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Long Water Waves as a Structure Fluid Interaction Problem

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Abstract

The paper describes a new formulation of the theory of long shallow water waves, which is based on the fundamental assumption that vertical material lines of fluid remain vertical during the entire motion. To make the problem consistent from the point of view of physics the case of waves in a flume due to the motion of a piston type generator is considered. At the piston the material line of water particles remains vertical during the entire motion and thus the generation follows the assumption in the description of the motion of water in the flume. Wave equations are derived with the help of a variational formulation of the problem in a material description. The Lagrangian is the difference between the kinetic and potential energies of the fluid and the mechanical system that describes a very simplified wave generator. The basic assumption simplifies the geometry of the displacement field. The definitions of generalized forces follow from variational calculus. The procedure ensures that the energy is preserved. A simple discrete formulation of the problem is based on the finite element method and the corresponding approximate expressions for energies.

Key words: long waves, Lagrangian description, Hamiltonian principle

1. Introduction

The theory of long water waves deals with waves the lengths of which are much greater than the depth of water in which the waves are propagated. Particular forms depend on assumptions and simplifications introduced into equations of fluid motion. The formulations include the Boussinesq theory of long waves. The original Boussinesq equations correspond to the irrotational one-dimensional motion of an incompressible fluid over a horizontal bed. In derivation of the equations, the dependent variables can be chosen in different ways and a variety of different forms of the Boussinesq equations exist in the literature on the subject (Madsen, Murray and Sorensen 1991). For example, as typical velocity variable one may use surface velocity, depth averaged velocity or bottom velocity. The simplest formulation is based on the assumption that the pressure at any point in the fluid is equal to the hydrostatic head of water above this point. A better description of the phenomenon is obtained by taking into account some vertical acceleration of the fluid particle. In the latter case one of the simplest and most widely applied formulations is based on the assumption that the vertical component of the velocity increases linearly from zero at the bed to a maximum value at the free surface (Abbot 1979). Particular forms of vertical acceleration terms used in the momentum equations lead to different forms of the long wave equations. In general, different approximations lead to different properties of solutions of the derived differential equations. This leads to such paradoxes as the one described by Ursell (1953). In most formulations of the long wave equations encountered in the literature on the subject, Eulerian variables have been exploited (Stoker 1957, Whitham 1974, Mei 1983). In our case, when the waves in the flume are considered, it is easier to describe the piston motion in Lagrangian variables.

In the present paper a new formulation of the long wave description which is based on the fundamental assumption that the vertical material lines of fluid remain vertical during entire motion is presented. It is assumed that the fluid is non-viscous and incompressible. The equations are derived with the help of a variational formulation in the Lagrangian variables. Two dimensional plane waves are considered as they are generated in a standard wave flume. A mechanical system corresponding to a simplified generator is considered. The application of standard variational calculus leads to a partial differential equation of the problem and natural initial and boundary conditions. The variational formulation leads to solutions that conserve the mechanical energy of the interacting fluid and generator. Material description is natural when kinetic and potential energies of fluid particles and the elements of the simplified generator are considered. Such a formulation is especially important in approximate numerical solutions formulated within the finite element method.

From the condition of incompressibility and assumed simplified geometry of deformation, it follows that the vertical material lines are uniformly strained and the vertical coordinates of displacements of the free surface are determined by horizontal strains. Equations describing the kinetic energy contain terms corresponding to the horizontal, as well as the vertical components of the fluid velocity. If the vertical term is neglected as a negligible quantity, then the description leads to the standard theory of linear non-dispersive waves. Thus, the vertical term is responsible for dispersion of the long waves. As in the classic Hamiltonian mechanics of a system of material points, the Lagrangian function in the action integral is equal to the difference between the kinetic and potential energies of the fluid and the mechanical system. Applying the variational calculus to the action integral, the partial differential equations of the problem, together with the natural initial and boundary conditions, are established. The boundary conditions correspond to the standard ordinary linear differential equation for forced vibrations of mechanical systems with one degree of freedom. The forcing term leads to the definition of the resultant horizontal component of the hydrodynamic force. The corresponding relations for the linear theory supplement all the expressions.

A simple discrete model based on the finite element method is presented. The continuum is replaced by a countable set of nodal points in the distance-time space by assuming a simple shape function for the horizontal displacement fields. The result is a set of algebraic equations. Within the FEM the expressions for the kinetic and potential energies depend upon the assumed shape functions for the element. The energy is preserved in the analysis within the assumed approximation.

The results of numerical calculations are compared with results of experiments carried out at our institute. P. Wilde et al (2001) describe the experiments. A detailed description of these and the sets of measured data is contained in an internal report available in the library of the institute (Wilde et al 2000). In this paper the results of measurements of surface elevations by gauges along the flume are used to verify the results of the numerical solutions. Comparison of the amplitudes of the basic, double and triple frequency terms in a Fourier expansion of values obtained from experiments and calculations show good conformity for the length to depth ratio equal to twelve. Where the double frequency term in the motion of the piston is exaggerated, the agreement is not as good. It must be remembered that for a regular Stokes wave, the components move with the same velocity. By exaggeration of the second term the physics of the problem is an interaction of waves with two frequencies, thus the assumption that vertical material lines remain vertical does not have to be justified.

2. General Variational Formulation in Material Descriptions

2.1. The Simplified Geometry of Displacements in Material Descriptions

Let us consider a two-dimensional problem in a rectangular region of fluid of depth H, width B and length L. Let us introduce time variable t and Cartesian coordinate system in space in the fluid at rest $Z^1, Z^2, 0 \leq Z^1 \leq L, 0 \leq Z^2 \leq H$. These horizontal and vertical coordinates are names of the material points in the fluid. They define the reference system.

Let us assume that vertical material lines $Z^1 = \text{const}$ remain vertical during the deformation and that they are uniformly stretched while displacements are zero at the bottom $Z^2 = 0$. Thus, in the Cartesian coordinate system in space z^1, z^2, the actual positions of material points are

$$\begin{aligned} z^1\left(Z^1, Z^2, t\right) &= Z^1 + u\left(Z^1, t\right), \\ z^2\left(Z^1, Z^2, t\right) &= Z^2 + w\left(Z^1, t\right) Z^2/H, \end{aligned} \tag{1}$$

where $u(Z^1, t)$ is the horizontal displacement and $w\left(Z^1, t\right)$ the vertical displacement of the free surface as functions of the material points, names and time.

The functions (1) are the mapping of the positions at rest (names) into the actual positions in space. The deformation gradient (corresponds to a square matrix with elements equal to the partial derivatives of the positions with respect to the variables $Z^\alpha, \alpha = 1, 2$) is

$$z^i_\alpha = \begin{bmatrix} 1 + u'\left(Z^1, t\right) & 0 \\ w'\left(Z^1, t\right) Z^2/H & 1 + w\left(Z^1, t\right)/H \end{bmatrix} \tag{2}$$

where $()'$ denotes the partial derivative with respect to the variable Z^1. The Jacobian is the determinant of the matrix (2) and the condition of incompressibility leads to the following expression for the function w

$$w = -H \frac{u'\left(Z^1, t\right)}{1 + u'\left(Z^1, t\right)}. \tag{3}$$

Thus the kinematics of the displacement field for the considered incompressible case is described by one function $u\left(Z^1, t\right)$ that corresponds to the horizontal displacement. It should be noted that according to the relation (2) the mapping is isochoric for any value of Z^2, and thus the condition that the vertical lines are uniformly stretched is equivalent to the assumption of incompressibility along the depth.

The velocity field is defined as the partial derivative with respect to time. Thus it follows from relations (1)

$$\begin{aligned} v^1\left(Z^1, Z^2, t\right) &= u'\left(Z^1, t\right), \\ v^2\left(Z^1, Z^2, t\right) &= -Z^2 \frac{\dot{u}'\left(Z^1, t\right)}{\left[1 + u'\left(Z^1, t\right)\right]^2}. \end{aligned} \tag{4}$$

The dot over a symbol denotes the time derivative above and in other expressions.

