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A Note on Discrete Descriptions of Water Flows in Material Variables

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Abstract

The paper describes the problem of discrete formulation of plane fluid flows in material description. The investigation is confined to chosen cases of stationary potential and vortex motion of an incompressible inviscid fluid within circular domains with perfect boundaries. The paths of fluid particles are obtained by numerical integration of momentum equations within a discrete time space. Brownian type random disturbances are attached to the displacement field obtained by the integration. It has been shown, that the discrete formulation may lead to solutions in which a small distance between two material points may grow to a relatively large value after a finite elapse of time. The last feature of the procedure may be a serious drawback of the discrete formulation in the material variables.

Key words: potential motion, vortex flow, material variables, discrete integration

1. Introduction

The theory of water flows is based on the fundamental assumption that the fluid is a continuous medium. This means that the space (three dimensional Euclidean space) is occupied by continuously distributed fluid particles i.e. we have space points with attached mass of fluid. Accordingly, all parameters associated with fluid motion are continuous functions, to the desired order, in a mathematical sense, and thus, the functions can be expanded into Taylor series with respect to any surroundings of a given point. In particular, within the surrounds of the point, an infinite number of material points exists. The governing equations of fluid dynamics expressing the conservation of mass and momentum, as well as the balance of energy, assume the form of partial differential equations for the unknown fields of velocity and pressure depending on the space, or material, variables and time. In describing the problem of water waves, it is more common to use the space variables, but in some cases it is more convenient to apply the material ones.

For example, the latter description is simpler in solving boundary conditions especially on moving boundaries of the fluid domain. Both descriptions are equally admissible and should lead to the same results. In order to describe a state of the fluid we have to find a solution to the aforementioned differential equations satisfying given boundary and initial conditions. In principle, for a well posed problem, a solution to the equations exists (Pogorzelski 1962). When applying the material description we follow individual particles of fluid and thus, the problem of a surrounding of a given material point is of practical importance. The latter problem closely relates to the Reynolds experiments (Troskolański 1962, Sawicki 1998), which have shown, that laminar flow of fluid may exist only for small Reynolds numbers. With growing values of the numbers a phenomenon of turbulent motion of the fluid occurs, for which, the fundamental assumption on surroundings of a point in the material description does not hold. Such a situation is similar to mixing motion of a granular medium, where we cannot speak on a surrounding of a given material point in the mathematical sense. The last feature is especially important in discrete descriptions of the phenomenon, when we follow chosen particles representing the whole fluid domain. With the discrete approach, a path of a chosen particle is calculated approximately at chosen instants of time and therefore for a finite elapse of time the particle may jump from one path to another. Such a situation may happen especially in the vicinity of singular points where a concentration of the fluid paths may be observed. Since a general solution of the equations of fluid motion is not known, in order to get a solution of a given problem we are forced to resort to certain approximation of the equations. Such a procedure is frequently justified and enables us to obtain solutions of acceptable accuracy. For example, in order to simplify the equations for water waves, an assumption that the velocity field is a potential field, is frequently introduced into descriptions of the wave phenomenon. Another way is to resort to discrete formulations of the original task formulated in continuum. In the discrete formulations we operate with small, but finite, increments of space variables and time and thus the notion of the fluid continuity is lost. Moreover, with the discrete approach we have a finite dynamical system for which a deterministic chaos may occur (Baker and Gollub 1998, Schuster 1993). In the latter case we cannot predict the future state of the fluid. As far as the discrete formulations are concerned, the finite difference method is of practical importance. With this method the derivatives entering the fundamental equations of the problem considered are substituted by finite difference quotients. In this way the differential equations for continuum are substituted by a system of algebraic equations written for a set of chosen discrete points. Such a procedure is not unique because one can choose different schemes for a derivative at hand (Szymkiewicz 2000). In particular, with the discrete formulation, we cannot speak on a surrounding of a given material point, however we can describe the distance between chosen points. The accuracy of the discrete approximation of the differential equations

depends on many factors. For instance, the first finite difference may be close to the first differential in a continuum, but with growing order of differentiation, the discrepancy between the two formulations will increase. At the same time, when paths of fluid particles calculated in a continuum are very close to each other in a certain area of the fluid, the discrete formulation may lead to an uncontrolled change of paths of the fluid particles.

