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On the Transformation of Long Gravitational Waves in a Region of Variable Water Depth: a Comparison of Theory and Experiment

Kazimierz Szmidt, Benedykt Hedzielski

Institute of Hydro-Engineering, Polish Academy of Sciences,
ul. Waryńskiego 17, 71-310 Szczecin, Poland, e-mail: jks@ibwpan.gda.pl

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Abstract

The paper describes investigations on transformation of long gravitational waves in water of variable depth with reflection of the waves from a shelf barrier. In the model considered, a long water wave arrives from an area of constant water depth to an area of constant, smaller water depth, where it reflects at a vertical wall. The analysis is confined to a finite fluid domain, relevant to experimental investigations in a laboratory flume. In theoretical analysis of the phenomenon, we follow a non-linear shallow water approximation to the problem considered. The fundamental equations of fluid motion are derived with the help of a standard variational procedure in a material system of coordinates. The equations proved to be a reasonable approximation to a description of the long waves propagating in fluid with small variation of its depth. In the discussed case of reflection of such waves from a vertical barrier, however, the motion of the fluid is more complicated and therefore the long water wave theory does not deliver as good results as in the case of pure propagation of the waves. The primary objective of this paper is thus to compare the theoretical solution proposed with data obtained in experiments, and to answer the question about accuracy and applicability of the theoretical model in the description of the problem investigated.

Key words: shallow water, non-linear wave, non-uniform water depth, unsteady motion, wave reflection

1. Introduction

In theoretical description of long water waves, i.e. when their wave-length is much greater than the water depth, a vertical momentum equation is usually assumed in such a form that the equation may be integrated independently from equations corresponding to horizontal variables. In this way, the three-dimensional flow problem is reduced to a two-dimensional one. In the case of a plane problem, considered in this paper, the reduction leads to one-dimensional in space problem of the fluid flow. A particular assumption in a description of the vertical acceleration term in the vertical momentum equation leads to specific equations describing the waves. For example, the assumption that the fluid pressure is hydrostatic leads to the so-called Airy theory of simple wave (Stoker 1948). Another important formulation of the shallow water theory belongs to Boussinesq, and Korteweg and de Vries (Mei 1983, Whitham 1974). Following the assumption on the potential motion of the fluid, the authors derived equations describing evolution of solitary waves (Ursell 1953). In the case of a non-uniform bottom, a derivation of the relevant shallow water equations becomes more complicated. In the case of a constant bottom slope however, Stoker (1948) obtained a closed form solution to a simple non-linear wave on a sloping beach. Carrier and Greenspan (1958) gave an exact solution to a time dependent non-linear problem of waves of finite amplitude climbing on a sloping beach, by means of a modified hodograph transformation (Whitham 1979). A review of the Boussinesq-type equations for surface waves may be found in Madsen and Schäffer (1999). In particular, it has been found that the Boussinesq-type equations considered have led to solutions of acceptable accuracy of the problem of waves propagating in water with slow variation of its depth. All the above formulations have been based on the Eulerian approach with space coordinates as independent variables. With this formulation it is relatively easier to derive fundamental equations of the fluid motion. A certain drawback of the approach is a solution to boundary conditions, especially conditions on the free surface. On the other hand, the formulation of the problem within material coordinates, as independent variables, ease the description of boundary conditions, but, at the same time, it results in a more complicated structure of equations of conservation of mass and momentum. As concerns the material approach, Shuto (1967) applied the Lagrangian description to the problem of run-up of long waves on a sloping beach. The discussion was confined to the first order approximation of basic equations, for which a vertical acceleration term does not appear in momentum equations. Theoretical results obtained were compared with experimental data. A non-linear long water theory in the Lagrangian description has been developed by Goto (1979). The basic equations, for the uniform and non-uniform water depth, have been derived under assumption of hydrostatic pressure. A detailed discussion of a number of problems associated with the long water waves propagating over uneven bottoms together with an extensive bibliography of the subject may be found in the Dingemans monograph (1997). In most of the classical theories, the continuity and momentum equations correspond to an average, over the water depth, horizontal component of the velocity field. This means that all fluid particles distributed along the water depth above a chosen point of the bottom, at a given instant of time, are subject to a common infinitesimal displacement within an infinitesimal change of time. This kinematical assumption has been taken as the starting point in description of shallow water waves propagating in water of constant depth developed by Wilde and Chybicki (2004). The authors assumed that vertical material lines of fluid particles remain vertical during the entire motion of the fluid. An extension of the latter formulation to the case of small variation of the water depth (partially sloping bottom) may be found in

Chybicki (2006) and Szmidt (2006). It has been found (Chybicki 2006) that the theoretical formulation is good enough for description of long waves propagating in fluid of non-uniform depth (the difference between theoretical and experimental results for the long waves considered was less than 5%). With respect to the above, in the present paper the problem of reflection of long water waves from a vertical barrier is investigated. In the model considered, a long wave arrives at the area of non-uniform water depth where it reflects from a vertical barrier. Before reaching the barrier, the wave undergoes changes resulting from variation of the water depth. The main objective of the present paper is to examine accuracy of the theoretical approach to the problem of transformation and reflection of long waves. In accordance with this objective, we will consider a motion of a finite fluid domain, as part of a laboratory flume, with linear variation of its depth. The fluid motion is induced by a piston-type wave maker starting to move at a certain moment of time. After a finite elapse of time from the starting point, the generated waves will reflect at a rigid vertical wall installed a certain distance from the wave-maker, in the area of a smaller water depth. The problem considered corresponds to real conditions when long water waves arrive from the sea to the area of diminishing water depth, where they reflect at a vertical barrier. One may expect that the formulation presented in this paper may be good enough for some waves while, at the same time, it may fail to deliver results of acceptable accuracy for other waves. The investigations are assumed to allow us to answer the question about the range of application of the formulation in description of the afore-mentioned problems. The analysis is performed with the help of the material description of the phenomenon, and, in a sense, it is a continuation of the problem discussed in Szmidt (2006). To make the discussion clear, some of the results obtained in that earlier work will be summarized below.

2. Theoretical Solution to the Initial Value Problem of Waves in Water of Non-Uniform Depth

We consider a plane problem of fluid motion in the finite domain as shown schematically in Fig. 1. The motion is induced by the piston-type generator placed on the left hand side of the domain (the vertical wall AF in the figure). The generator starts to move at a certain moment of time. Our aim is to solve the initial value problem of the fluid flow and find the free surface elevation at chosen space points of the domain as functions of time. The motion of the generator is assumed in advance as approaching the harmonic motion within a finite elapse of time, measured from the starting point. In order to describe the generator motion, we apply here the results developed in Wilde and Wilde (2001). Accordingly, the displacement of the piston-type generator is described by the formula

$$u_0(t) = G[A(\tau)\cos(\omega t) + D(\tau)\sin(\omega t)], \tag{1}$$

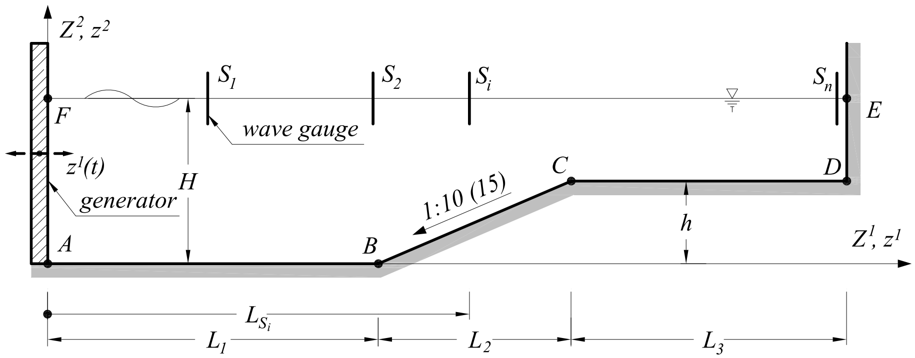

Fig. 1. A finite fluid domain with experimental equipment

where G is the amplitude of the generation, ω is the angular frequency, and

$$\begin{gathered} A(\tau) = \frac{1}{4!}\tau^4 \exp(-\tau), \\ D(\tau) = 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3 + \frac{1}{4!}\tau^4\right)\exp(-\tau), \quad \tau = \eta t, \end{gathered} \tag{2}$$

where $\eta[s^{-1}]$ is a parameter describing a growth in time of the generator displacement, and τ is the non-dimensional time factor.

