrequencies exists. The frequency of the case shown in the figure is close to the eigenfrequency of the fluid domain; therefore, the generation leads to increase of the displacement field with growing time. Examination of the remainder plots in Figures 3 and 4 reveals that for a certain elapse of time measured from the starting point, a rapid increase of the onshore displacement of the shoreline points may be observed. Such an increase is a result of another phenomenon emerging when time exceeds a certain value. Investigation of the plots in Figs. 2b, 3B and 4B shows that the topology of the fluid domain is not preserved through the entire motion of the fluid i.e., the simply connected domains become multiply connected ones. This means that at those moments of time, the solutions obtained are not proper solutions of the problem considered because the main assumption on continuity of the fluid medium is lost at those moments. The numerical integration of the momentum equations should be terminated before such moments of time. Although the numerical computations breakdown, the formulation of the problem in material coordinates has allowed us to calculate the evolution in time of a shoreline till the breaking point and indicates a reason for the breaking. Theoretical solutions obtained reflect a phenomenon frequently observed in natural conditions, where waves approaching a shoreline may induce a return stream resulting from mixing water particles. From the analysis it follows that a proper formulation of the problem in material variables, by means of the Hamilton's principle is not sufficient to deliver proper solutions to arbitrary possible cases of shallow water flows. It should be stressed that most of the theoretical models of solutions to the aforementioned problems are based on the fundamental assumption on the flow continuity, which is frequently lost in many practical cases; therefore, it is not possible to develop a reliable general computational model providing proper solutions to arbitrary fluid flows.

a)

b)

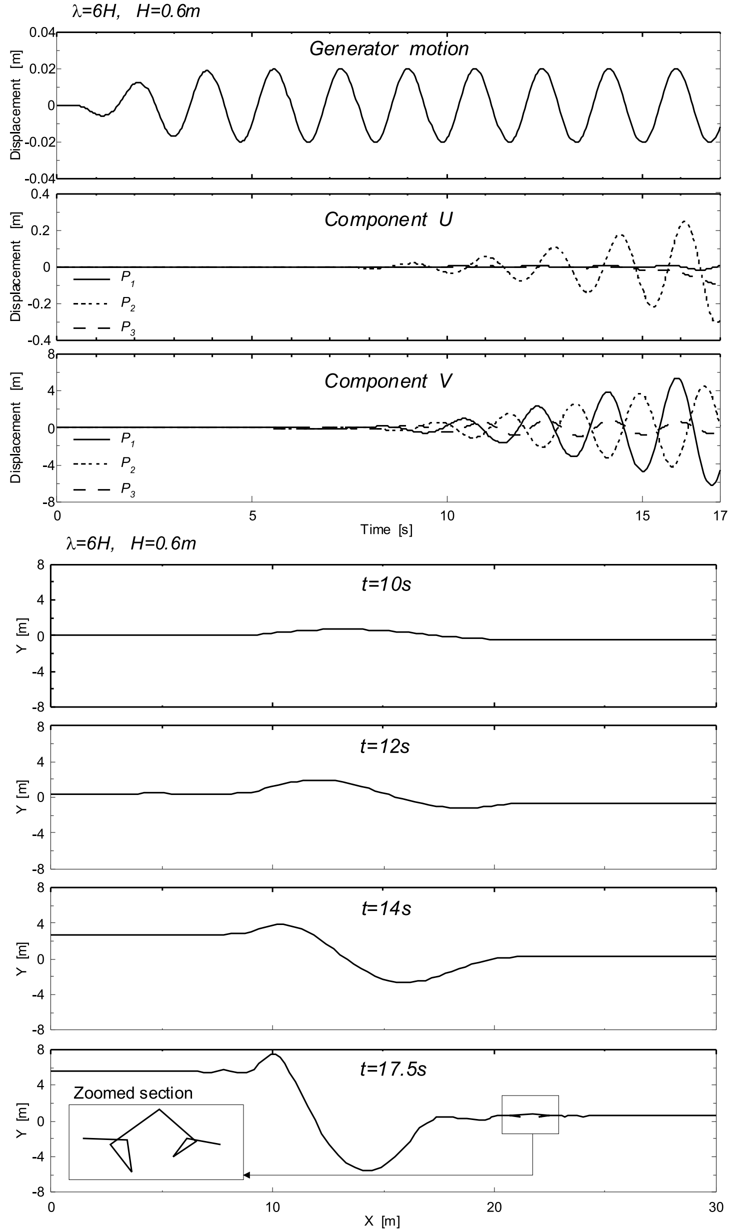

Fig. 2. Evolution in time of the displacements of the material points P_1, P_2, P_3 for the wave of length $\lambda = 6H$(a) and the distribution of the shoreline for chosen moments of time (b)