In the present paper a discrete description of the fluid flow in material variables is considered. We focus our attention on the problem of accuracy of the discrete calculation of the fluid paths within a finite range of time. In order to estimate a possible departure of the discrete solution from an analytical one, Brownian type disturbances are added to the displacement field of the fluid. In this way the state of the fluid influences the final departure of individual particles. In what follows we confine our attention to plane problems of steady fluid flows in bounded domains on the assumption that the gravitational acceleration is perpendicular to the plane. Two examples of the fluid motion are examined in detail. The first one is a potential velocity field within a plane ring. The second one is a rotational motion of the fluid in a circular domain. For the assumed velocity fields satisfying given boundary conditions the Brownian type motion of chosen particles is considered. In calculations of the discrete paths of individual particles a finite number of fluid particles is assumed to be dropped into the fluid at a chosen point. In this way, a sequence of individual drops with equal time elapse between the subsequent drops is formed. The fluid drops are assumed to have the same velocity as the fluid passing through the space point.

2. Steady Velocity Fields in Circular Domains

Let us consider a plane problem of fluid flow in circular regions as shown in Fig. 1. With respect to the shape of the regions it is convenient to introduce the polar system of coordinates r and φ. In the first case (Fig. 1a) we have the circular ring $r_0 \leq r \leq R$, $0 \leq \varphi \leq 2\pi$, and in the second one (Fig. 1b) the circular region $0 \leq r \leq R$, $0 \leq \varphi \leq 2\pi$ is considered. In accordance with the Cartesian $(x = X^1, y = X^2)$, and polar (r, φ), systems of coordinates, the velocity vector is written in the form

$$\dot{\vec{\mathbf{r}}} = \vec{\mathbf{v}} = \dot{x}\vec{\mathbf{i}} + \dot{y}\vec{\mathbf{j}} = \dot{X}^i \vec{\mathbf{e}}_i = \dot{r}\vec{\mathbf{e}}_r + r\dot{\varphi}\vec{\mathbf{e}}_\varphi, \tag{1}$$

where $\vec{\mathbf{e}}_r = \dfrac{\partial \vec{\mathbf{r}}}{\partial r}$, $r\vec{\mathbf{e}}_\varphi = \dfrac{\partial \vec{\mathbf{r}}}{\partial \varphi}$ and $|\vec{\mathbf{e}}_r| = |\vec{\mathbf{e}}_\varphi| = 1$.

For the case of potential motion, with the velocity potential $\phi(r, \varphi)$, the following relations hold

$$\vec{\mathbf{v}} = \operatorname{grad} \phi = v^r \vec{\mathbf{e}}_r + v^\varphi \vec{\mathbf{e}}_\varphi = \frac{\partial \phi}{\partial r}\vec{\mathbf{e}}_r + \frac{1}{r}\frac{\partial \phi}{\partial \varphi}\vec{\mathbf{e}}_\varphi. \tag{2}$$

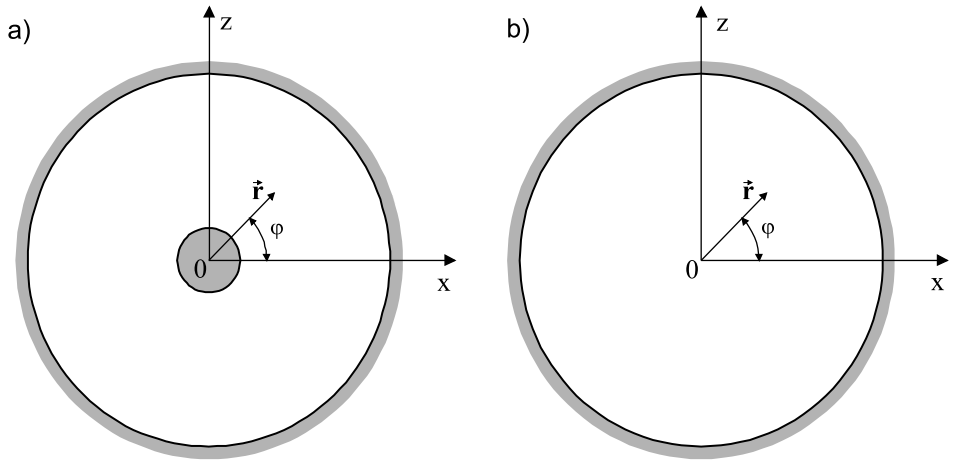

Fig. 1. Circular regions of fluids

At the same time the Laplace equation for the velocity potential assumes the form

$$\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} + \frac{1}{r^2}\phi_{,\varphi\varphi} = 0. \tag{3}$$

Let us consider now the case of potential flow within the ring shown in Fig. 1a. To make the discussion clear we confine our attention to the following velocity potential

$$\phi = d\varphi + \text{ const}, \tag{4}$$

where d is a constant.