One can check that, with increasing time, the generation will approach a simple harmonic motion with the prescribed frequency and amplitude. The generated waves will reflect from the right boundary. In order to describe the fluid motion, we introduce the Cartesian system of co-ordinates in an actual configuration ($z^r, r = 1, 2$), and a similar system in a reference configuration denoted by capital letters ($Z^\gamma, \lambda = 1, 2$). The co-ordinates of the latter system define names of the fluid particles (positions of the particles at the initial moment of time). Moreover, it is convenient to introduce a common Cartesian system of co-ordinates. The motion of the fluid is described as the mapping of the names into actual positions occupied by the material points

$$\begin{aligned} z^1(Z^\alpha, t) &= Z^1 + u(Z^1, t), \\ z^2(Z^\alpha, t) &= Z^2 + v(Z^\alpha, t), \end{aligned} \tag{3}$$

where $\alpha = 1, 2$, and $u\left(Z^1, t\right)$ and $v\,(Z^\alpha, t)$ are horizontal and vertical components of a displacement vector, respectively.

For the case of an uneven bottom, it is assumed that the vertical component of the displacement field is described by the formula:

$$v\left(Z^{\alpha},t\right)=h\left(Z^{1}+u\right)-h\left(Z^{1}\right)+\frac{w\left(Z^{1},t\right)}{H-h\left(Z^{1}\right)}\left[Z^{2}-h\left(Z^{1}\right)\right]. \quad (4)$$

In the equation $w(Z^1, t)$ denotes the vertical displacement of material points of the free surface, H is the water depth, and and $h(Z^1)$ describes the bottom (see Fig. 1). In what follows we confine our attention to the approximation

$$f\left(Z^{1},t\right)=h\left(Z^{1}+u\right)-h\left(Z^{1}\right)\cong u\left(Z^{1},t\right)h'\left(Z^{1}\right)+\frac{1}{2}u^{2}h''\left(Z^{1}\right), \quad (5)$$

where the primes denote differentiation with respect to Z^1. For example, the derivative $h'(Z^1) = dh\big/dZ^1$ defines the slope of the bottom. Having components of the displacement field for the incompressible fluid, we can calculate the Jacobian of the transformation (1)

$$J=\det\left[z^{i}_{,\alpha}\right]=(1+u')\left(1+\frac{w}{H-h}\right)=1. \quad (6)$$

From the equation it follows that

$$w(Z^{1},t)=-(H-h)\frac{u'}{1+u'}, \quad (7)$$

and finally

$$v(Z^{\alpha},t)=f(Z^{1},t)-\frac{u'}{1+u'}(Z^{2}-h). \quad (8)$$

Having the displacement field, it is a simple task to calculate the vertical velocity

$$\dot{v}\left(Z^{\alpha},t\right)=\dot{f}\left(Z^{1},t\right)-\frac{\dot{u}'}{(1+u')^{2}}\left(Z^{2}-h\right), \quad (9)$$

where the dots denote differentiation with respect to time.

In a similar way, the vertical acceleration of fluid particles is obtained

$$\ddot{v}=\ddot{u}\left(h'+uh''\right)+(\dot{u})^{2}h''-\frac{1}{(1+u')^{2}}\left[\ddot{u}'-2\frac{(\dot{u}')^{2}}{(1+u')}\right]\left(Z^{2}-h\right). \quad (10)$$

Knowing the acceleration, one may calculate the fluid pressure from the momentum equation

$$\frac{\partial p}{\partial z^{2}}=-\rho[g+\ddot{v}(Z^{\alpha},t)]. \quad (11)$$

Following the displacement field, we can calculate the potential energy of the fluid

$$E_{pot.} = \rho g \int\limits_0^L \int\limits_h^H z^2(Z^\alpha, t) J dZ^2 dZ^1 = \\ = \frac{1}{2}\rho g H \int\limits_0^L \left[H(1-\alpha^2) + 2h'(1-\alpha) - H(1-\alpha)^2 \frac{u'}{1+u'} \right] dZ^1, \tag{12}$$

where

$$\alpha = \alpha\left(Z^1\right) = \frac{h\left(Z^1\right)}{H} \tag{13}$$

describes the bottom change. At the same time, the kinetic energy of the fluid is described by the formula

$$E_{kin.} = \frac{1}{2}\rho \int\limits_0^L \int\limits_h^H \left[(\dot{u})^2 + (\dot{v})^2\right] J dZ^2 dZ^1. \tag{14}$$

By substituting of the velocity component (8) into the last equation, the following relation results

$$E_{kin.} = \frac{1}{2}\rho H \int\limits_0^L \left[\left(1 + h'^2\right)(1-\alpha)(\dot{u})^2 - Hh'(1-\alpha)^2 \frac{\dot{u}\dot{u}'}{(1+u')^2} + \right. \\ \left. + \frac{1}{3}H^2 (1-\alpha)^3 \frac{(\dot{u}')^2}{(1+u')^4} \right] dZ^1. \tag{15}$$

Fundamental equations of the problem are derived by means of a standard variational procedure. For the conservative system considered, the variation of the action integral reads

$$\delta I = \delta \int\limits_0^{t_k} \left(E_{kin.} - E_{pot.}\right) dt. \tag{16}$$

Substitution of equations (12) and (15) into the last relation gives

$$\delta I = \frac{1}{2}\rho H \int\limits_0^{t_k} \int\limits_0^L [R_1 \delta\dot{u} + R_2 \delta u' + R_3 \delta\dot{u}'] dZ^1 dt + \\ + \frac{1}{2}\rho g H^2 \int\limits_0^{t_k} \int\limits_0^L \left[G_7 \delta u' - \frac{2h'}{H}(1-\alpha)\delta u \right] dZ^1 dt = 0, \tag{17}$$

where

$$
\begin{gathered}
R_1 = 2(1+h'^2)G_1 - Hh'G_3, \quad R_2 = 2Hh'G_2 - \frac{4}{3}H^2G_6, \\
R_3 = \frac{2}{3}H^2G_5 - Hh'G_4,
\end{gathered}
\tag{18}
$$

and

$$
\begin{gathered}
G_1 = (1-\alpha)\dot{u}, \quad G_2 = (1-\alpha)^2\frac{\dot{u}\dot{u}'}{(1+u')^3}, \quad G_3 = (1-\alpha)^2\frac{\dot{u}'}{(1+u')^2}, \\
G_4 = (1-\alpha)^2\frac{\dot{u}}{(1+u')^2}, \quad G_5 = (1-\alpha)^3\frac{\dot{u}'}{(1+u')^4}, \quad G_6 = (1-\alpha)^3\frac{(\dot{u}')^2}{(1+u')^5}, \\
G_7 = (1-\alpha)^2\frac{1}{(1+u')^2}.
\end{gathered}
\tag{19}
$$