2.2. Potential and Kinetic Energy and the Action Integral

Now let us calculate the potential energy in the gravitational field. The value of Jacobian equal to is one and thus it is easy to calculate the integral in the reference configuration. The potential energy is

$$E_p(t) = \rho g B \int_0^L \int_0^H z^2\left(Z^1, Z^2, t\right) J dZ^2 dZ^1, \tag{5}$$

where g is the acceleration in the gravitational field and ρ the density of the fluid. Substitution of the second relation in (1) upon integration leads to the following expressions for the potential energy of the water in the considered rectangular region

$$E_{pw}(t) = \frac{1}{2}\rho g H^2 B \int_0^L \frac{1}{1+u'(Z^1,t)} dZ^1,$$
$$E_{pwl}(t) = \frac{1}{2}\rho g H^2 B \int_0^L \left\{1 - u'\left(Z^1,t\right) + \left[u'\left(Z^1,t\right)\right]^2\right\} dZ^1, \tag{6}$$

where the second expression corresponds to the linear theory that is a power series expansion up to the square term.

In the problem discussed the kinetic energy of the water is

$$E_{kw}(t) = \frac{1}{2}\rho B \int_0^L \int_0^H \left\{\left[v_1\left(Z^1,t\right)\right]^2 + \left[v_2\left(Z^1,t\right)\right]^2\right\} dZ^2 dZ^1. \tag{7}$$

Upon substitution of the expressions (4) and integration along the vertical direction it follows that

$$E_{kw}(t) = \frac{1}{2}\rho H B \int_0^L \left\{\left[\dot{u}\left(Z^1,t\right)\right]^2 + \frac{1}{3}H^2 \frac{\left[\dot{u}'(Z^1,t)\right]^2}{\left[1+u'(Z^1,t)\right]^4}\right\} dZ^1,$$
$$E_{kwl}(t) = \frac{1}{2}\rho H B \int_0^L \left\{\left[\dot{u}\left(Z^1,t\right)\right]^2 + \frac{1}{3}H^2\left[\dot{u}'\left(Z^1,t\right)\right]^2\right\} dZ^1, \tag{8}$$

where the second expression corresponds to the linear theory. In the expression for kinetic energy the first term under the integral is due to the horizontal component of velocity and the second due to the vertical one.

Let us now introduce the idea of a simplified mechanical system based on the piston type generator of a standard wave flume. There is a rectangular block of mass M_l supported by a linear spring with elastic coefficient k_l on the left and a corresponding one with mass M_r and coefficient k_r on the right. The blocks can move frictionless in horizontal directions. The flume is empty and the forces in the horizontal springs are zero. Now the flume is filled with water so slowly that the velocities are negligible and the dynamic effects may be disregarded. The final result corresponds to a uniform depth of fluid denoted by H. Let us denote the displacement of the right end of the left spring due to the process of filling by u_{l0} and the displacement of the right spring by u_{r0}. It follows from hydrostatics that

$$u_{l0} = -\frac{1}{2k_l}\rho H^2 B, u_{r0} = \frac{1}{2k_r}\rho H^2 B. \tag{9}$$

This state corresponds to the initial state for the wave motion and we commence our consideration at this state. Let us assume that additional displacements introduced on the left end of the left spring are denoted by $u_{la}(t)$ while displacements of the right end (and the displacement of the block) is $u(0,t)$. The corresponding quantities at the right spring are $u_{ra}(t)$ and $u(L,t)$ on the right by $u_{r0}(t)$, where L is the distance of the blocks in the initial state. The expressions for the potential and kinetic energies of the two mechanical systems are

$$
\begin{aligned}
E_{pl}(t) &= \frac{1}{2}k_l[u(0,t) - u_{la}(t)]^2, \\
E_{pr}(t) &= \frac{1}{2}k_r[u(L,t) - u_{ra}(t)]^2, \\
E_{kl}(t) &= \frac{1}{2}M_l[\dot{u}(0,t)]^2, \\
E_{kr}(t) &= \frac{1}{2}M_r[\dot{u}(L,t)]^2.
\end{aligned}
\tag{10}
$$

Hamilton's action integral is the time integral of the difference between the kinetic and potential energies given by the relations (6), (8), (10)

$$
I = \int_0^{t_k} [E_k - E_p]dt. \tag{11}
$$

The total energy is the sum of energies of subsystems such as the fluid, the mechanical systems on the left and right. The action integral based on the expressions (6), (8), (10) does not take into account all possible physical boundary and initial conditions of the problem.

2.3. The Differential Equation, the Initial and Boundary Conditions Derived by Variational Calculus

The solution has to be calculated by standard variational methods. Let us consider the family of functions from the infinitesimal neighborhood

$$
u\left(Z^1, t\right) \to u\left(Z^1, t\right) + \varepsilon\xi\left(Z^1, t\right), \tag{12}
$$

where ε is an infinitesimal parameter and ξ is an arbitrary function having the necessary continuous derivatives. In our consideration the function $u\left(Z^1, t\right)$ is the only unknown function. Functions $u_{la}(l)$ and $u_{ra}(t)$ are assumed to be known as corresponding to the functions fed into the control system of the generator. Let us introduce this expression into the action integral based on the relations (6) and (8) and calculate the derivative with respect to ε and then substitute ε equal to zero. It follows that

$$
\begin{aligned}
\left.\frac{\partial I}{\partial \varepsilon}\right|_{\varepsilon=0} = & \frac{1}{2}\rho H B \int_0^{t_k}\int_0^L \left\{2\dot{u}\dot{\xi} + \frac{2}{3}H^2 \frac{\dot{u}'}{[1+u']^4}\dot{\xi}' - \frac{4}{3}H^2 \frac{[\dot{u}']^2}{[1+u']^5}\xi'\right\} dZ^1 dt + \\
& + \frac{1}{2}\rho H B \int_0^{t_k}\int_0^L \left\{\frac{1}{2} g H \frac{1}{[1+u']^2}\xi'\right\} dZ^1 dt + \\
& - \int_0^{t_k} \left\{M_l \ddot{u}(0,t) + k_l\left[u(0,t) - u_{la}(0,t)\right]\right\} \xi(0,t) + \\
& + M_l \dot{u}(0,t)\,\xi(0,t)\big|_0^{t_k} + \\
& - \int_0^{t_k} \left\{M_r \ddot{u}(L,t) + k_r\left[u(L,t) - u_{ra}(L,t)\right]\right\} \xi(L,t)\, dt + \\
& + M_r \dot{u}(L,t)\,\xi(L,t)\big|_0^{t_k} = 0,
\end{aligned}
\tag{13}
$$

where in the expressions for the discrete mechanical system the formulae for integration by parts have been applied. This is the variational formulation in terms of the independent unknown horizontal displacements in the fluid.

The partial derivatives of the function $\xi\left(Z^1, t\right)$ appear in the double integral in the expression (13). They are not independent functions and thus the basic lemma of variational calculus cannot be applied. For the one-dimensional integrals, differentiation by parts has been used to eliminate the derivative with respect to time. For the double integral, appropriate identities based on Green's Theorem have to be applied. Finally the transformations lead to the following expressions for the double integral in the relation (13) (this part is denoted by a subscript w):

$$
\begin{aligned}
\left.\frac{\partial I_w}{\partial \varepsilon}\right|_{\varepsilon=0} = & -\rho H B \int_0^{t_k}\int_0^L \ddot{u}\left(Z^1,t\right)\xi\, dZ^1 dt + \rho H B \int_0^L \dot{u}\left(Z^1,t\right)\xi\, dZ^1 \Bigg|_0^{t_k} + \\
& + \frac{1}{3}\rho H^3 B \int_0^{t_k}\int_0^L \dot{G}_1' \xi\, dZ^1 dt - \frac{1}{3}\rho H^3 B \int_0^L G_1' \xi\, dZ^1 \Big|_0^{t_k} - \frac{1}{3}\rho H^3 B \int_0^{t_k} \dot{G}_1 \xi\, dt \Bigg|_0^L + \\
& + \frac{1}{3}\rho H^3 B G_1 \xi \Big|_0^L \Big|_0^{t_k} + \frac{2}{3}\rho H^3 B \int_0^{t_k}\int_0^L G_2' \xi\, dZ^1 dt - \frac{2}{3}\rho H^3 B \int_0^{t_k} G_2 \xi\, dt \Bigg|_0^L + \\
& + \rho g H^2 B \int_0^{t_k}\int_0^L \frac{u''\left(Z^1,t\right)}{[1+u']^3}\xi\, dZ^1 dt + \frac{1}{2}\rho g H^2 B \int_0^{t_k} \frac{1}{[1+u']^2}\xi\, dt \Bigg|_0^L = 0,
\end{aligned}
\tag{14}
$$

where $G_1 = \dfrac{\dot{u}'\left(Z^1,t\right)}{\left[1+u'\left(Z^1,t\right)\right]^4}$, $G_2 = \dfrac{\left[\dot{u}'\left(Z^1,t\right)\right]^2}{\left[1+u'\left(Z^1,t\right)\right]^5}$.