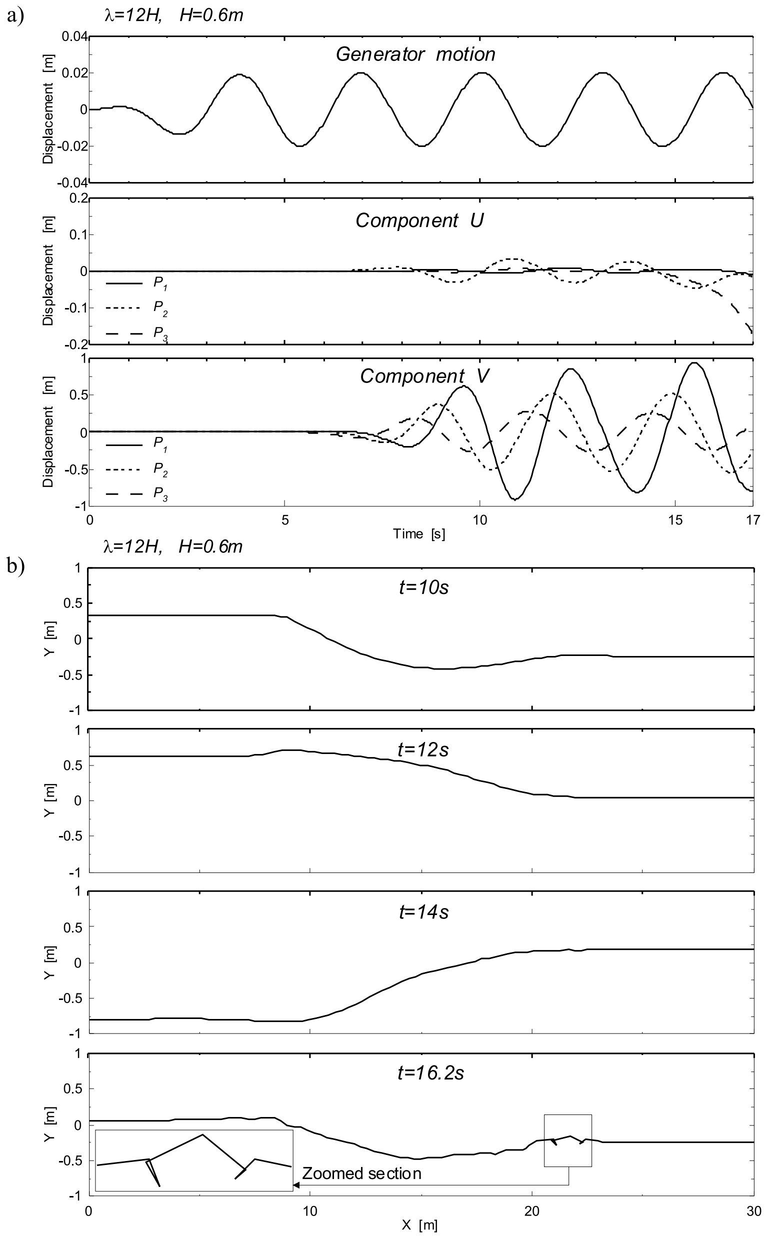

Fig. 3. Evolution in time of the displacements of the material points P_1, P_2, P_3 for the wave of length $\lambda = 12H$(a) and the distribution of the shoreline for chosen moments of time (b)