Finally, simple manipulations of the integrands in equation (17) lead to the equation

$$
\begin{gathered}
-\int_0^{t_k}\int_0^{L}\left[\dot{R}_1 + R_2' - \dot{R}_3' + gHG_7' + 2gh'(1-\alpha)\right]\delta u dZ^1 dt + \left. R_3\delta u\right|_0^{t_k}\Big|_0^{L} + \\
+\int_0^{L}\left[R_1 - R_3'\right]\delta u|_0^{t_k} dZ^1 + \int_0^{t_k}\left[R_2 - \dot{R}_3 + gHG_7\right]\Big|_0^{L}\delta u dt = 0.
\end{gathered}
\tag{20}
$$

For the discussed case of fluid motion starting from rest, the arbitrary variation δu vanishes at the end time points, i.e. for $t = 0$ and $t = t^k$, and the variation disappears at the end points of the domain i.e. for $Z^1 = 0$ and $Z^1 = L$ (point D in Fig. 1), respectively. At the same time, we require (20) to vanish for all $\delta u(Z^1, t)$, which implies

$$
-R_0 + \dot{R}_1 + R_2' - \dot{R}_3' + gHG_7' + 2g(1-\alpha)(h' + h''u) = 0. \tag{21}
$$

The equation obtained is the horizontal momentum equation for the water motion within the finite domain. Substitution of the descriptions (18) and (19) into the equation provides a relatively complex non-linear partial differential equation with respect to the material variable and time.

3. Approximate Solutions to the Momentum Equation

In order to find a solution of the momentum equation (21) we resort to approximate formulation in which it is assumed that the displacement $u(Z^1, t)$ possesses the power series expansion with respect to a parameter ε (Stoker 1957, Wehausen and Laitone 1960)

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \ldots. \tag{22}$$

Substituting (22) into equation (21) and collecting terms with the same power of the parameter, a system of linear equations is obtained. In order to simplify the analysis we limit our consideration to the two lowest powers of the expansion. The first order approximation of the equation reads

$$\begin{aligned} &\left[1 + \frac{1}{2}Hh''(1-\alpha)\right]\ddot{u}_1 - \frac{1}{3}H^2(1-\alpha)^2\ddot{u}_1'' + Hh'(1-\alpha)\ddot{u}_1' + \\ &\quad - gH(1-\alpha)u_1'' + 2gh'u_1' + gh''u_1 = 0. \end{aligned} \tag{23}$$

For $h = 0$ the equation reduces to the case of constant water depth (Wilde 1999). Similarly, the second power terms in the expansion lead to the following equation

$$\begin{aligned} &\left[1 + \frac{1}{2}Hh''(1-\alpha)\right]\ddot{u}_2 - \frac{1}{3}H^2(1-\alpha)^2\ddot{u}_2'' + Hh'(1-\alpha)\ddot{u}_2' + \\ &\quad - gH(1-\alpha)u_2'' + 2gh'u_2' + gh''u_2 + NL = 0, \end{aligned} \tag{24}$$

where the non-linear term (NL term) in the equation depends on the first order solution:

$$\begin{aligned} NL \cong\ & \frac{1}{3}H^2(1-\alpha)^2[-\ddot{u}_1''u_1' + 4\ddot{u}_1'u_1'' + 4\dot{u}_1'\dot{u}_1'']+ \\ & + Hh'(1-\alpha)\left[\ddot{u}_1' - (\dot{u}_1')^2 - \ddot{u}_1u_1''\right]+ \\ & + \left[5 + 2(h')^2 + 2Hh''(1-\alpha)\right]\ddot{u}_1u_1' + \\ & + Hh''(1-\alpha)\dot{u}_1\dot{u}_1' + h'h''u_1\ddot{u}_1 + \\ & + 7gh'(u_1')^2 - 2gH(1-\alpha)u_1''u_1' + 5gh''u_1u_1'. \end{aligned} \tag{25}$$

Although the equations (23) and (24) are linear, they have variable coefficients and, thus, they are still difficult to solve analytically. Therefore, in order to get solutions of the equations we resort to a discrete formulation by means of the finite difference method. With the discrete approach, the space derivatives with respect to the independent variable Z^1 are substituted by finite difference quotients. The equations differ by the NL term, and therefore, it is convenient to omit the lower indices of the dependent variables in the further discussion. For the assumed spacing a = const. of nodal points within the fluid domain, a finite difference analogue of equation (23) is written for all the points $k = 1, 2, \ldots, N$ as continuous in time and discrete in space system of differential equations. For a typical point k ($Z^1 = ka$) within the fluid domain, the differential equation is written in the form

$$-W_1\ddot{u}_{k-1} + W_2\ddot{u}_k - W_3\ddot{u}_{k-1} - S_1u_{k-1} + S_2u_k - S_3u_{k+1} = 0, \tag{26}$$

where:

$$\begin{aligned} W_1 &= \frac{1}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2 + \frac{1}{2}h'\frac{H}{a}(1-\alpha), \\ W_2 &= 1 + \frac{1}{2}Hh''(1-\alpha) + \frac{2}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2, \\ W_3 &= \frac{1}{3}\left(\frac{H}{a}\right)^2(1-\alpha)^2 - \frac{1}{2}h'\frac{H}{a}(1-\alpha), \end{aligned} \tag{27}$$

and

$$\begin{aligned} S_1 &= \frac{1}{a}\left[\frac{gH}{a}(1-\alpha) + gh'\right], \\ S_2 &= 2\frac{gH}{a^2}(1-\alpha), \quad S_3 = \frac{1}{a}\left[\frac{gH}{a}(1-\alpha) - gh'\right]. \end{aligned} \tag{28}$$

The final set of equations (26) is written in the matrix form

$$[\mathbf{AM}]\left(\ddot{\mathbf{U}}\right) + [\mathbf{BM}](\mathbf{U}) + (\mathbf{P}) = \mathbf{0}, \tag{29}$$

where:

$$\begin{aligned} (\mathbf{U})^T &= (u_1, u_2, \ldots, u_N), \\ \left(\ddot{\mathbf{U}}\right)^T &= (\ddot{u}_1, \ddot{u}_2, \cdots, \ddot{u}_N), \\ (\mathbf{P})^T &= (-W_1\ddot{u}_0 - S_1u_0, 0, 0, \cdots, 0)\,. \end{aligned} \tag{30}$$

The vector $(\mathbf{P})^T$ in the equations depends on the generator motion.

With respect to the notations (27) and (28), the matrix $[\mathbf{AM}]$ assumes the following form

$$[\mathbf{AM}] = \begin{bmatrix} W_2 & -W_3 & & & & \\ -W_1 & W_2 & -W_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & -W_1 & W_2 & -W_3 \\ & & & & -W_1 & W_2 \end{bmatrix}. \tag{31}$$

In a similar way, the matrix [**BM**] reads

$$[\mathbf{BM}] = \begin{bmatrix} S_2 & -S_3 & & & & \\ -S_1 & S_2 & -S_3 & & & \\ & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot \\ & & & -S_1 & S_2 & -S_3 \\ & & & & -S_1 & S_2 \end{bmatrix}. \tag{32}$$

It should be noted that the non-zero elements of the matrices depend on the independent variable Z^1 of the considered point, but do not depend on time. It may be seen that the first equation of the set (29) contains terms $\ddot{u}_0$ and u_0 which are known functions of time prescribed by the boundary conditions at $Z^1 = 0$. In order to perform integration of equation (29) with respect to time, we introduce the discrete time and make use of the Wilson θ method. In this method, the acceleration between the subsequent time steps is approximated by a linear function of time. For a mechanical system, the procedure is unconditionally stable for $\theta > 1.37$ (Bathe 1982). Having the displacement vector $(\mathbf{U})^T = (u_1, u_2, \ldots, u_N)$ we can calculate its approximate derivative with respect to the horizontal variable. Then, with the help of equation (7), one can calculate the vertical displacements of the free surface.