Now we have to group terms corresponding to double integrals, integrals with respect to time, integrals with respect to space and to the corners of the $\left(Z^1,t\right)$ rectangular region. Application of the fundamental lemma of variational calculus to the group of double integrals leads to the following differential equation for the action integral of the problem

$$\begin{gathered} -\ddot{u}\left(Z^1,t\right) + \frac{1}{3}H^2\dot{G}_1'\left(Z^1,t\right) + \frac{2}{3}H^2G_2'\left(Z^1,t\right) + gH\frac{u''\left(Z^1,t\right)}{\left[1+u'\right]^3} = 0, \\ 0 < Z^1 < L, \quad 0 < t < t_k, \\ -\ddot{u}\left(Z^1,t\right) + \frac{1}{3}H^2\ddot{u}''\left(Z^1,t\right) + gHu''\left(Z^1,t\right) = 0. \end{gathered} \tag{15}$$

The first equation describes the non-linear case and the second its linear approximation. Such a description is used in the following relations.

Analysis of the group of integrals with respect to time t leads to the following boundary conditions as functions in time for the boundaries $Z^1 = 0, Z^2 = 0$ and $Z^1 = L$ respectively

$$\begin{gathered} \left\{-M_l\ddot{u} - k_l\left[u - u_{la}\right] + \frac{1}{3}\rho H^3B\left[\dot{G}_1 + 2G_2\right] + \right. \\ \left. -\frac{1}{2}\rho gH^2B\frac{1}{\left[1+u'\right]^2}\right\}\xi\left(Z^1,t\right) = 0, Z^1 = 0, \\ \left\{-M_r\ddot{u} - k_r\left[u - u_{ra}\right] - \frac{1}{3}\rho H^3B\left[\dot{G}_1 + 2G_2\right] + \right. \\ \left. +\frac{1}{2}\rho gH^2B\frac{1}{\left[1+u'\right]^2}\right\}\xi\left(Z^1,t\right) = 0, Z^1 = L, \\ \left\{-M_l\ddot{u} - k_l\left[u - u_{la}\right] + \frac{1}{3}\rho H^3B\ddot{u}' + \right. \\ \left. -\frac{1}{2}\rho gH^2B\left[1 - 2u'\right]\right\}\xi\left(Z^1,t\right) = 0, Z^1 = 0, \\ \left\{-M_r\ddot{u} - k_r\left[u - u_{ra}\right] - \frac{1}{3}\rho H^3B\ddot{u}' + \right. \\ \left. +\frac{1}{2}\rho gH^2B\left[1 - 2u'\right]\right\}\xi\left(Z^1,t\right) = 0, Z^1 = L. \end{gathered} \tag{16}$$

These expressions have to be satisfied for arbitrary functions ξ and thus the expressions in the largest brackets that correspond to differential equations have to be zero. If the springs are very stiff then the displacements of both ends are almost equal, but such terms as $k_l[u(0,t) - u_{la}(t)]$ and $k_r[u(0,t) - u_{ra}(t)]$ retain their physical meaning as forces in the springs. If the displacements of both ends of a spring are equal then the displacement of the block is known and thus it is not subject to variation. The boundary conditions may be used to calculate the forces in the springs.

The analysis of the group of integrals with respect to the variable Z^1 leads to the following initial condition for $t = 0$

$$\begin{aligned} -\rho HB\Big[\dot{u}\left(Z^1,0\right) - \frac{1}{3}H^2G_1'\left(Z^1,0\right)\Big]\xi\left(Z^1,0\right) = 0, \quad 0 < Z^1 < L, \\ -\rho HB\Big[\dot{u}\left(Z^1,0\right) - \frac{1}{3}H^2\dot{u}''\left(Z^1,0\right)\Big]\xi\left(Z^1,0\right) = 0, \quad 0 < Z^1 < L. \end{aligned} \tag{17}$$

It should be noted that the initial conditions correspond to ordinary differential equations defining the velocities as functions of Z^1 for the fixed initial time. The physical meaning of this condition will be explained later.

Finally the conditions at corners $(0,0)$ and $(L,0)$ are

$$\begin{aligned} \left\{-M_l\dot{u}(0,0) + \frac{1}{3}\rho H^3BG_1(0,0)\right\}\xi(0,0) = 0, \\ \left\{-M_r\dot{u}(L,0) - \frac{1}{3}\rho H^3BG_1(L,0)\right\}\xi(L,0) = 0, \\ \left\{-M_l\dot{u}(0,0) + \frac{1}{3}\rho H^3B\dot{u}'(0,0)\right\}\xi(0,0) = 0, \\ \left\{-M_r\dot{u}(L,0) - \frac{1}{3}\rho H^3B\dot{u}'(L,0)\right\}\xi(L,0) = 0. \end{aligned} \tag{18}$$

In all the relations (16)–(18) $\xi\left(Z^1,t\right)$ is in terms of mechanics a virtual displacement.

To discuss the physical meaning of the derived initial conditions, let us consider the case that at times $t < 0$ the fluid is at rest, but at the time $t = 0^+$ the displacements are zero, but there may be finite velocities and accelerations. This is the problem of sudden application of acceleration or velocity. Let us assume that the motion of the block on the left side induces motion of the fluid and that the ratio L/H is so large that there is only negligible motion close to the block on the right side. For very small values of times $t > 0$ the displacements may be represented by the following series

$$u\left(Z^1,t\right) = v_0tf_1\left(Z^1\right) + \frac{1}{2}a_0t^2f_2\left(Z^1\right) + \frac{1}{6}c_3t^3f_3\left(Z^1\right) + \dots, \tag{19}$$

where the coefficients in the terms correspond to the velocity, acceleration and the values of third and consecutive derivatives with respect to time at time $t = 0$. If the velocity and acceleration coefficients are zero it means the velocity and acceleration fields are continuous in time, then for $0^+ \leftarrow t$ the displacement field satisfies the differential equation (15), the boundary conditions (16), the initial conditions (17) and the conditions at the corners (18). Thus it is convenient in generation of a wave to assume that the initial velocity and accelerations are zero in the motion of the piston.

Let us consider a second case where the velocity of the piston at the time $t = 0^+$ is zero, but there is a jump in accelerations ($v_0 = 0, a_0 \neq 0, c_3 = 0, ...$). Substitution of the corresponding equation (19) into the differential equation (15) and approaching the limit leads to the following expression for the displacement field in the fluid

$$u\left(Z^1, 0^+\right) = \frac{1}{2} a_0 t^2 \exp\left(-\sqrt{3} Z^1 / H\right). \tag{20}$$

When this solution is substituted into the first boundary condition (16) and the behaviour around positive but very small values of time are considered, it follows that the differential equation may be simplified to

$$\left[M_l + \frac{1}{\sqrt{3}} \rho H^2 B\right] \ddot{u}\,(0,t) - k_l\,[u(0,t) - u_{la}\,(t)] = 0, \quad t > 0, \quad t \approx 0^+. \tag{21}$$

The spring has no mass and thus the differential equation corresponds to the standard equation of linear vibrations with one degree of freedom initiated by a sudden application of force. It is easy to check that the initial condition (17) and the conditions at the corners (18) are satisfied. It should be noted that a situation of the sudden application of a force is an approximation of real behaviour and is plausible only for very stiff springs. The solution given by the expressions (20) and (21) are based on the assumptions that the blocks are rigid bodies and the fluid is incompressible – it means the velocity of dilatational waves is infinite. In reality the velocity of dilatational waves is of the order of 1500 m/s and thus in experiments the real behaviour is far from the solution given by the relation (20).

