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Technical Note

3D and 2D Formulations of Incremental Stress-Strain Relations for Granular Soils

Andrzej Sawicki

Institute of Hydro-Engineering PAS, ul. Kościerska 7, 80-328 Gdańsk-Oliwa, Poland,
e-mail: as@ibwpan.gda.pl

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Abstract

3D formulation of incremental relations, describing pre-failure deformations of granular soils, is presented. The starting point are respective equations formulated previously for the axi-symmetrical configuration, as that in the tri-axial apparatus. These relations, proposed for particular configuration, are generalized in the form of tensor equations for the strain increments. Similarly, the loading/unloading criterion and the instability line have been generalized for 3D conditions. A kind of cross-isotropy of granular soil is taken into account. Then, the incremental stress-strain relations for the plane strain state are re-derived from general equations, as such conditions are most often used for simulations of practically important problems. The procedure proposed in this paper is practically oriented, as the soil parameters can be determined just from the tri-axial tests.

Key words: granular soils, pre-failure deformations, instability, anisotropy, constitutive equations

1. Introduction

The mechanical behaviour of granular soils is still not well understood. There exist tens of various models, but none of them has become a standard in geotechnical engineering for many reasons, see Sawicki (2007). Therefore, there is still a need for a relatively simple model, that would enable an estimation of pre-failure deformations of granular soils. In order to achieve this goal, respective research programme has been carried out in the Institute of Hydro-Engineering for many years. This programme includes extensive experimental investigations, using modern laboratory equipment as, for example, tri-axial apparatuses enabling measurement of lateral strains. A large amount of experimental data has been collected, that have enabled formulation of semi-empirical constitutive equations, valid for tri-axial conditions, see Sawicki (2007), Sawicki and Świdziński (2007). The aim of this paper is to generalize these equations for 3D conditions, and then to re-derive respective relations for the plane strain conditions, as such a state is most often used for simulation of practically important geotechnical problems. The method applied in this paper is based on straightforward generalization of "tri-axial equations" to 3D conditions, by using techniques of tensor calculus.

2. Summary of Equations for Tri-Axial Conditions

In previous papers, e.g. Sawicki (2007), the following shapes of incremental equations, for the tri-axial conditions, have been proposed:

$$d\varepsilon_v = Mdp' + Ndq, \tag{1}$$

$$d\varepsilon_q = Pdp' + Qdq, \tag{2}$$

where:

$d\varepsilon_v = d\varepsilon_1 + 2d\varepsilon_3$	–	increment of the volumetric strain,
$d\varepsilon_q = \frac{2}{3}(d\varepsilon_1 - d\varepsilon_3)$	–	increment of the deviatoric strain,
$\varepsilon_1, \varepsilon_3$	–	vertical and horizontal strains respectively, positive in compression,
$dp' = \frac{1}{3}(d\sigma'_1 + 2d\sigma'_3)$	–	increment of the mean effective stress,
$dq = d\sigma'_1 - d\sigma'_3$	–	increment of the stress deviator,
σ'_1, σ'_3	–	vertical and horizontal effective stresses respectively, positive in compression,
M, N, P, Q	–	constitutive functions.

The constitutive functions M, N, P and Q have been determined experimentally from the tri-axial tests, for specific stress paths: the functions M and P from the pure compression tests ($dq = 0$); the functions N and Q from the pure shearing tests at constant mean effective stress ($dp' = 0$). Some of the constitutive functions (particularly N) have different shapes for the initially dilative and contractive soils. The constitutive functions have also different shapes for loading and unloading. These important processes have been defined separately for the changes of spherical and deviatoric parts of the effective stress tensor, which differs from commonly accepted definitions, see Sawicki and Świdziński (2008):

$dp' > 0$	–	spherical loading,
$dp' < 0$	–	spherical unloading,
$d\eta > 0$	–	deviatoric loading,
$d\eta < 0$	–	deviatoric unloading,

where: $\eta = q/p'$ is a new non-dimensional shear stress variable.