4. Experiments in a Laboratory Flume

The experimental set-up together with an arrangement of wave gauges is shown schematically in Fig. 1. The laboratory experiments were conducted in the wave flume of the Institute of Hydro-Engineering of PAS in Gdańsk. The wave channel is 64 m long with a cross section of 0.6 m broad and 1.4 m high and is equipped with a programmable wave-maker. The undisturbed water depth was chosen to be 0.6 m. The water waves were generated by a piston-type wave maker placed at the beginning of the flume. A rigid, plane inclined ramp was installed at a distance of 6 m from the generator. The ramp was made of waterproof plywood, supported by a steel structure. Two slopes of the inclined ramp were considered (height to length ratio of the slope was chosen to be 1:10 or 1:15). The right vertical wall was placed 4 m from the right end of the ramp. The experiments were conducted for chosen sets of amplitude and frequency of the generator motion. The motion approaches the case of steady state harmonic motion within a few first periods inherent to an assumed frequency. The frequency had been calculated from fundamental dispersion relation for water of constant depth equal to 0.6 m for a set of the assumed wave lengths. The experiments were performed for the wave length ranging from $\lambda = 2H$ ($\omega_g = 7.1494$ s^{-1}) up to $\lambda = 16H$ ($\omega_g = 1.5461$ s^{-1}) with the length step $\Delta\lambda = 2H$, where $H = 0.6$ m is the calm water depth in the generator vicinity. The amplitudes of the generation changed from 1.2278 cm for the highest frequency up to 4.6413 cm for the smallest frequency of the generation. The generator stroke was measured by means of a horizontal displacement gauge. The wave-maker motion was measured for each run. The measurements of the free surface elevation at chosen space points have been carried out by means of water wave gauges installed at chosen distances from the wave-maker (see Fig. 1). The first water wave gauge was installed at a distance of $L_{S_1} = 3$ m from the generator. The successive gauges were installed at points $L_{S_2} = L_1$, $L_{S_3} = L_1 + L_2/2$, $L_{S_4} = L_1 + L_2$, $L_{S_5} = L_1 + L_2 + L_3/2$ and $L_{S_6} = L_1 + L_2 + L_3$. For the slope $1 : 10$, $L_1 = 6$ m and $L_2 = 3$ m, while for the smaller slope $L_1 = 4.5$ m and $L_2 = 4.5$ m, respectively. Segment L_3 was the same for both slopes of the ramp. The data obtained in experiments was recorded by means of a PC computer with sampling frequency equal to 200 Hz. Some of the results obtained in experiments are presented in the subsequent Figs. 2 and 3, where the plots represent the distribution in time of the free surface elevations measured by the set of wave gauges. Obviously, all the plots shown in the figures depend on the frequency and the amplitude of the generator motion. From comparison of the plots in the figures it may be seen that the distribution in time of the elevation at different horizontal points changes significantly. For instance, the amplitude of the free surface elevation at point B (see Fig. 1) is much smaller than the amplitude measured at other points of the horizontal coordinate. Moreover, as compared to the first gauge (S_1 in Fig. 1), higher order components have an important share in the distribution of the elevation in time. It may be seen that, within the first elapse of time (approximately 15 s from the starting point) there are no significant differences between the two first plots. With the passage of time, however, the difference between the two plots increases. The latter results from the reflection of the water waves from the right and lower boundaries of the fluid domain. The reduction of the wave height at point B is a characteristic feature of the experiments conducted in the laboratory flume. Moreover, from the plots it may be seen that the free surface elevations grow in time. The observed growth of the free surface elevation results from a wave reflection together with a resonance phenomenon, which may occur for the finite fluid domain considered. As a matter of fact, the finite fluid domain forms a mechanical system with its own set of eigenfrequencies. Therefore, in general, one may expect that some of the components of the fluid motion may fall into a resonance range, and may thus be strengthened. The strengthening of the elevation will lead to a breaking phenomenon which necessarily emerges within a relatively large lapse of time measured from the starting point. In order to avoid such a possibility, the experiments have been terminated at a proper moment of time. With the passing time, in addition to the components corresponding to the generator frequency, higher order components, corresponding to higher frequencies, emerge. In order to extract main components of the time records, a digital Kalman filter method has been applied. With the help of the filtration one can decompose an original record into components corresponding to a multiple of the leading frequency. For illustration, the decomposition of the free surface elevation, recorded