The relevant velocity components are

$$v_r = \frac{\partial\phi}{\partial r} = 0,\ v_\varphi = \frac{1}{r}\frac{\partial\phi}{\partial\varphi} = \frac{d}{r}. \tag{5}$$

It is seen that the chosen velocity field does not depend directly on time. From the relations (1) and (5) it follows

$$\begin{aligned} &\dot{r} = 0, \rightarrow r = \text{ const} = r_s, \\ &\dot{\varphi} = \frac{d}{r^2}, \ \rightarrow \varphi = \frac{d}{r_s^2}t + \varphi_s. \end{aligned} \tag{6}$$

The equations describe a set of circles defined by the parameter r_s. From the second equation in (6) the period for a given particle is obtained

$$T = \frac{2\pi r_s^2}{d}. \tag{7}$$

As the second case, let us consider the rotational flow within the circle (Fig. 1b), described by the assumed velocity field (Szmidt 2001)

$$\begin{aligned} \dot{r} &= dr \sin\frac{r\pi}{R} \sin 2\varphi, \\ \dot{\varphi} &= d\left[2\sin\frac{r\pi}{R} + \frac{r\pi}{R}\cos\frac{r\pi}{R}\right]\cos^2\varphi. \end{aligned} \tag{8}$$

One can check, that the velocity field (8) satisfies the incompressibility condition div $\vec{\mathbf{v}} = 0$. With respect to the Cartesian system of coordinates, the relevant components $(\dot{x}, \dot{y})$ of the velocity vector (1) read

$$\begin{aligned} \dot{x} &= \dot{r}\cos\varphi - r\sin\varphi\dot{\varphi}, \\ \dot{y} &= \dot{r}\sin\varphi + r\cos\varphi\dot{\varphi}. \end{aligned} \tag{9}$$

The equations can be transformed into the following form

$$\begin{aligned} v_x &= \frac{dx}{dt} = \frac{\dot{r}x}{r} - y\dot{\varphi}, \\ v_y &= \frac{dy}{dt} = \frac{\dot{r}y}{r} + x\dot{\varphi}. \end{aligned} \tag{10}$$

Integration of the velocity field in the time domain leads to the components of the displacement vector. In the case of discrete time step $\Delta t > 0$, the displacement may be described by means of the difference equation

$$\Delta\mathbf{X} = \mathbf{X}(t + \Delta t) - \mathbf{X}(t) = \vec{\mathbf{v}}\Delta t, \tag{11}$$

where $\mathbf{X}^{\mathbf{T}} = (x, y)$ is the matrix containing the Cartesian coordinates of a given material point.

With respect to the notations (10) the last equation gives

$$\begin{aligned} x^{n+1} &= x^n + \left(\frac{\dot{r}x}{r} - y\dot{\varphi}\right)\Delta t, \\ y^{n+1} &= y^n + \left(\frac{\dot{r}y}{r} + x\dot{\varphi}\right)\Delta t. \end{aligned} \tag{12}$$

where the superscript n means the level of time ($t^n = n\Delta t$) and $r = \sqrt{x^2 + y^2}$.

A random displacement of an individual fluid particle may be described by an equation similar to equation (12), namely

$$\Delta\mathbf{X} = \mathbf{X}(t + \Delta t) - \mathbf{X}(t) = \vec{\mathbf{v}}(\mathbf{X}, t)\Delta t + \sigma(\mathbf{X}, t)\sqrt{\Delta t}\mathbf{U}(t + \Delta t), \tag{13}$$

where σ is a diagonal matrix of positive numbers and $\mathbf{U}(t + \Delta t)$ describes the vector of independent random variables having the normal distribution $N(0, 1)$.

The difference equation corresponds to Itô stochastic differential equation (Jaźwiński 1970, Wilde and Paczos 1988)

$$d\mathbf{X} = \mathbf{v}(\mathbf{x}, t)dt + \sigma(\mathbf{x}, t)d\mathbf{B}, \tag{14}$$

which describes the difference of a displacement field with random disturbances.

In the equation, the $d\mathbf{B}$ term means the differential of the Brownian motion process in the Itô sense (the Brownian motion process is not differentiable in a conventional sense).