Let us consider a third special case (19) – the velocity is not zero, but the acceleration is ($v_0 \neq 0, a_0 = 0, c_3 = 0, ...$). For this displacement field the differential equation (15) and the boundary condition (16) are satisfied for $t = 0^+$. From the initial condition (17) for the same time it follows that:

$$u_0\left(Z^1, 0^+\right) = v_0 t \exp\left(-\sqrt{3} Z^1 / H\right). \tag{22}$$

For a particle with mass m a finite jump of momentum from 0 at time 0 to mv at time Δt, due to a collision or impact, the impulse is defined as

$$\Pi = \int_0^{\Delta t} F\,(t)\,dt = mv\,(\Delta t) - mv\,(0), \tag{23}$$

where Π is the impulse with dimension Ns and $F(t)$ is the applied force in N. In some cases we have little knowledge about the applied forces. For example, by blowing with a hammer we observe that a motion starts and observe a jump in velocity, but we are not interested in the mechanics of contact of bodies. In our problem, the boundary condition at the corner (0,0) leads to the following expression for the corresponding impulse

$$\Pi\,(0,0) = \left[M_l + \frac{1}{\sqrt{3}}\rho H^2 B\right] v_0, \tag{24}$$

where the expression in the square brackets represents the mass of the block supplemented by the added mass of water. Comparison with the corresponding expression for the sudden application of acceleration (21) shows that the total masses have the same values. It should be stressed that the condition at the corner (18) means that the velocity v_0 has to be zero according to the variational procedure based on the assumed action integral (11).

Let us now examine the case in which the spring coefficients k_l, k_r are so large that it is possible to assume that the connection is rigid. In such a case the extension of the springs is almost zero during motion. Let us look in detail into the behaviour for the left spring. The dynamic equation is contained in the first equation in the boundary condition (16). In the limit the product of the coefficient and displacements becomes an undefined symbol, but its product is always equal to the tensile force in the spring $N_{ls}\,(t)$. Thus it follows

$$\begin{aligned} N_{ls}\,(t) &= -M_l\ddot{u} + \frac{1}{3}\rho H^3 B\big[\dot{G}_1 + 2G_2\big] - \frac{1}{2}\rho g B\frac{H^2}{[1+u']^2}, Z^1 = 0, \\ N_{ls}\,(t) &= -M_l\ddot{u} + \frac{1}{3}\rho H^3 B\ddot{u}' - \frac{1}{2}\rho g BH^2\big[1 - 2u'\big], Z^1 = 0. \end{aligned} \tag{25}$$

The physical meaning of the last term is clear when we take into account the following identity $H + w\left(Z^1, t\right) = H\big[1 + u'\left(Z^1, t\right)\big]^{-1}$ that results from the relation (3). The term corresponds to the force that is due to the hydrostatic pressure with actual depth of water at the block.

2.4. Properties of the Solutions of the Linear Theory

The linear approximation of the differential equation and the initial and boundary conditions gives an insight into the properties of the solution. It also establishes the first step in a nonlinear solution obtained by perturbation method. The basic relations for the linear theory are given in the preceding paragraph.

Adopting the standard method of separation of variables a particular solution may be found. A periodic solution of the linear differential equation given in (15) that corresponds to a harmonic wave moving to the right is

$$u\left(Z^1, t\right) = C\exp\left[i\left(k_w Z^1 - \omega_w t\right)\right], \tag{26}$$

where $k_w = 2\pi/L_w$ is the wave number, L_w is the length of the wave, $\omega_w = 2\pi/T_w$ is the angular frequency, T_w is the wave period, and the differential equation is satisfied if the following dispersion relation is satisfied

$$\omega_w^2 = gH\frac{k_w^2}{1 + H^2k_w^2/3}, \quad k_w^2 = \frac{\omega_w^2}{gH - H^2\omega^2/3}, \tag{27}$$

where the second relation corresponds to the inverse of the dispersion relation. The phase velocity c for very long waves should be close to its limit value when the ratio L/H tends to infinity $c = \sqrt{gH}$. Thus it follows from the relations (27) that

$$H^2k_w^2/3 << 1, \quad \omega_w^2 H^2/3 << gH. \tag{28}$$

The expansion of the dispersion relation (27) in power series leads to the expression

$$\omega_w^2 = k_w^2 gH\left[1 - \frac{1}{3}(k_w H)^2 + \frac{1}{9}(k_w H)^4 \ldots\right]. \tag{29}$$

It is worthwhile noting that in the corresponding expansion of the general dispersion relation of the linear wave theory, the first two terms are the same and thus for sufficiently small values of $k_w H = 2\pi H/L_w$ the assumed simplification of the geometry of displacements is reasonable.

3. A Finite Element Numerical Model – Discrete in Space and Time

3.1. The Approximated Displacement Field

Let us consider a finite rectangular element with one side corresponding to the length ΔZ and the second side corresponding to the time interval of length Δt. The unknown function of horizontal displacements $u(Z, t)$ in the space time rectangle is approximated by the two-dimensional function

$$\begin{aligned} u(Z,t) = {} & u(r,s)\left[\frac{1}{2} - \frac{t}{\Delta t}\right]\left[\frac{1}{2} - \frac{Z}{\Delta Z}\right] + u(r+1,s)\left[\frac{1}{2} - \frac{t}{\Delta t}\right]\left[\frac{1}{2} + \frac{Z}{\Delta Z}\right] + \\ & + u(r,s+1)\left[\frac{1}{2} + \frac{t}{\Delta t}\right]\left[\frac{1}{2} - \frac{Z}{\Delta Z}\right] + u(r+1,s+1)\left[\frac{1}{2} + \frac{t}{\Delta t}\right]\left[\frac{1}{2} + \frac{Z}{\Delta Z}\right], \end{aligned} \tag{30}$$

where in the local coordinate system Z is measured from the midpoint of the element $-\Delta Z/2 \leq Z \leq \Delta Z/2$, the time t is measured from the midpoint of the interval $-\Delta t/2 \leq t \leq \Delta t/2$ and $u(r, s)$ is a short notation for $u(r\Delta Z, s\Delta t)$ in the global coordinate system and denotes the value of the horizontal displacements at the specified corner of the space time element (Z corresponds to Z^1 in the preceding paragraph).

The derivative of the horizontal displacements function (30) with respect to the space variable Z is

$$\frac{\partial u\,(Z,t)}{\partial Z}=\frac{\alpha}{\Delta Z}+\frac{\beta}{\Delta Z\Delta t}t, \tag{31}$$

where

$$\alpha=[u\,(r+1,s+1)+u\,(r+1,s)-u\,(r,s+1)-u\,(r,s)]/2,$$

$$\beta=u\,(r+1,s+1)-u\,(r+1,s)-u\,(r,s+1)+u\,(r,s)\,,$$

and function α corresponds to the finite difference in the Z direction and function β corresponds to the second mixed finite differences (in space and time) The derivative of the horizontal displacements function (30) with respect to the time variable t is

$$\frac{\partial u\,(Z,t)}{\partial t}=\frac{\gamma}{\Delta t}+\frac{\beta}{\Delta Z\Delta t}Z, \tag{32}$$

where

$$\gamma=[u\,(r+1,s+1)-u\,(r+1,s)+u\,(r,s+1)-u\,(r,s)]/2,$$

and the function γ corresponds to the finite difference in the t direction.

3.2. The Potential and Kinetic Energies and the Action Integral

According to the relation (6) the expression for the potential energy of the water for an element after substitution of the expression (31) and integration in the limits $-\Delta Z/2$ to $\Delta Z/2$ is

$$E_{pw}=\frac{1}{2}\rho g H^2\Delta Z\frac{1}{1+\alpha/\Delta Z+t\beta/\left(\Delta Z\Delta t\right)}. \tag{33}$$

The potential energy of the left and right springs is given in the first row of the relations (10). In a local coordinate system, for a line element $-\Delta t/2<t<\Delta t$ the expression for the potential energy of the left spring is

$$E_{pl}=\frac{1}{2}k_l\left[\frac{1}{2}\lambda_0+\lambda_1\frac{t}{\Delta t}\right]^2, \tag{34}$$

where

$$\lambda_0=[u\,(0,s+1)+u\,(0.s)]-[u_{la}\,(s+1)+u_{la}\,(s)]-4u_{l0},$$

$$\lambda_1=[u\,(0,s+1)-u\,(0.s)]-[u_{la}\,(s+1)-u_{la}\,(s)].$$

For the right spring the expressions are similar.