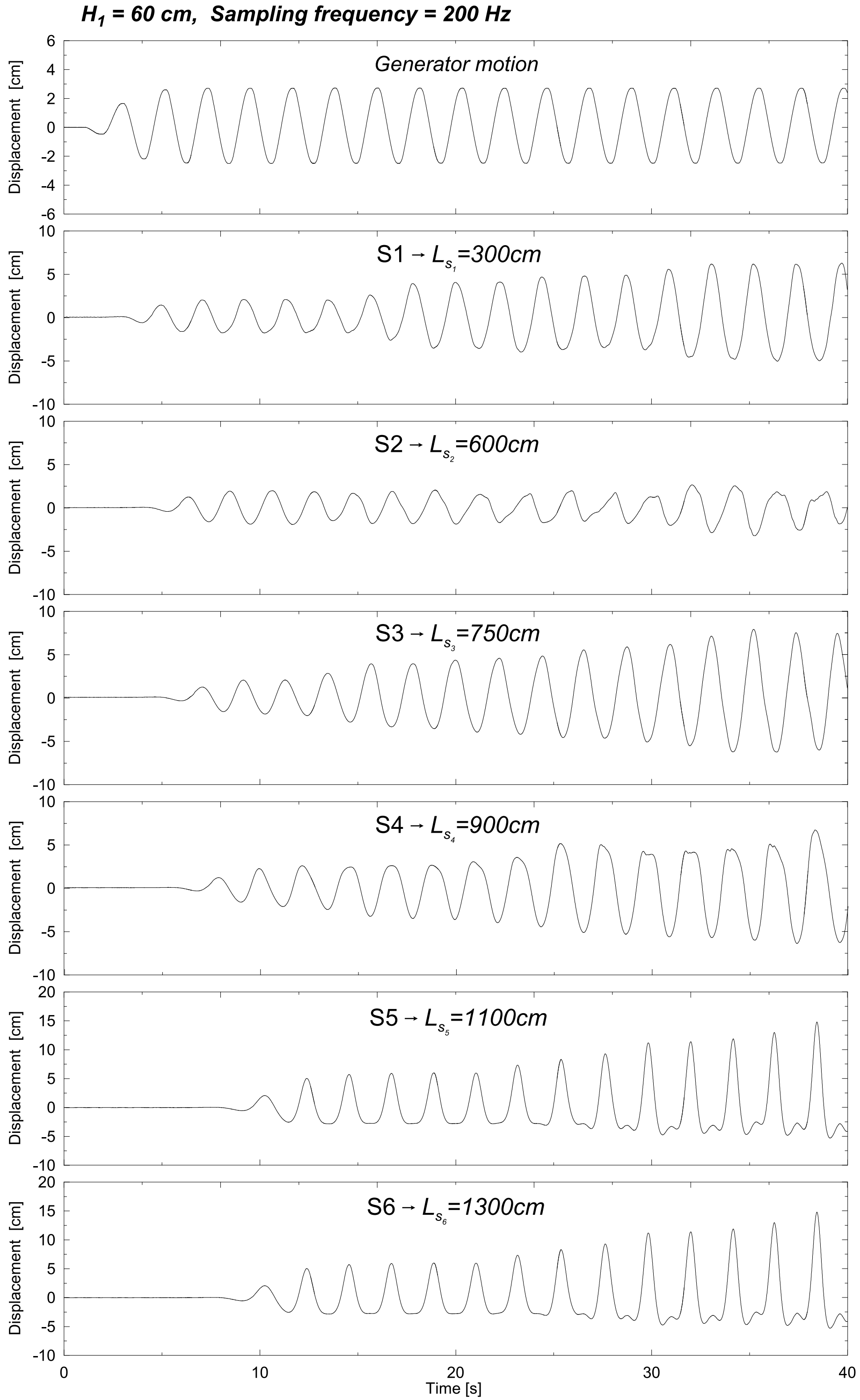

Fig. 2. Surface elevations for the bottom slope 1 : 10, recorded in laboratory experiments

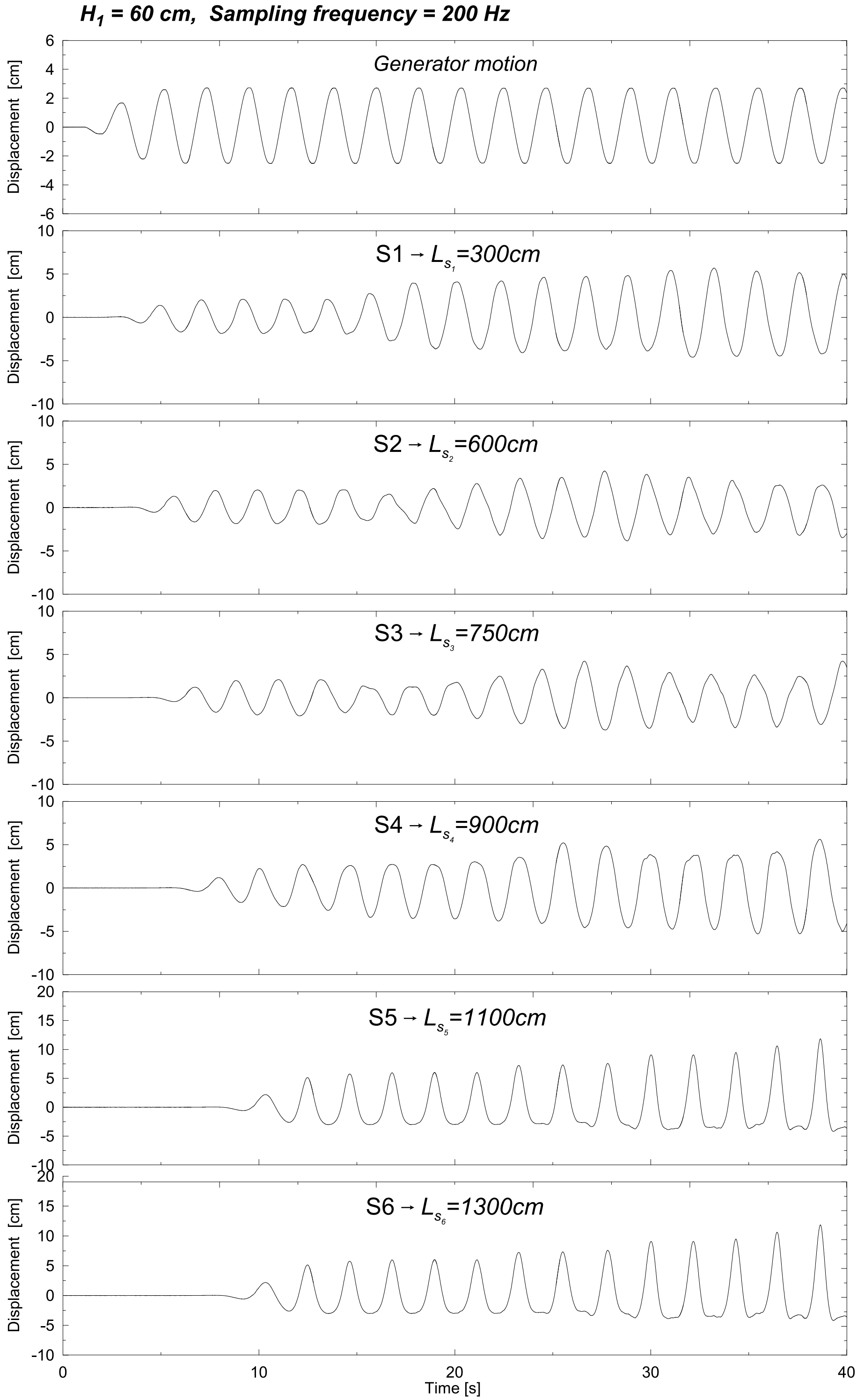

Fig. 3. Surface elevations for the bottom slope 1 : 15, recorded in laboratory experiments