The basic shape of incremental equations (1) and (2) reflects the experimentally observed behaviour of granular soils, that takes into account some specific features of that behaviour. Firstly, the function N describes the volumetric changes of granular soils due to shearing, i.e. the phenomena of compaction and dilation, which are not so pronounced in classical materials. For example, in classical elasticity there should be $N = 0$. The other feature deals with anisotropic behaviour of granular soils. For isotropic materials there should be $P = 0$, as the mean stress should not influence the deviatoric strains. In the case of granular soils, it is difficult to prepare ideally isotropic samples, and the experimental stress-strain relations always display some kind of anisotropy. Therefore, this effect has been included into the constitutive relation (2) through the function P. The functions M, N, P and Q are presented in detail in previous publications, see Sawicki (2007), Sawicki and Świdziński (2007).

The incremental relations (1) and (2) take into account the unstable behaviour of granular soil before failure. In the case of initially dilative soils, this instability is characterized by the change of sign of the volumetric strain increments, i.e. during pure shearing the soil first compacts and then dilates. In the case of initially contractive soils, fully saturated and tested in undrained conditions, the unstable behaviour is characterized by rapid generation of pore pressure and reduction of effective stresses, which leads to liquefaction. These effects are related to a certain object, designated as the instability line. In the stress space $p' - q$, this object is defined by the following equation:

$$q = \Psi p', \tag{3}$$

where Ψ is a number, which should be determined experimentally. For details and respective literature see Świdziński (2006).

3. General Form of Incremental Equations

In the incremental equations (1) and (2), the independent variables are the mean effective stress p' and the stress deviator q. These quantities are related directly to the first invariant of the stress tensor:

$$I_1 = p' = \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma} \tag{4}$$

and the second invariant of the stress deviator:

$$J_2 = \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\sigma}^{dev}\right)^2, \tag{5}$$

$$\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \frac{1}{3}\,(\mathrm{tr}\,\boldsymbol{\sigma})\,\mathbf{1}, \tag{6}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor and $\mathbf{1}$ is the unity tensor. It can be easily checked that

$$q = \sqrt{3J_2}. \tag{7}$$

Similarly, one can relate the volumetric strain ε_v and the deviatoric strain ε_q to respective invariants of the strain tensor $\boldsymbol{\varepsilon}$. The first invariant of the strain tensor is the following:

$$K_1 = \mathrm{tr}\,\boldsymbol{\varepsilon} = \varepsilon_v, \tag{8}$$

and the second invariant of the strain deviator is:

$$K_2 = \frac{1}{3}\,\mathrm{tr}\left(\boldsymbol{\varepsilon}^{dev}\right)^2, \tag{9}$$

$$\boldsymbol{\varepsilon}^{dev} = \boldsymbol{\varepsilon} - \frac{1}{3}\,(\mathrm{tr}\,\boldsymbol{\varepsilon})\,\mathbf{1}. \tag{10}$$

The tri-axial deviatoric strain is related to this invariant by the following relation:

$$\varepsilon_q = \sqrt{\frac{4}{3}K_2}. \tag{11}$$

The above relations suggest that the incremental equations (1) and (2) can be generalized formally to the 3D case. Some comments about such generalization are necessary, before introducing respective equations. There are two distinct approaches in the mechanics of continuous media. The first one is based on straightforward generalization of 1D approaches, as most of experiments can be performed only in such reduced conditions. The most pronounced example is the Hooke's law. Robert Hooke performed his fundamental experiments in uni-axial conditions and, in subsequent years, his results have been generalized, in an elegant manner, to 3D conditions. The classical elasticity is still the best example of a good theory that still is applied in many technical problems. The other approach, originated by the so-called rational mechanics, suggests that the constitutive equations should be presented in the most general form, and then possibly reduced, according to circumstances. In this paper, the first approach will be followed.

The starting point to generalization are the incremental equations (1) and (2). Assume that these equations can be written in the following general form:

$$d\varepsilon_v = Adp' + BdJ_2, \tag{12}$$

$$d\boldsymbol{\varepsilon}^{dev} = \boldsymbol{C}dp' + Dd\boldsymbol{\sigma}^{dev}, \tag{13}$$

where A, B and D are some scalar functions, which possibly depend on the invariants of the effective stress tensor. $\boldsymbol{C}$ is a certain tensor, which may depend on the current stress state and the kind of initial anisotropy of soil. Experimental results indicate that the soil samples display a cross-isotropic behaviour, with the vertical axis indicating the privileged direction. The stresses and strains corresponding to this direction have customarily been distinguished by the subscript “1”, so this notation will also be used in the present paper. Also note that soils in the field conditions probably also display similar behaviour as the gravity influences their structure. Because this cross-isotropy should be included into the constitutive equations, let us introduce the following object, designated as the structural tensor:

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

The above tensor shows that the vertical direction x_1 is privileged, and that the soil behaviour in the horizontal $x_2 - x_3$ planes is isotropic. Assume that

$$\boldsymbol{C} = C\boldsymbol{S}^{dev}, \tag{15}$$

where C is a scalar function, and

$$\boldsymbol{S}^{dev} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}. \tag{16}$$

The functions A, B, C and D will be determined from the condition that Eqs. (12) and (13) reduce to Eqs. (1) and (2) in the case of tri-axial compression tests. Consider first the scalar equation (12). It follows from Eq. (7) that

$$dq = \frac{\sqrt{3}}{2\sqrt{J_2}} dJ_2. \tag{17}$$

Substitution of relation (17) into Eq. (1) leads to the following formula:

$$B = \frac{N\sqrt{3}}{2\sqrt{J_2}}. \tag{18}$$

Obviously, the following identity also holds:

$$A = M. \tag{19}$$

In order to derive Eq. (13), consider first the deviators of the effective stress and strain tensors, which are of the following forms in the case of tri-axial tests:

$$\boldsymbol{\sigma}^{dev} = q\begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & & -1/3 \end{bmatrix}, \tag{20}$$

$$\boldsymbol{\varepsilon}^{dev} = \frac{3}{2}\varepsilon_q \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}. \tag{21}$$

Comparison of Eqs. (2), (13), (16), (20) and (21) leads to the following formulae:

$$C = \frac{3P}{2}, \quad D = \frac{3Q}{2}. \tag{22}$$

Therefore, Eqs. (18), (19) and (22) relate respective functions, determined from the tri-axial experiments, to more general functions appearing in general constitutive relations. As already mentioned, the "tri-axial" functions have already been presented in previous publications, see Sawicki (2007), Sawicki and Świdziński (2007). In order to include these functions into a general framework presented in this paper, one should replace respective quantities appearing there by more general objects. For example, p' should not be replaced by the other object, as it has already a general meaning. But a new variable $\eta = q/p'$, should be replaced by $\sqrt{3J_2}\big/ p'$, according to Eq. (7).

4. Loading, Unloading and the Instability Condition

In the same way, we can generalize the conditions defining loading and unloading, introduced for tri-axial conditions in Section 2. In the case of spherical loading and unloading, the definition becomes unchanged as the mean effective stress is the invariant. In the case of deviatoric loading and unloading, respective equations should be re-arranged. Simple manipulations lead to the following formulae:

$$d\eta = \frac{1}{p'}\left(dq - \eta dp'\right) = \frac{\sqrt{3}}{p'}\left(\frac{1}{2\sqrt{J_2}}dJ_2 - \frac{\sqrt{J_2}}{p'}dp'\right). \tag{23}$$

Respective conditions, introduced in Section 2, apply for the non-dimensional deviatoric stress increment (23).

The instability condition (3) can also be generalized, and after some simple manipulations, takes the following form:

$$J_2 = \frac{1}{3}\Psi^2\left(p'\right)^2. \tag{24}$$

5. Plane Strain State

The plane strain conditions are most often analysed in geotechnical engineering for practical reasons. Assume that x_1 denotes the vertical and privileged direction, and that $\varepsilon_{22} = 0$. This latter condition is commonly applied in continuum mechanics, for the plane strain problems, in order to determine the respective stress, i.e. σ_{22}. Many practical examples can be found in the books devoted to elasticity. The other approach is based on the assumption that the intermediate stress, σ_{22} in the case considered, does not influence the overall behaviour of the soil. Respective examples may be found in almost all textbooks dealing with the limit states of soils. In this paper, this second approach will be adopted for practical reasons. General equations, derived in the previous sections, will serve only as guidelines. They are very difficult to be applied in practice for their entangled form. In order to obtain equations that are useful in practical applications, it is necessary to re-define respective relations. Therefore, it is assumed that the intermediate stress does not influence the soil behaviour, and will not be taken into account in the following relations. This means that the purely 2D problem will be considered. The strain and effective stress tensors are of the following form:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{bmatrix}, \tag{25}$$

$$\boldsymbol{\sigma}' = \begin{bmatrix} \sigma'_{11} & \sigma'_{13} \\ \sigma'_{13} & \sigma'_{33} \end{bmatrix}. \tag{26}$$