Substitution of the preceding expressions into the expression for the kinetic energy of the fluid and integration leads to the formula for the kinetic energy of the element

$$\begin{aligned} E_{kw} = {} & \frac{1}{2\,(\Delta t)^2}\rho H \Delta Z\left[\gamma^2 + \frac{\beta^2}{12}\right]+ \\ & +\frac{1}{6\,(\Delta t)^2}\rho H^3 \Delta Z \frac{(\beta/\Delta Z)^2}{[1+\alpha/\Delta Z + t\beta/\,(\Delta Z \Delta t)]^4}. \end{aligned} \tag{35}$$

The first term on the right side corresponds to the kinetic energy due to the horizontal components of velocity and the second one due to the vertical component. For very shallow water the second term may be neglected and the problem may be simplified and reduced to the classic problem of solution of a hyperbolic partial differential equation. The expressions for the kinetic energy of the blocks result immediately from the relations in the second row when the velocities are expressed by finite difference ratios

$$\begin{aligned} E_{kl} = {} & \frac{1}{2\,(\Delta t)^2} M_l [u\,(0, s+1) - u\,(0, s)]^2,\; E_{kr} = \\ = {} & \frac{1}{2\,(\Delta t)^2} M_r [u\,(L, s+1) - u\,(L, s)]^2. \end{aligned} \tag{36}$$

The action integral is defined by the integral in time (11). Integration in the local coordinate system with respect to time from $-\Delta t/2$ to $\Delta t/2$ leads to the expression

$$\begin{aligned} I_w = {} & \frac{1}{18\Delta t}\rho H^3 \Delta Z\left\{\frac{\beta/\Delta Z}{[1+\alpha/\Delta Z - \beta/\,(2\Delta Z)]^3} - \frac{\beta/\Delta Z}{[1+\alpha/\Delta Z + \beta/\,(2\Delta Z)]^3}\right\}+ \\ & +\frac{1}{2\Delta t}\rho H \Delta Z^3\left[\frac{\gamma^2}{\Delta Z^2} + \frac{1}{12}\frac{\beta^2}{\Delta Z^2}\right] - \frac{1}{2}\rho g \frac{\Delta t}{\beta} H^2 \Delta Z^2 \ln\frac{1+\alpha/\Delta Z + \beta/\,(2\Delta Z)}{1+\alpha/\Delta Z - \beta/\,(2\Delta Z)}. \end{aligned} \tag{37}$$

It should be noted that when β tends to zero the ln function goes to zero and the expression (37) becomes an undetermined symbol 0/0. When the following expansion in power series is used

$$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{k=1}^{\infty}\frac{1}{2k-1}x^{2k-1}, \quad \left[x^2 < 1\right], \tag{38}$$

where $x = \beta/[2\,(\Delta Z + \alpha)]$, the last term in the relation (37) becomes

$$I_{pw} = -\rho g H^2\,(\Delta Z)^2\,\Delta t \sum_{k=1}^{\infty}\frac{1}{2k-1}\frac{\beta^{2k-2}}{[2\,(\Delta Z + \alpha)]^{2k-1}}. \tag{39}$$

The first three terms of the expansion are

$$I_{pw} \approx -\rho g H^2 (\Delta Z)^2 \Delta t \times$$
$$\times \left\{ \frac{1}{2(1+\alpha/\Delta Z)} + \frac{1}{3} \frac{\beta^2/\Delta Z^2}{2^3 (1+\alpha/\Delta Z)^3} + \frac{1}{5} \frac{\beta^4/\Delta Z^4}{2^5 (1+\alpha/\Delta Z)^5} \right\}. \tag{40}$$

There is no singularity at point $\alpha = 0$, $\beta = 0$ in the expression (39) and the expansion in terms of a polynomial in α and β may be done and is simple.

The integration in time for an element, of the difference between the kinetic and potential energies of the left mechanical system (36) and (34) leads finally to the relation

$$I_l = \frac{1}{2\Delta t} M_l [u(0, s+1) - u(0, s)]^2 - \frac{1}{2} k_l \Delta t \left[\frac{1}{4}\lambda_0^2 + \frac{1}{12}\lambda_1^2 \right]. \tag{41}$$

3.3. The Algebraic Equations Derived by Variational Calculus

The action integral for a finite element of the fluid is a function of quantities α, γ and β that are expressed in terms of the displacements at the corners. The action integral of the block and spring systems is also a function of the corresponding horizontal displacements. For a specific problem, the value of the action integral over the whole space and time domain is equal to the sum of values over the finite elements. Thus it is a function of all the known and unknown displacements. There are no derivatives of displacements in the expressions as in the case of continuum. Thus the variational procedure reduces to the statement that the action integral should attain an extreme value and thus the partial derivative with respect to the unknown displacements should be zero. The chain rule of differentiation yields the final set of algebraic equations. These relations lead to a set of algebraic equations in the unknown displacements in a short notation written as $u(r, s)$.

For a typical point in the fluid the displacement $u(r, s)$ enters expressions only in the four neighboring elements as is show in Fig. 1, where the elements are denoted by Roman numerals. For element I the functions α, γ, β are

$$\begin{aligned}
&\alpha^{(I)} = \\
&= \frac{1}{2}[u(r+1, s+1) + u(r+1.s+0) - u(r+0, s+1) - u(r+0, s+0)], \\
&\gamma^{(I)} = \\
&= \frac{1}{2}[u(r+1, s+1) - u(r+1.s+0) + u(r+0, s+1) - u(r+0, s+0)], \\
&\beta^{(I)} = u(r+1, s+1) - u(r+1.s+0) - u(r+0, s+1) + u(r+0, s+0)\,.
\end{aligned} \tag{42}$$

The corresponding expressions for the other three elements can be written without difficulty.

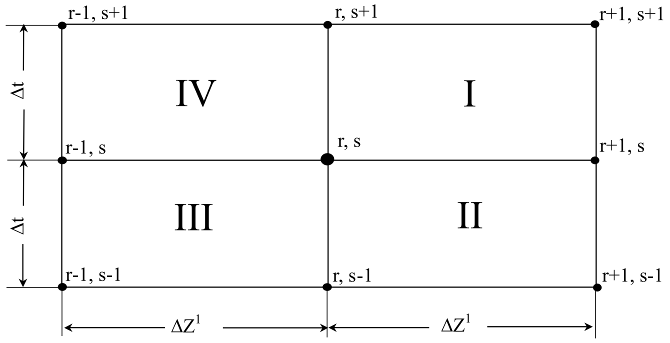

Fig. 1. The typical internal nodal point $r\Delta Z^1$, $s\Delta t$ and the adjanced finite elements I, II, III, IV

For a typical point (r, s) the unknown displacement $u\,(r, s)$ enters only in the four finite elements of its neighborhood. By I let us denote the finite element with the corners (r, s), $(r + 1, s)$, $(r + 1, s + 1)$ and $(r, s + 1)$. By II, III, IV let us denote the consecutive finite elements when one moves clock-wise around the point (r, s). In the general case the algebraic equation corresponding to the typical point (r, s) is obtained by substitution of the corresponding expressions $\alpha^{(I)}, ..., \alpha^{(IV)}, \beta^{(I)}, ..., \gamma^{(IV)}$ into the expressions for the action integral of the fluid and the use of the chain rule of differentiation to obtain the following set of non-linear equations in the unknown displacement $u\,(p, q)$

$$\sum_{J=I}^{J=IV} \frac{\partial I_w}{\partial \alpha^{(J)}} \frac{\partial \alpha^{(J)}}{\partial u\,(r, s)} + \sum_{J=I}^{J=IV} \frac{\partial I_w}{\partial \gamma^{(J)}} \frac{\partial \gamma^{(J)}}{\partial u\,(r, s)} + \sum_{J=I}^{J=IV} \frac{\partial I_w}{\partial \beta^{(J)}} \frac{\partial \beta^{(J)}}{\partial u\,(r, s)} = 0. \tag{43}$$