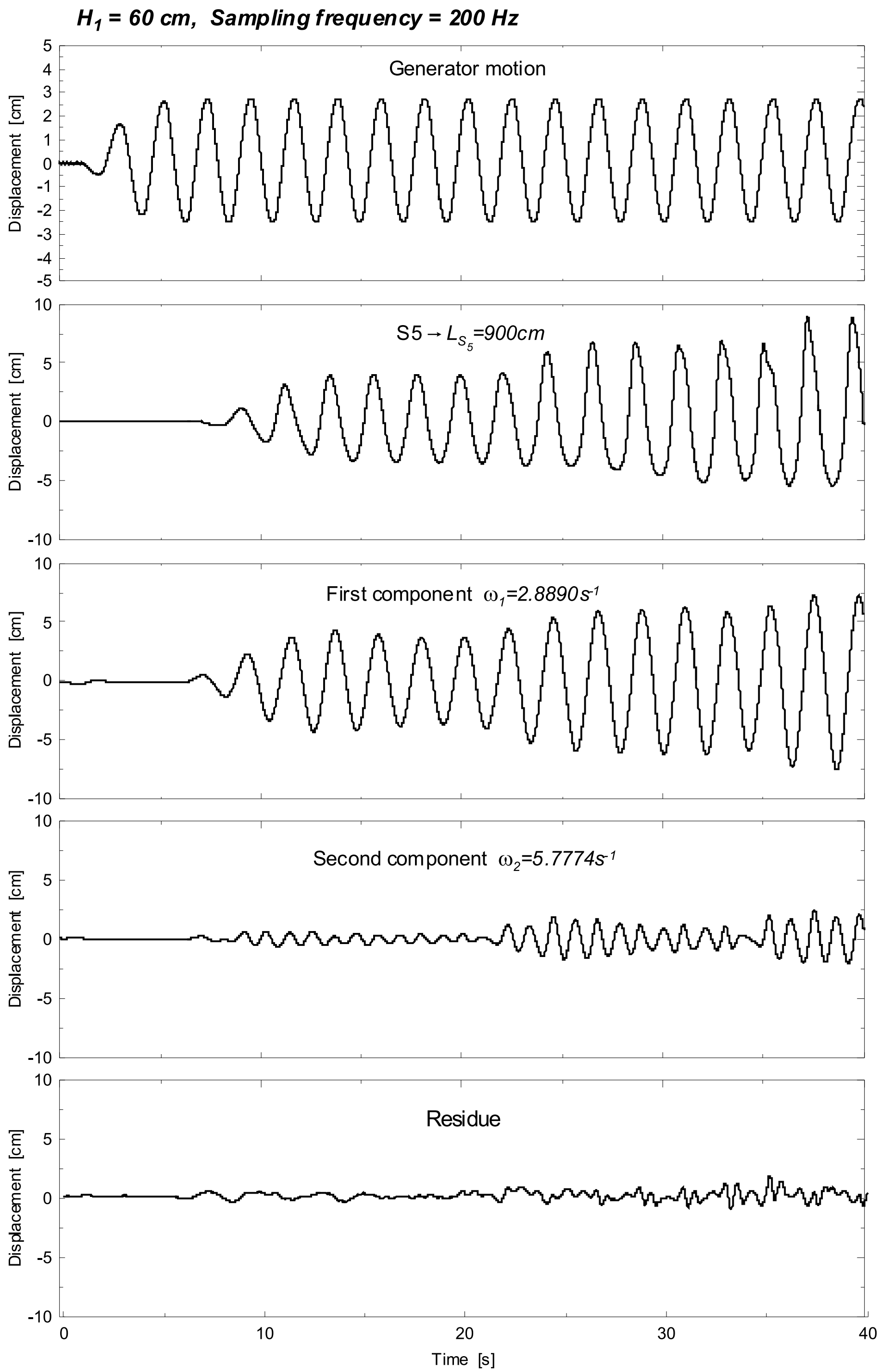

Fig. 4. Decomposition of the original record into components corresponding to leading frequency and its double for the S_5 wave gauge

by the wave gauge S_5, into components is shown in Fig. 4. It may be seen that, with the passage of time, due to the reflection phenomenon, an influence of the higher order components on final results becomes more important.

5. A Comparison of Theory and Experiments

The main objective of the research is to answer the question about accuracy and a range of applicability of the theoretical description of the phenomenon. Therefore, in what follows, we will focus our attention on theoretical solutions to cases chosen in the laboratory experiments. The data obtained in experiments is compared with theoretical results. In this way judgment on the applicability of the description is formulated. At the same time, it is perhaps important to emphasise here that the theory presented above has been derived under the assumption of a moderate wave height and sufficient length of a generated wave. Therefore, one may expect higher discrepancies between calculated and measured parameters for waves of higher amplitudes. The equations of fluid motion derived in the preceding sections enable us to obtain a linear solution to the problem and estimate a second order solution by means of the perturbation scheme applied. But, even in the linear case, the resulting momentum equation is a partial differential equation with variable coefficients depending on the space coordinate, and thus, in order to integrate the equation, we have to resort to the finite difference method. With this method, the momentum equation has been substituted by a system of ordinary differential equations with respect to time. Some of the results obtained in calculations are shown in Figs. 5 and 6. The theoretical results shown in the figures are compared with those obtained in experiments, which have been shown earlier in Figs. 2 and 3. The differences between the two sets of plots depend on the amplitude and length of a generated wave as well as the shape of the fluid domain. From the plots it may be seen that the differences between them depend also on the horizontal coordinates, where the free surface elevations were measured and calculated. The shift in the phase of the plots, especially within the first range of time, say up to 15 s from the starting point, is a result of a hydraulic oil servomechanism system responsible for the wave-maker motion. Therefore, in order to obtain a qualitative measure of comparison of the records, one may calculate squares of the elevations, or absolute values of them, and take their averages within an assumed range of time. Such a comparison will depend on the wave amplitude. Another possibility, chosen in the present paper, is a comparison of spectral densities of the records resulting from the finite Fourier transforms of them. Some of the results obtained in this way are shown in Figs. 7 and 8. The graphs in Fig. 7 represent the lowest transform amplitudes for the free surface elevation records corresponding to the successive wave gauges. All the cases presented in the figure are associated with the generation frequency $\omega = 3.666$ s^{-1} relevant to the wave of length $\lambda = 8H$, where H is the calm water depth. Like

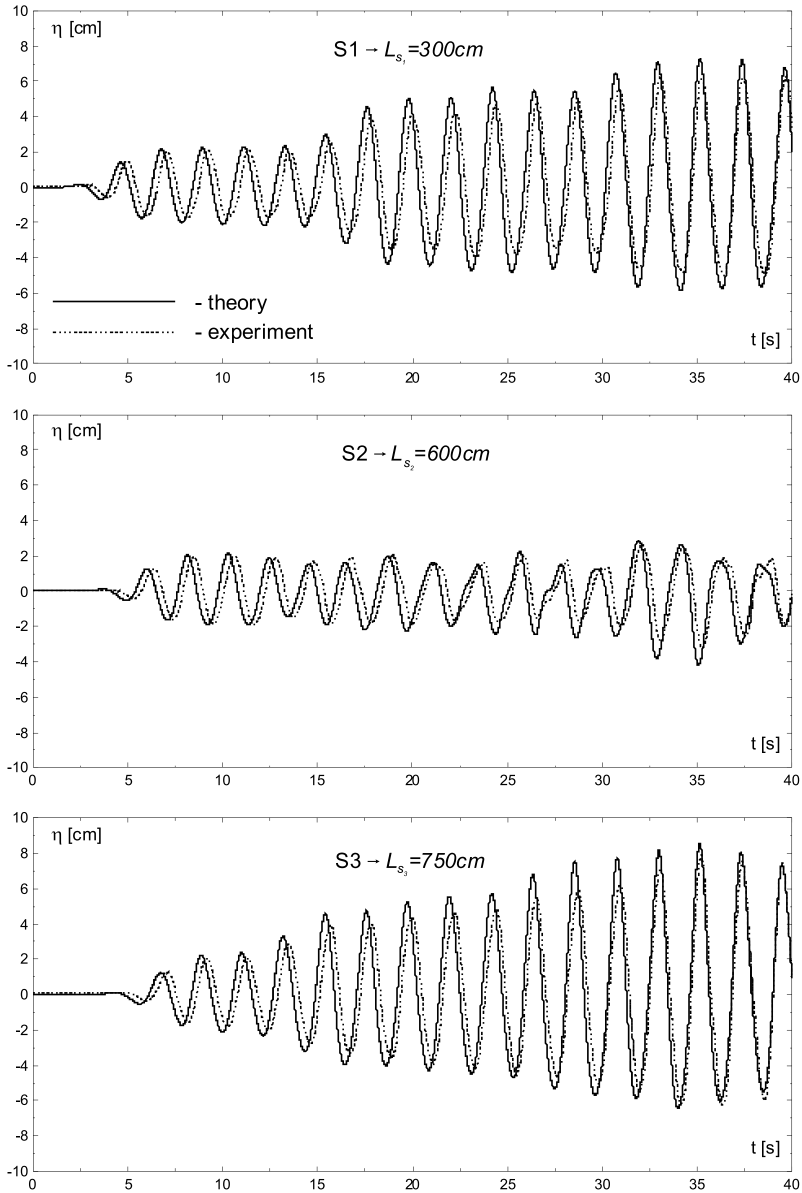

Fig. 5. Comparison of theoretical results with data obtained in experiments for the bottom slope 1 : 10

in the previous figure, the plots in Fig. 8 represent also the lowest amplitudes of the Fourier transforms, but now all the plots correspond to the point $x = x(S_5)$. The figure illustrates the accuracy of the theoretical solution for waves of different lengths. From the plots it may be seen that the theory provides reliable results for water waves of lengths ranging from $\lambda = 4H$ to $\lambda = 16H$. At the same time,