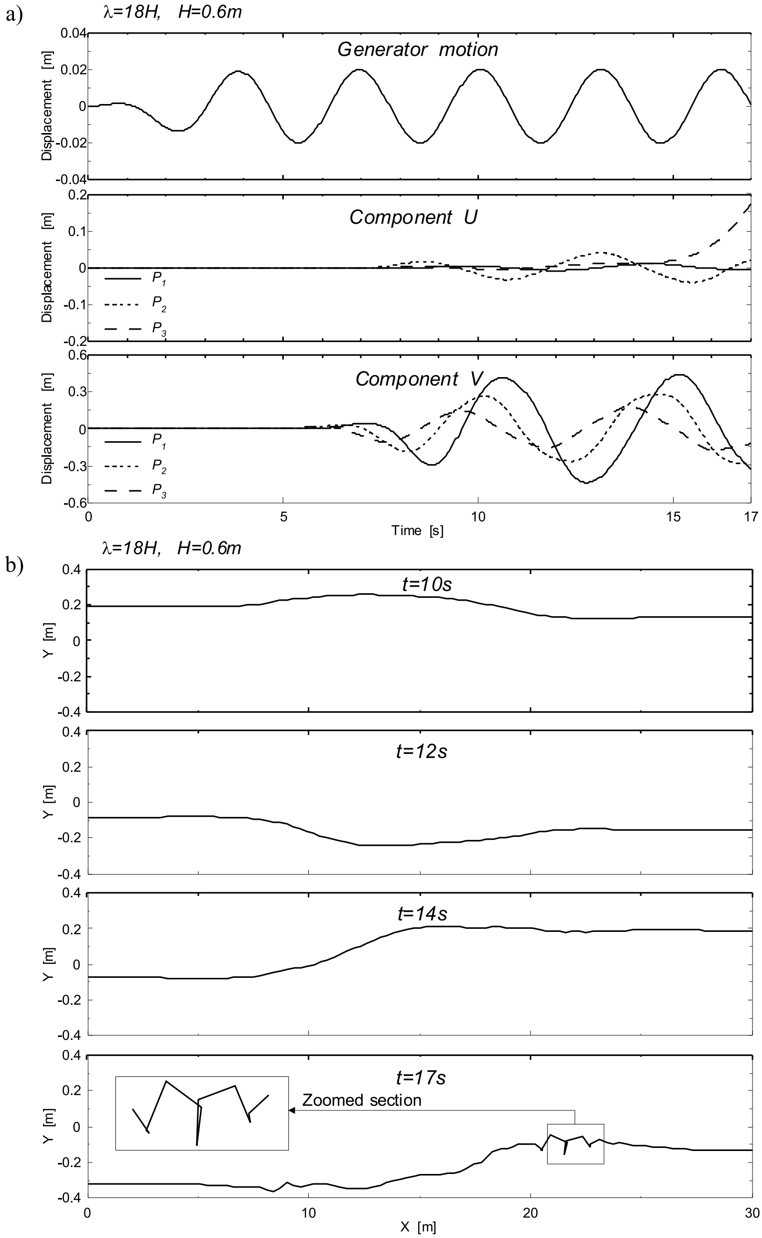

Fig. 4. Evolution in time of the displacements of the material points P_1P_2, P_3 for the wave of length $\lambda = 18H$(a) and the distribution of the shoreline for chosen moments of time (b)

5. Concluding Remarks

In order to describe a transformation of two-dimensional surface waves propagating in water of variable depth, material coordinates of fluid particles have been chosen as independent variables of the problem considered. With the material approach it is much easier to solve boundary conditions at moving boundaries of the fluid domain. The basic equations of the problem have been derived by means of the Hamilton's variation procedure. The discussion has been confined to linear momentum equations which are sufficiently accurate in describing the main features of the phenomenon. The equations have been solved for chosen particular problems of long waves approaching a sloping shore. The computations reveal that for the finite fluid domain considered, there exists an inherent set of eigenfrequencies of the system which may influence numerical results significantly. The assumed bathymetry of the fluid domain admits formation of fluid streams at the shoreline. It was shown that the bathymetry is a very important factor, which strongly influences numerical solutions. It may happen that arriving waves induce a fluid stream which corresponds to singular points emerging in the computational fluid domain. With the singularities, the topology of the domain is changed and the numerical computations break. Such cases have been indicated in the numerical solutions presented above. This means that, in principle, it is not possible to construct a general numerical scheme providing reliable results for waves approaching a fluid domain with arbitrary variations of its bottom.

References

Abbott M. B. (1979) *Computational Hydraulics*, Pitman, London.

Bathe J. (1982) *Finite Element Procedures in Engineering Analysis*, Prentice-Hall Inc., Englewood Cliffs, New Jersey.

Carrier G. F., Greenspan H. P. (1958) Water waves of finite amplitude on a sloping beach, *J. Fluid Mechanics*, **4**, 97–109.

Chan R. K. C., Street R. L. (1970) A computer study of finite-amplitude water waves, *J. Computational Physics*, **6**, 68–94.

Chybicki W. (2006) *Surface Wave Propagation over Uneven Bottom*, Wydawnictwo IBW PAN, Gdańsk (in Polish).

Chybicki W. (2007) Two-dimensional non-linear long waves in non-uniform water areas, *Inżynieria Morska i Geotechnika*, (4), 215–219 (in Polish).

Dingemans M. W. (1997) *Water Wave Propagation over UnevenBottoms*, World Scientific, Singapore, New York.

Goto C. (1979) Nonlinear Equation of Long Waves in the Lagrangian Description, *Coastal Engineering in Japan*, **22**, 1–9.

Kapiński J. (2004) Two-dimensional modelling of wave motion in shallow-water areas, *Archives of Hydro-Engineering and Environmental Mechanics*, **51** (1), 3–24.

Madsen P. A., Schäffer H. A. (1999) A review of Boussinesq-type equations for surface gravity waves, in *Advances in Coastal and Ocean Engineering*, Vol. 5, Ed. P. L.-F. Liu, World Scientific, Singapore.

Miles J., Salmon R. (1985) Weakly dispersive nonlinear gravity waves, *J. Fluid Mech.*, **157**, 519–531.

Shuto N. (1967) Run-up of long waves on a sloping beach, *Coastal Engineering in Japan,* **10**, 23–37.

Stoker J. J. (1948) The formation of breakers and bores, *Comm. Pure Applied Math.,* **1** (1), 1–87.

Szmidt K. (2006) Modelling of non-linear long water waves on a sloping beach, *Bull. of the Polish Academy of Sciences, Technical Sciences*, **54** (4), 381–389.

Wilde P., Chybicki W. (2004) Long water waves as a structure – fluid interaction problem, *Archives of Hydro-Engineering and Environmental Mechanics*, **51** (2), 95–118.

Wilde P., Wilde M. (2001) On the generation of water waves in a flume, *Archives of Hydro-Engineering and Environmental Mechanics*, **48** (4), 69–83.