For the isotropic cases considered in this paper the diagonal of the matrix σ in (13) is a constant. Because of the random term $\mathbf{U}(t + \Delta t)$ on the right hand side of the equation, numerical computations for a set of individual particles provide a family of realizations of the stochastic process describing the motion of the particles. With respect to the above, the path of a given particle is described by the difference formulae

$$\begin{aligned} x^{n+1} &= x^n + \left(\frac{\dot{r}x}{r} - y\dot{\varphi}\right)^n \Delta t + \sigma\sqrt{\Delta t}U_1(n+1), \\ y^{n+1} &= y^n + \left(\frac{\dot{r}y}{r} + x\dot{\varphi}\right)^n \Delta t + \sigma\sqrt{\Delta t}U_2(n+1), \end{aligned} \tag{15}$$

where $\sigma\sqrt{\Delta t}$ denotes the standard deviation, and $U_1(n+1)$ and $U_2(n+1)$ are random sequences with the normal distribution $N(0, 1)$. In calculations of the deterministic terms entering the equations it is convenient to use the Runge-Kutta methods (Björck, Dahlquist 1983). In such a case the discrete deterministic solution within the range $(t, t + \Delta t)$ is supplemented with the random impulse attached to the solution at the time $(t + \Delta t)$.

3. Numerical Examples

In what follows we confine our attention to calculations of chosen fluid paths in the finite regions of fluids shown in Fig. 1. As the first, let us consider the potential motion of the fluid within the ring shown in Fig. 1a. It is assumed that at the point $P(x = x_0, z = 0)$ a sequence of fluid particles is dropped into the fluid. The starting velocity of the particles is equal to the velocity of the fluid at this point. Following the equations (15) the path of each particle is calculated as a sequence of numbers for equal time steps. The procedure is repeated for a chosen number of the particles. In the discussed case, the paths for $N = 2000$ particles have been obtained. The positions of the particles at chosen moments of time were calculated and kept in a computer memory. The results of computations are shown in Fig. 2, where the pictures display the distribution of the particles at chosen instants of time. The particles form a ring whose thickness depends on the standard deviation of the random sequences. It may be seen that the average path of the set is close to that obtained from deterministic equations of motion. The second example shown in Fig. 3 is more complicated. First of

all, the equations of the velocity field (10) describe the rotational motion for which, in some regions of the fluid domain a concentrations of the fluid paths occur. Therefore, in discrete integration of the equations of motion together with the attached random disturbances, a given particle may jump from one path to another. The result is a relatively large spread of the fluid particles within the fluid area. As compared with the previous, potential motion, a departure of the particles from their average path is of course greater. The last feature is a serious drawback in the discrete description of the problem formulated in the material variables.

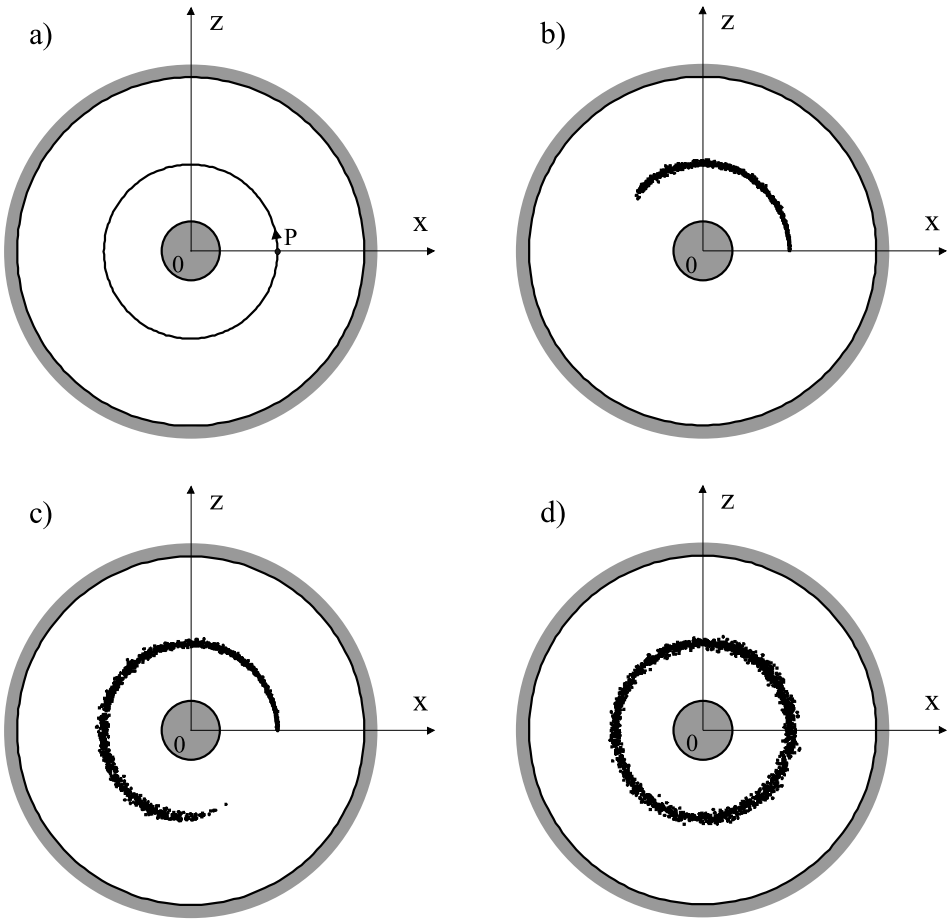

Fig. 2. Potential motion of fluid within a circular ring: deterministic path of chosen particles (a), and random distribution of particles at $t = 60$ s (b), $t = 120$ s (c) and $t = 200$ s (d)