For a typical point on the left boundary $(0, s)$ the neighboring finite elements correspond to the elements denoted above by I and II. One also has to take into account the action integral for the left block and spring. The differentiation with respect to the unknown displacement leads to the following typical equation

$$\begin{aligned} &\frac{M_l}{\Delta t}\left[u\,(0, s+1) - 2u\,(0, s) + u\,(0, s-1)\right] + \\ &+ \frac{k_l \Delta t}{6}\left\{\left[u\,(0, s+1) - u_{la}\,(s+1)\right] + 4\left[u\,(0, s) - u_{la}\,(s)\right] + \right. \\ &\left. \quad + \left[u\,(0, s-1) - u_{la}\,(s-1)\right] - 6u_{l0}\right\} + \\ &- \sum_{J=I}^{J=2} \frac{\partial I_w}{\partial \alpha^{(J)}} \frac{\partial \alpha^{(J)}}{\partial u\,(r, s)} - \sum_{J=I}^{J=2} \frac{\partial I_w}{\partial \gamma^{(J)}} \frac{\partial \gamma^{(J)}}{\partial u\,(r, s)} - \sum_{J=I}^{J=2} \frac{\partial I_w}{\partial \beta^{(J)}} \frac{\partial \beta^{(J)}}{\partial u\,(r, s)} = 0. \end{aligned} \tag{44}$$

The term with the mass of the left block corresponds to the inertia force, the second term corresponds to the tensile force in the spring and the rest of the terms is due to the hydrodynamic force of the wave. Note that the equation (44) has to be divided by Δt to obtain the finite difference expression for the acceleration and the averaged displacements. The relation (44) may be used to calculate the forces even in the case of the spring constant being infinite.

3.4. Properties of Solutions of the Linear Theory

In a linear approximation the action integral is a second order expression in α, γ and β. Thus it follows from (37) and (39)

$$I_w = \frac{1}{2\Delta t}\rho H \Delta Z B\left\{\left(\gamma^2+\frac{\beta^2}{12}\right)+\left(\frac{H}{\Delta Z}\right)^2\frac{\beta^2}{3}-\frac{g\,\Delta t^2 H}{\Delta Z^2}\left(\alpha^2+\frac{\beta^2}{12}\right)\right\}, \tag{45}$$

where in the expression, due to the potential energy, the constant term has no influence and the linear term in α leads to the expression for u_{0l} and therefore these two terms are omitted.

For the typical point (r,s) within the linear theory, the differentiation according to the relation (43) leads to the following linear equation

$$\begin{aligned}
&\frac{gH}{6\,(\Delta Z)^2}\begin{bmatrix} +1u\,(r-1,s+1) & -2u\,(r+0,s+1) & +1u\,(r+1,s+1)\\ +4u\,(r-1,s+0) & -8u\,(r+0,s+0) & +4u\,(r+1,s+0)\\ +1u\,(r-1,s-1) & -2u\,(r+0,s-1) & +1u\,(r+1,s-1)\end{bmatrix}+\\
&-\frac{1}{6\,(\Delta t)^2}\begin{bmatrix} +1u\,(r-1,s+1) & +4u\,(r+0,s+1) & +1u\,(r+1,s+1)\\ -2u\,(r-1,s+0) & -8u\,(r+0,s+0) & -2u\,(r+1,s+0)\\ +1u\,(r-1,s-1) & +4u\,(r+0,s-1) & +1u\,(r+1,s-1)\end{bmatrix}+\\
&+\frac{H^2}{3\,(\Delta Z\Delta t)^2}\begin{bmatrix} +1u\,(r-1,s+1) & -2u\,(r+0,s+1) & +1u\,(r+1,s+1)\\ -2u\,(r-1,s+0) & +4u\,(r+0,s+0) & -2u\,(r+1,s+0)\\ +1u\,(r-1,s-1) & -2u\,(r+0,s-1) & +1u\,(r+1,s-1)\end{bmatrix}=0,
\end{aligned} \tag{46}$$

where this expression has been divided by $\rho H \Delta Z B \Delta t$. It is easy to see that the finite difference equation corresponds to the linear differential equation given in the relation (15).

For the partial differential equation (15) many finite difference approximate expressions may be written. The variational approach leads to the algebraic expression (46), which is consistent with the assumed relation for the action integral with the chosen shape function.

Let us consider the typical initial conditions that at the time $t = 0$ the displacements and velocities are zero. The finite elements formulation corresponds to the conditions that the displacements for the time zero and Δt are zero for all considered points in space (the first two rows are zero). Thus the equations written for the first three rows on the basis of (46) form a tri-diagonal regular set. The first term may be omitted. Finally the equation that corresponds to the first rows multiplied by $3\Delta Z^2 \Delta t^2/H^2$ in the equation (46) is

$$\frac{g\,\Delta t^2}{2H}\begin{bmatrix} u\,(r-1,2) \\ -2u\,(r,2) \\ +u\,(r+1,2) \end{bmatrix} - \frac{\Delta Z^2}{2H^2}\begin{bmatrix} u\,(r-1,2) \\ +4u\,(r,2) \\ +u\,(r+1,2) \end{bmatrix} + \begin{bmatrix} u\,(r-1,2) \\ -2\,(r,2) \\ +u\,(r-1,2) \end{bmatrix} = 0, \tag{47}$$

where the expressions in the square brackets have to be summed. If the time step tends to zero $u\,(r,2) = u\,(0,2)\exp(-rx)\,, r = 0, 1, \ldots$ then the equations are satisfied when

$$\cosh x = \frac{1 + (\Delta Z/H)^2}{1 - 0.5\,(\Delta Z/H)^2}. \tag{48}$$

In the limit when $\Delta Z/H$ tends to zero, the value of x approaches $\pm\sqrt{3}\Delta Z/H$. Thus this case corresponds to the case of a jump in acceleration at the time zero in continuum (21).

Let us consider a typical point $(0,s)$ on the left boundary of the fluid. We have to consider the two finite elements and as in the relation (44) the inertia force of the left block and the potential energy of the spring on the left. Upon tedious but simple calculation it follows that:

$$\begin{aligned} &\frac{M_l}{\Delta t^2}\begin{bmatrix} +1u\,(0,s+1) \\ -2u\,(0,s+0) \\ +1u\,(0,s-1) \end{bmatrix} + \frac{k_l}{6}\begin{bmatrix} u\,(0,s+1) - u_{la}\,(s+1) \\ +4u\,(0,s+0) - 4u_{la}\,(s+0) \\ +u\,(0,s-1) - u_{la}\,(s-1) \end{bmatrix} + \\ &-\frac{gH}{\Delta Z^2}\frac{M_w}{6}\begin{bmatrix} -u\,(0,s+1) + u\,(1,s+1) \\ -4u\,(0,s) + 4u\,(1,s) \\ -u\,(0,s-1) + u\,(1,s-1) \end{bmatrix} + \\ &-\frac{1}{\Delta t^2}\frac{M_w}{6}\begin{bmatrix} -2u\,(0,s+1) - u\,(1,s+1) \\ +4u\,(0,s) + 2u\,(1,s) \\ -2u\,(0,s+1) - u\,(1,s-1) \end{bmatrix} + \\ &-\frac{H^2}{\Delta Z^2}\frac{1}{\Delta t^2}\frac{M_w}{6}\begin{bmatrix} -2u\,(0,s+1) + u\,(1,s+1) \\ +4u\,(0,s) - 4u\,(1,s) \\ -2u\,(0,s-1) + u\,(1,s+1) \end{bmatrix} = 0, \end{aligned} \tag{49}$$

where $M_w = \rho H \Delta Z B$ is the mass of the column of water with width equal to ΔZ.