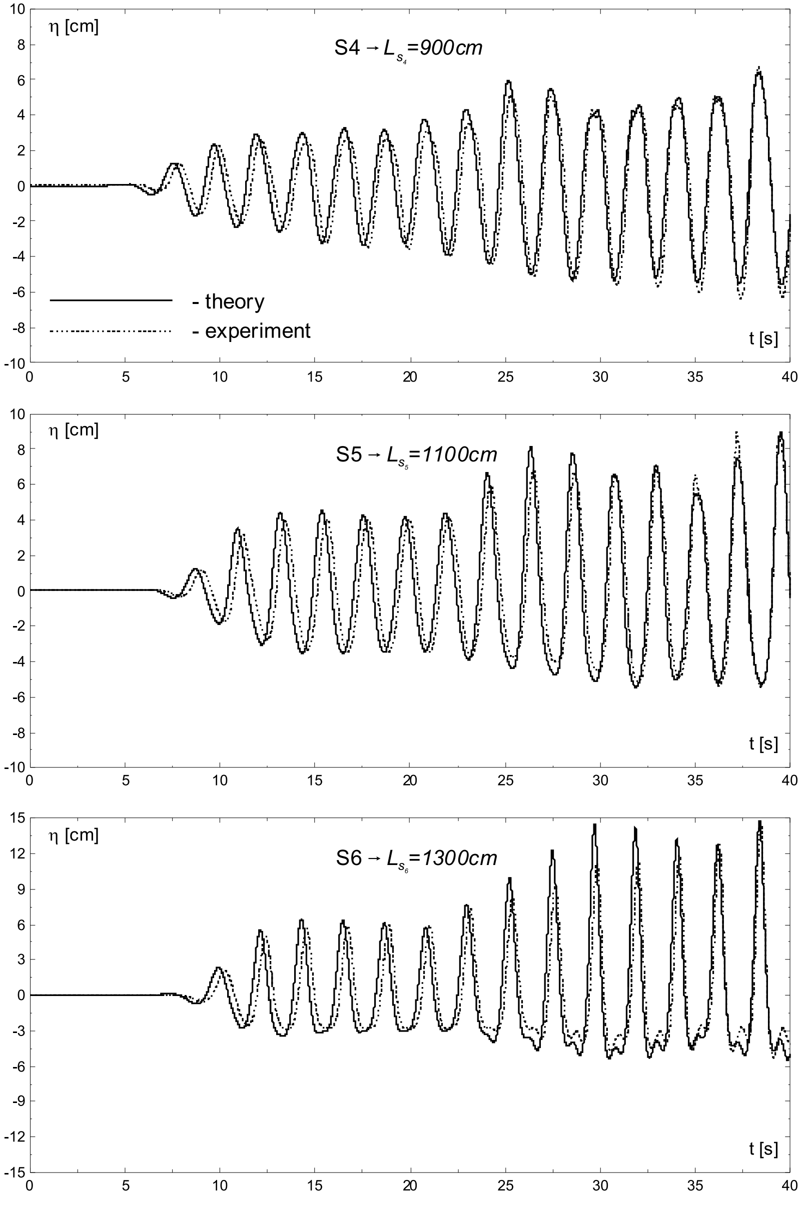

Fig. 5. Continued

the theory fails to deliver proper results for shorter waves, say waves of lengths $\lambda < 4H$. Thus, with respect to practical applications, the theory presented above is, in principle, accurate enough for estimation of transformation and reflection of long sea waves approaching a sea shore.

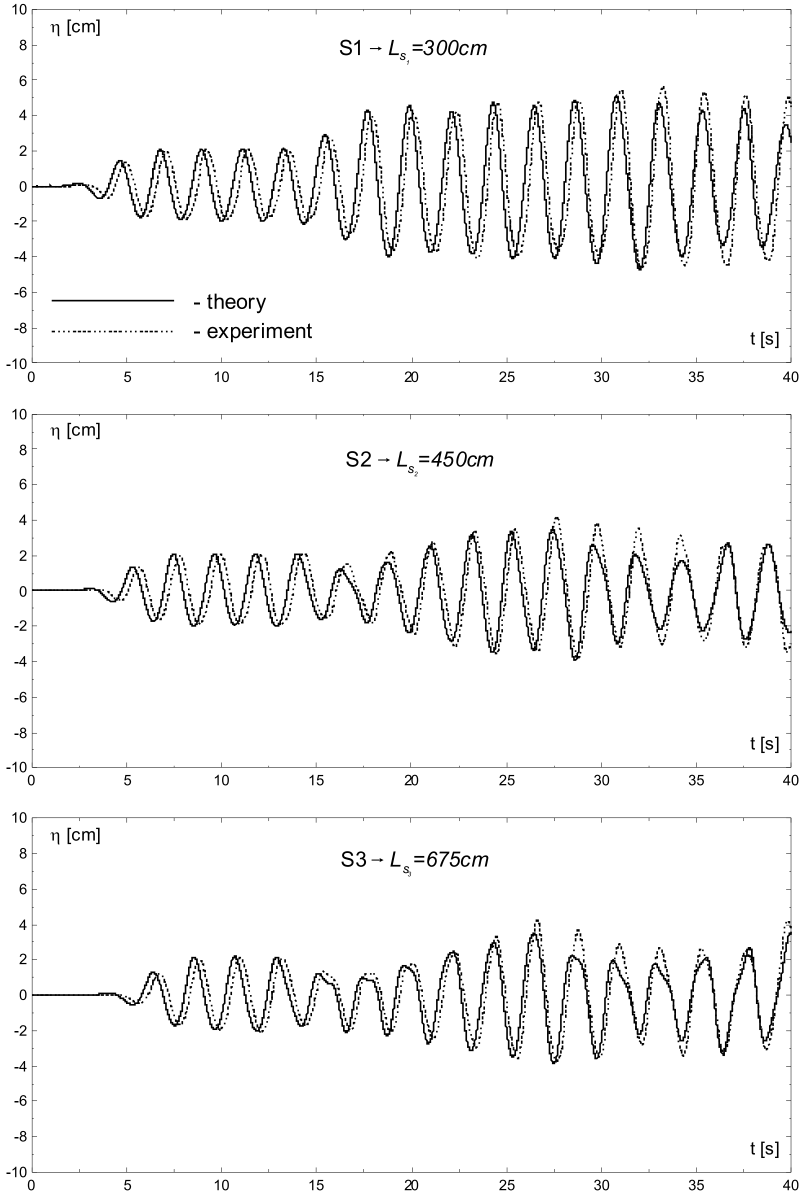

Fig. 6. Comparison of theoretical results with data obtained in experiments for the bottom slope 1 : 15