Their deviators are:

$$\boldsymbol{\varepsilon}^{dev} = \boldsymbol{\varepsilon} - \frac{1}{2}(\mathrm{tr}\,\boldsymbol{\varepsilon})\mathbf{1} = \frac{1}{2}\begin{bmatrix} \varepsilon_{11} - \varepsilon_{33} & 2\varepsilon_{13} \\ 2\varepsilon_{13} & -\varepsilon_{11} + \varepsilon_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \varepsilon & \gamma \\ \gamma & -\varepsilon \end{bmatrix}, \tag{27}$$

$$\boldsymbol{\sigma}'^{dev} = \boldsymbol{\sigma}' - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}')\mathbf{1} = \frac{1}{2}\begin{bmatrix} \sigma'_{11} - \sigma'_{33} & 2\sigma'_{13} \\ 2\sigma'_{13} & -\sigma'_{11} + \sigma'_{33} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sigma & 2\tau \\ 2\tau & -\sigma \end{bmatrix}. \tag{28}$$

The structural tensor and its deviator are the following:

$$\boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{29}$$

$$\boldsymbol{S}^{dev} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{30}$$

Note that:

$$J_2 = \frac{1}{4}\sigma^2 + \tau^2, \tag{31}$$

$$dJ_2 = \frac{\partial J_2}{\partial \sigma}d\sigma + \frac{\partial J_2}{\partial \tau}d\tau = \frac{1}{2}\sigma d\sigma + 2\tau d\tau. \tag{32}$$

The volumetric strain is the following in the 2D case:

$$\varepsilon_v = \varepsilon_{11} + \varepsilon_{33}. \tag{33}$$

The mean effective stress is defined as:

$$p' = \frac{1}{2}\left(\sigma'_{11} + \sigma'_{33}\right). \tag{34}$$

The above relations are also valid for the stress and strain increments. Note that the 2D definition of the mean effective stress differs from Eq. (4) which is valid for a general 3D state. From the practical point of view, these differences are not so large, which can be checked by elementary calculations.

In the case considered, the basic constitutive equations (12) and (13) take the following scalar forms:

$$d\varepsilon_v = Adp' + BdJ_2, \tag{35}$$

$$d\varepsilon = Cdp' + Dd\sigma, \tag{36}$$

$$d\gamma = 2Dd\tau. \tag{37}$$

Note that the strains ε_{11}, ε_{33} and ε_{13} can be easily calculated from respective equations (27) and (33), for a given loading path. The criteria of loading and unloading, introduced in Section 2, and the definition of non-dimensional shearing stress (23) remain unchanged. One should substitute into that relation Eqs. (31) and (32) respectively. The instability surface, given by Eq. (24), takes the following form:

$$\frac{1}{4}\sigma^2 + \tau^2 = \frac{1}{3}\Psi^2 \left(p'\right)^2. \tag{38}$$

The above equation represents a cone in the space σ, τ, p'.

In the special case of $\tau = 0$ and when $\sigma = \sigma_{11} - \sigma_{33} > 0$, Eq. (38) takes the following simple form:

$$\sigma = \frac{2}{\sqrt{3}}\Psi p', \tag{39}$$

and the deviatoric loading condition is:

$$d\sigma - \frac{\sigma}{p'}dp' > 0. \tag{40}$$

6. Discussion

1. The incremental equations, describing pre-failure deformations of granular soils for tri-axial configuration, have been generalized for the 3D case. The advantage of such a generalization is that the constitutive functions can be determined

from experiments performed in the tri-axial apparatus. One of original features of such an approach is that the cross-isotropic response of granular soils is taken into account. Note that most of the models of geomaterials are based on the assumption of isotropy, which generally is not supported by experimental data. The instability line and the loading and unloading criteria have also been generalized.

2. The general equations have been re-derived for the 2D case, see Eqs. (35), (36) and (37). It was shown, that the instability line for the tri-axial configuration has been generalized to the cone in this case, see Eq. (38). These results are also original. Of practical importance is a simple form of re-derived equations.
3. This paper supplements previous publications, see Sawicki (2007), Sawicki and Świdziński (2007). Experimental verification of the results obtained is planned as the next step of research, using small scale models.

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