In this relation only dynamic loads are considered. The hydrostatic force and the corresponding horizontal displacement u_{0l} is omitted. These equations may be used to calculate the tensile force in the spring even if the mass of the block is neglected and the spring has an infinite rigidity. In such a case the horizontal displacement of the block is not an unknown entering the set of equations, $u\,(0,s) = u_{la}\,(s)\,.$

Let us look for a solution of the discrete description, corresponding to the initial condition that in the continuum at time zero there are no displacements but there is a jump in velocity at the left boundary. Such a discrete solution should start from the situation that $u(r,0) = 0, r \geq 0, u(r,1) \neq 0, r \geq 0, u(r,s) = 0, r \geq 0, s \geq 2$. It is easy to verify that the second rows in the relation (46) lead to a similar solution as in the case of a jump in accelerations. The final result is that $u(r,1) = u(0,1)\exp(-rx)$, where x is given by the equation (48). Let us look for the dispersion relation in the discrete formulation. The substitution of the expression for a progressive harmonic wave (26) into the typical equation for the point $r = 0$, $s = 0$ yields

$$\begin{aligned}
&\frac{gH}{6\Delta Z^2}[-8 + 8\cos k_w \Delta Z - 8\cos \omega_w \Delta t - 4\cos k_w \Delta Z \cos \omega_w \Delta t]+ \\
&-\frac{1}{6\Delta t^2}[8 + 4\cos k_w \Delta Z - 8\cos \omega_w \Delta t - 4\cos k_w \Delta Z \cos \omega_w \Delta t]+ \\
&+\frac{H^2}{3\Delta Z^2 \Delta t^2}[4 - 4\cos k_w \Delta Z - 4\cos \omega_w \Delta t + 4\cos k_w \Delta Z \cos \omega_w \Delta t] = 0.
\end{aligned} \tag{50}$$

This equation may be used to calculate the angular frequency in terms of the wave number when the other parameters are known. It is possible to arrange the equation in a manner more suitable for numerical calculations. It is easy to calculate the limit when $\Delta Z \to 0$, $\Delta t \to 0$. When two terms of expansion of the cosine functions are substituted into (50) it follows

$$gH\left[-k_w^2 + k_w^2\omega_w^2\Delta t^2/6\right] + \left[\omega_w^2 - k_w^2\omega_w^2\Delta Z^2/6\right] + H^2 k_w^2 \omega_w^2/3 = 0. \tag{51}$$

When the time and space steps tend to zero, the expression becomes the dispersion relation in continuum (27). The discrete formulation has an almost equal dispersion relation if both conditions $\omega_w^2\Delta t^2/6 << 1$, $k_w^2\Delta Z^2/6 << 1$, are satisfied.

4. The Comparison of Calculated Values with Experimental Data

The experiments were performed in the wave laboratory of the Institute of Hydro-Engineering in the year 2000 and described in a paper written by Wilde et al (2001). The wave flume is 64 m long, 0.6 m wide and 1.4 m high and is equipped with a piston type generator. Control time series of piston motion, calculated on a computer, can be fed into the control system of the generator. In the above-mentioned experiments the water depth was $H = 0.6$ m. The position of the free surface was measured at six points along the flume as a function of time. The first gauge was placed 4 m from the piston at rest and the consecutive ones in spacing of 4 m. Thus the distance from the piston to the last one was 24 m. At the end of the flume a very effective wave-damper is installed, such that the reflection of waves is very small. Additionally the length of the wave group was chosen in such a way that the time of approach of the reflect wave is so large that it had no influence on the measured values.

In Fig. 2 typical graph of measured surface elevation as a function of time is depicted. The wave height increases slowly and then decreases. The interval denoted by vertical lines may be considered as an almost regular wave. The Fourier coefficients were calculated by a least square approximation. It was possible to estimate three components, the first corresponded to the basic one introduced by the control series and the double and triple frequencies were due to the nonlinear effects.

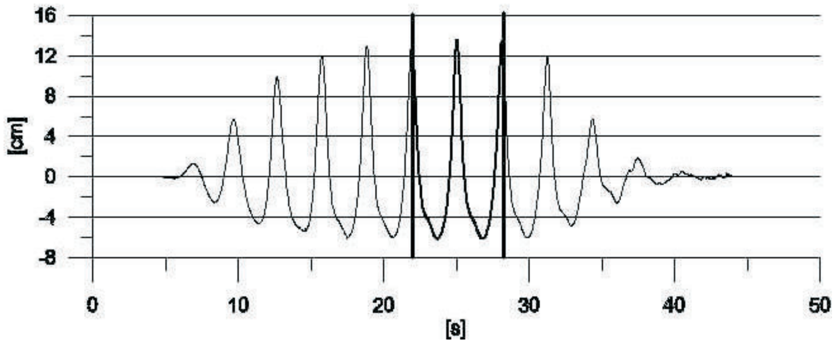

Fig. 2. The wavetrain as a function of time

From the large set of measured data included in the internal report, the measured data for long waves with dominant periods $T = 3.098$ s (dominant frequency $f = 0.3228$ Hz) were chosen. The waves with three wave heights $h = 0.0735$ m, $h = 0.135$ m, $h = 0.188$ m were considered. The Ursell dimensionless number U gives a good estimate of the influence of nonlinear effects and is defined by the formula

$$U = \frac{h}{H}\frac{L_w^2}{H^2},$$

where L_w is the wave length (its value in the first step may be calculated on the basis of the linear dispersion relation, a better estimation is obtained from the formulae of the Stokes wave). For the considered wave heights the Ursell numbers are 1) $U = 17.64$, 2) $U = 32.4$, 3) $U = 45$. In the first case the nonlinear effects are weak and the influence increases for the greater values of the Ursell number.

The measured displacements of the piston are used as the boundary condition in the numerical calculations. The differences between the control series that has the basic harmonic only and the measured series are not large. For example the amplitudes of the second harmonics for the consecutive Ursell numbers are 2.5%, 4%, 9% of the corresponding first ones. The corresponding triple frequency component is less than 3% of the first one. These values indicate that there is an interaction between the piston and the waves in the flume. The double and triple frequency components of the hydrodynamic loads introduce such components in the motion of the piston.

In Fig. 3 comparison of the amplitudes is depicted of components calculated by the numerical model and the corresponding empirical results as a function of distance from the piston at rest. In the calculation the nonlinear terms up to the eight order are taken into account. The graphs 3a, 3b, 3c correspond to the experiments with the consecutive Ursell numbers. Solid lines represent the calculated first components, the dotted the second and the dashed lines the third components. The diamonds represent the empirical values of the first components, circles the second and triangles the third components. It is seen that there is good agreement between the empirical values and the calculated functions. In the theoretical description the piston is perfect, there is no friction and no flow between the piston and the glass walls of the flume. In the real situation there is a flow and there are boundary layers that introduce turbulence into the neighborhood of the piston. Thus there must be a loss of energy in the generated waves.

In the experiments performed in the year 2000 the above-described experiments were supplemented by experiments with modified control time series. Exaggerated components with double and triple frequencies were added. In Fig. 4 the results of analysis are depicted. The notations and symbols are the same as in Fig. 3. In this case the agreement between the theoretical calculations and the experimental results is not good. In the paper P. Wilde et al (2001) the results of measurements of horizontal components of velocities are presented. It may be seen, as is expected from the theory of regular Stokes waves, that the assumption that the displacements and velocities along the depth are uniform is reasonable, but there is a substantial difference in the case of the second and third component. If the higher components are exaggerated in the control time series, an interaction problem has to be studied and it looks like the assumption of uniform displacements along the depth should be abandoned.

5. Results and Conclusions

For the waves in a standard wave flume equipped with a piston generator the introduced assumptions in the description of the geometry of motion lead to the expressions for the kinetic and potential energies as functions of Z^1 and time t.

Application of the standard variational calculus to the action integral leads directly to the partial differential equation of the problem and the natural initial and boundary conditions in a formal way.

The obtained description has a physical meaning that is facilitated by the simplified linear expressions. It should be noted that the assumption of incompressibility results in the assumption that the velocity of dilatational waves is infinite. Thus the expressions for a sudden application of a force do not conform to experiments that show dilatational waves with high frequencies.