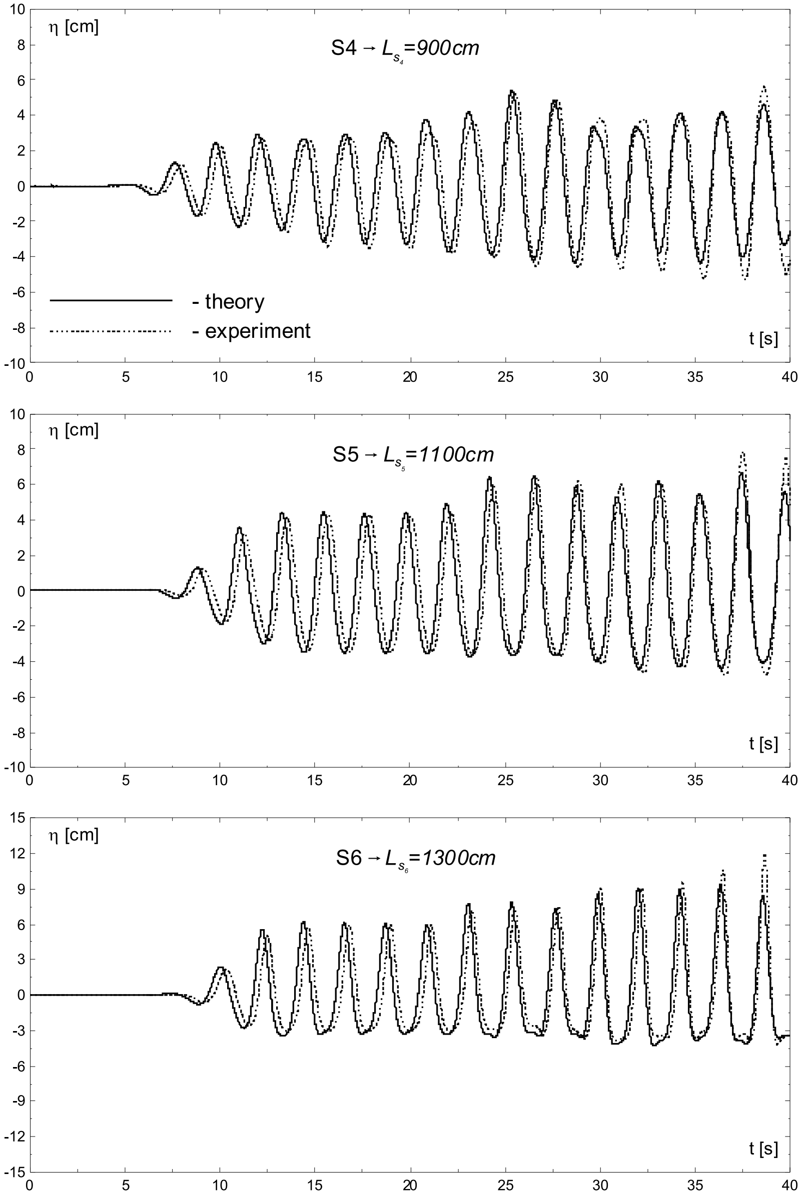

Fig. 6. Continued

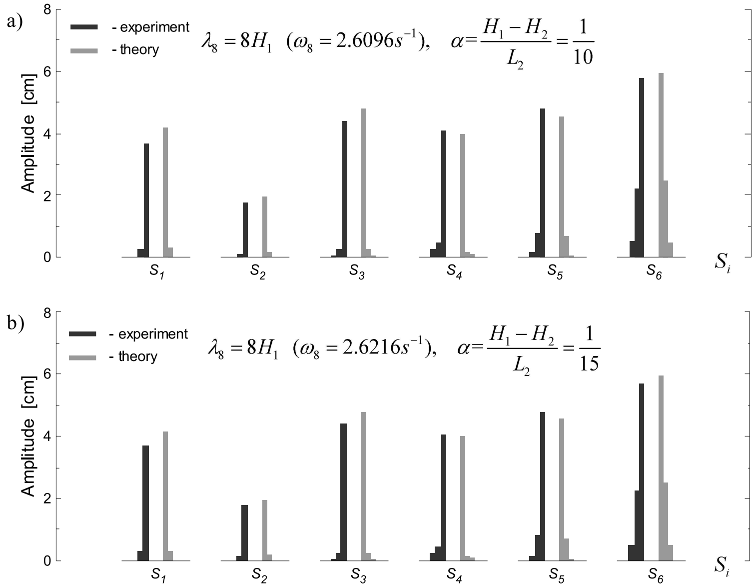

Fig. 7. Discrete spectral densities of the free surface elevations resulting from theoretical solutions and obtained in laboratory experiments for all the wave gauges

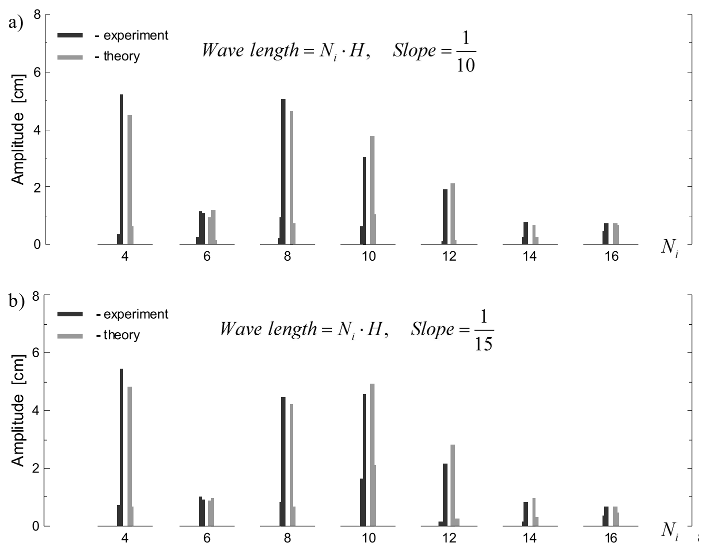

Fig. 8. Discrete spectral densities of the free surface elevations at the wave gauge S_5 resulting from theoretical solutions and obtained in laboratory experiments for a set of generation frequencies

6. Concluding Remarks

The transformation of water waves approaching the area of uneven bottom together with reflection of the waves from a barrier depends mainly on changes of the water depth and characteristics of the waves. In the considered cases, the water depth diminishes towards the barrier installed in an area of smaller water depth. In the laboratory experiments and theoretical approach presented, we have been dealing with the initial value problem of the finite fluid domain starting to move at a certain moment of time. A relatively complicated structure of the water flow is a result of geometry of the fluid domain and a resonance phenomenon which is an important factor especially for the harmonic generation of the water flow within finite domains. The resonance is responsible for a significant growth of wave height with passing time, till a breaking of the wave. With the breaking phenomenon, the condition of continuity of the free surface is lost, and thus displacements of the fluid particles forming the free surface are uncontrolled. As compared to classical formulations existing in the literature of the subject, the most important feature of the theoretical description of the phenomenon presented is a reduction of the description to the momentum equation for the horizontal displacement, instead of the vertical one. With respect to the kinematical assumption on average horizontal displacements of all fluid particles forming a vertical material line, the formulation allows us to estimate the fluid flow also for cases of relatively steep waves, even for breaking waves. Such an estimation is possible because the vertical displacements do not enter the fundamental equations as the dependent variable. The vertical displacement depends on the horizontal solution, and thus it may be calculated for all cases except $u' = -1$. Comparison of the theoretical results with data obtained in the laboratory experiments has shown that the average difference between the two sets of the dependent variables was less than about 10%. It should be noted here, that the comparison has been conducted for relatively high waves, i.e. for waves the height of which has reached the level of about 40% of the water depth in the neighbourhood of the right boundary (CD in Fig. 1). It means that the description model developed in the paper may be used for practical cases as a convenient tool in analysing the transformation and reflection of sea waves approaching sea shores.

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