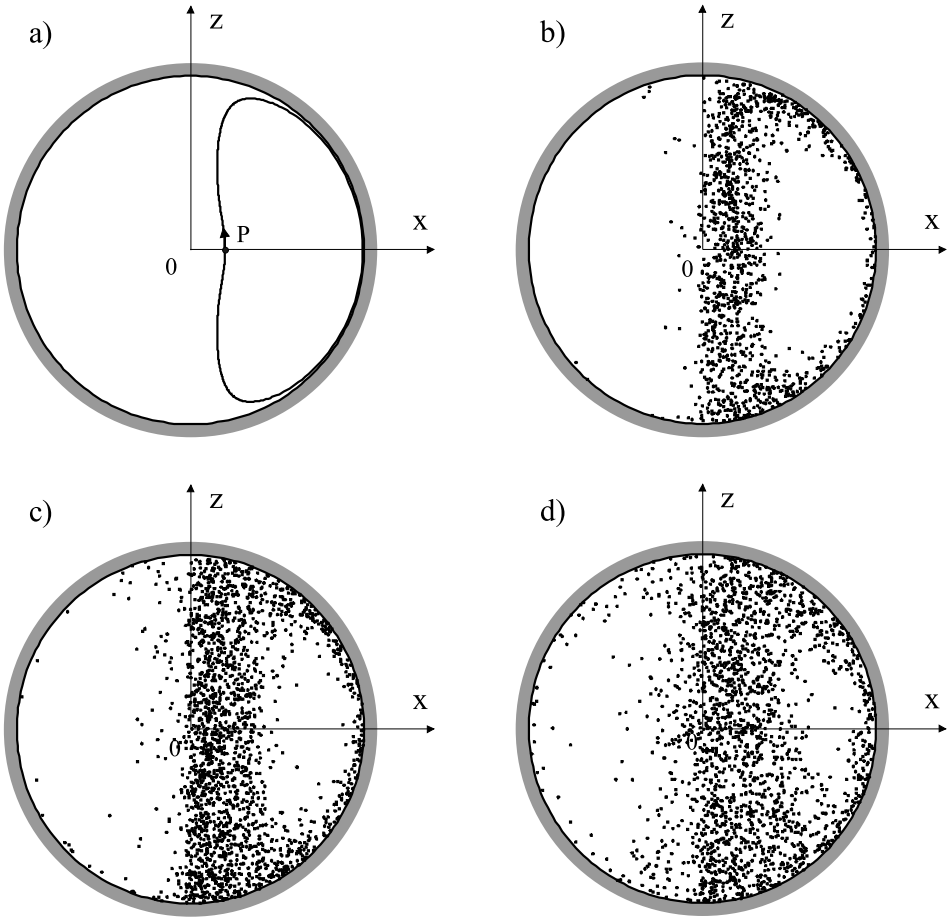

Fig. 3. Vortex motion of fluid within a circular domain: deterministic path of chosen particles (a), and random distribution of particles at $t = 60$ s (b), $t = 120$ s (c) and $t = 200$ s (d)

4. Concluding Remarks

In the above, we have obtained numerical solutions to the plane problems of potential and vortex motions of a perfect fluid in bounded domains. The investigations have been confined to chosen stationary velocity fields within circular regions. The deterministic fluid paths were supplemented with random disturbances attached to displacements of the fluid particles. In this way the distribution of a set of individual particles at chosen moments in time has been obtained. The distribution shows that a small distance between two particles at one moment in time may grow to a relatively large value at another moment in time. Such cases may occur especially in the vicinity of singular points where the fluid paths are so close to each other that in the numerical integration a given particle may

jump from one path to another. Such a phenomenon is a serious drawback of the discrete formulation in the material variables. For instance, in the finite element formulation with elements described by material nodal points, a large element of distortion may occur which leads to large errors, or even to loss of the solution uniqueness. An example of the later phenomenon is a distortion of elements describing the free surface of a gravitational wave with large steepness.

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TECHNICAL NOTES