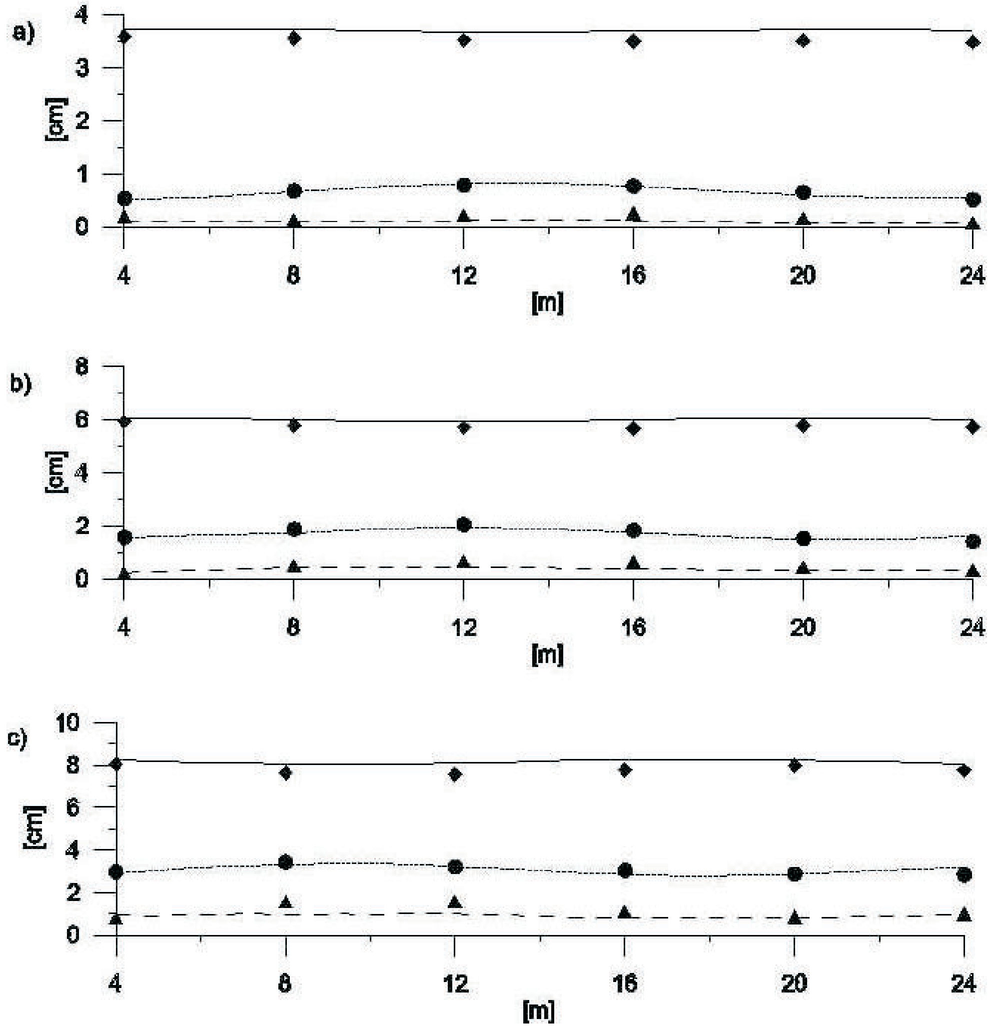

Fig. 3. The amplitudes of Fourier Components of the calculated time series as a function of distance from the piston at rest: solid lines – basic frequency, dotted lines – double frequency, dashed lines – triple frequency. The amplitudes of Fourier Component of data measured at the positions of gauges: diamonds – basic frequency, circles – double frequency triangles – triple frequency. The graphs correcpond to Ursell numbers a) $U = 17.64$, b) $U = 32, 4$, c) $U = 45$

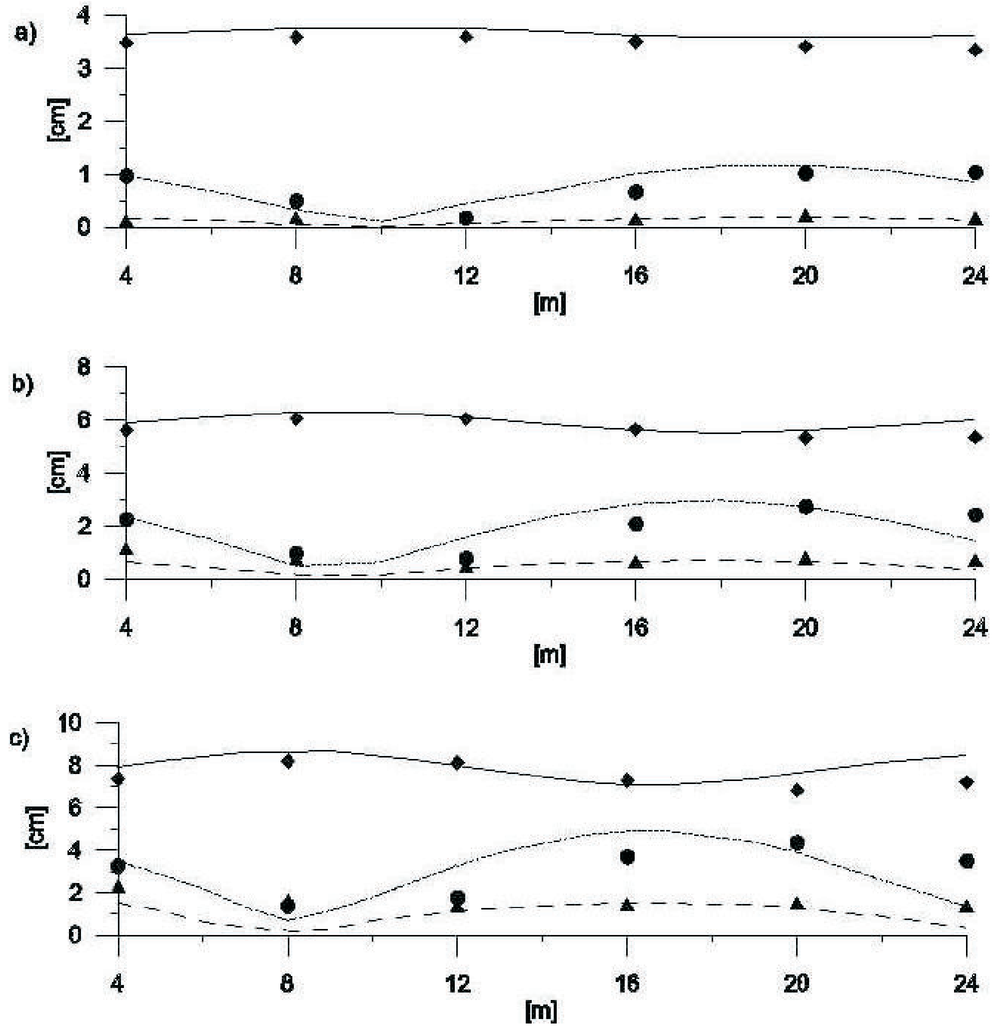

Fig. 4. Modified control time series with exaggerated components. The amplitudes of Fourier Components of the calculated time series as a function of distance from the piston at rest: solid lines – basic frequency, dotted lines – double frequency, dashed lines – triple frequency. The amplitudes of Fourier Component of data measured at the positions of gauges: diamonds – basic frequency, circles – double frequency triangles – triple frequency. The graphs correcpond to Ursell numbers a) $U = 17.64$, b) $U = 32, 4$, c) $U = 45$

The simplified description of motion, the incompressibility condition and the applied variational calculus enable us to define the resultant horizontal force only and not the pressure field in the fluid.

The substitution of an expression for a stationary wave moving to the right into the simplified linear differential equation leads to the dispersion relation. It is worthwhile to note that the general dispersion relations of the linear wave theory equations and the derived relation have the same two first terms in a power series expansion.

In the numerical finite element method a simple shape function is assumed and the expressions for the potential and kinetic energies are derived as functions of values at the nodal points. The action integral is an algebraic equation in terms of the unknown horizontal displacements. The set of the algebraic equations for typical interior points and points on the boundaries are obtained by differentiating with respect to displacements at these points. All the finite differences are consistent within the same discrete formulation.

The linear case that is described by a set of linear equations is studied. It is easy to see that the sets of linear equations correspond exactly to the linear differential equations of the problem in continuum. There are many possibilities to approximate differential equations by finite differences. The variational calculus furnishes a straightforward method that yields the corresponding schemes for the interior and boundary points that preserve the mechanical energy.

The comparison of the experimental results with the calculated values shows good agreement for the case of a wavetrain with a dominant frequency that corresponds to the case of a shallow water wave. In case the control series is modified by assuming an exaggerated double frequency term the agreement is not so good.

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