Archives of Hydro-Engineering and Environmental Mechanics
Vol. 50 (2003), No. 3, pp. 269–285

An Approximate Solution to the Boussinesq Type Equations Describing Periodic Waves

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(Received July 23, 2003; revised September 16, 2003)

Abstract

The paper concerns the non-linear problem of description of shallow-water waves of finite amplitude. The description is based on the conservation-law formulation, which seems to be particularly convenient in constructing an approximate solution of the problem considered. The analysis is confined to the one-dimensional case of waves propagating in water of constant depth. In the model considered, vertical acceleration of the fluid is taken into account, and thus, the fundamental equations of the problem are similar to those given by Boussinesq (Abbott 1979). The equations differ from those frequently used in shallow-water hydrodynamics which are based on the assumption of hydrostatic pressure. An approximate solution of the problem is constructed by means of a perturbation scheme with the third order expansion of the equations with respect to a small parameter. It is demonstrated that the solution procedure may be successfully applied only within a certain range of the two ratios defining wave height to water depth and the depth to wave length, respectively.

Key words: shallow water, water wave, Boussinesq approximation

1. Introduction

In describing a flow in shallow water, it is frequently justified to assume that the flow is almost horizontal and thus it is reasonable to integrate the fundamental equations of the problem with respect to the vertical direction, and finally, to consider a system of averaged equations corresponding to horizontal directions only. In this way, a system of so-called shallow-water equations is obtained. In the case of water waves, one can speak of the theory of waves in shallow water. It is understood that in the latter case the approximate description is justified only for relatively long waves of small amplitude. A classification of the description depends on two length ratios: the amplitude of the wave to the water depth, and the water depth to the wave length. In the literature on the subject two non-linear theories of shallow-water waves exist depending on the level of approximation (Mei 1983). The first one developed by Airy and the other by Boussinesq and Korteweg and de Vries. In the Airy approach the fluid pressure is hydrostatic, while in the second formulations a vertical acceleration of the fluid is taken into account. Obviously, the second formulation is a more general one. The Airy approximation may be applied to very long waves, while the Boussinesq theory serves for moderately long waves in shallow water. In the Boussinesq approach the vertical acceleration of the fluid is described as an approximate. One of the simplest description is based on the assumption that the magnitude of the vertical component of the velocity varies from zero at the bottom to a maximum value at the free surface. A particular choice of the description of the vertical velocity leads to a particular form of the Boussinesq equations describing shallow water waves (Abbott 1979). In shallow-water hydrodynamics there are also possible other restrictions which can be imposed on the flow. For instance, the Korteveg and de Vries equation has been derived on the assumption of the irrotational motion of the fluid corresponding to one-dimensional propagation of a surface wave (van Groesen and de Jager 1994). Most of the formulations of shallow water hydrodynamics mentioned above, lead to a system of non-linear differential equations of fluid motion. In order to integrate the equations, further simplifying assumptions are necessary. For example, the system of differential equations for the one dimensional water waves formulated by Boussinesq (Whitham 1974, Abbott 1979) has the form

$$\begin{aligned} &h_t + (uh)_x = 0, \\ &u_t + uu_x + gh_x + \frac{1}{3}h_0 h_{xtt} = 0, \end{aligned} \tag{A}$$

where:

- h – the water depth (h_0 is the fluid depth at rest),
- u – the average horizontal velocity,
- t, x – subscripts denoting the partial derivatives with respect to time and space coordinates, respectively.

In the literature on the subject (Ursell 1953, Whitham 1974, van Groesen and de Jager 1994) one can find differential equations describing the wave profile $\eta(x,t)$. An example is the equation obtained by linearization of the system (A) (Whitham 1974)

$$\eta_{tt} - gh_0\eta_{xx} - \frac{1}{3}h_0^2\eta_{xxtt} = 0. \tag{B}$$

A more general, non-linear equation describing the wave profile, resulting from the system (A), may be found in Ursell (1953). At the same time, a particular approximation of the shallow water equations leads to the non-linear Korteveg and de Vries equation (Whitham 1974)

$$\eta_t + gh_0\left(1 + \frac{3}{2}\frac{\eta}{h_0}\right)\eta_x + \frac{1}{6}gh_0^3\eta_{xxx} = 0. \tag{C}$$

In general, it is not possible to find closed form analytical solutions of the non-linear shallow water equations and therefore we have to resort to approximate methods. In the approximate description of the equations it is convenient to use the so-called conserved variables, namely, mass density, momentum and energy per unit mass, instead of the physical variables i.e. mass density, pressure, velocity vector (Toro 1997). The system of equations obtained in this way describes the conservation of fluid mass, fluid momentum and its energy, respectively. In the case of adiabatic flow of a perfect incompressible fluid, the pressure is not a thermodynamic variable, therefore the energy equation may be excluded from the system of the fundamental equations. When the pressure is described by a hydrostatic formula and effects of body forces, viscous stresses and heat flux are neglected, the Navier-Stokes equations expressed in the new variables form a system of non-linear conservation laws known in the literature of the subject as the Euler equations (Toro 1997). For long waves and nearly horizontal flows the assumption of hydrostatic pressure is justified and therefore the Euler equations are frequently used to describe the shallow water phenomena. The Euler equations, in their conservation slightly modified shallow water form, are even used to describe a rapidly varied flow, as, for example, the flood flows (Jha et al. 2000) and the dam break problem (Ambrosi 1995, Szydłowski 2001). In the present paper a conservative system of equations corresponding to the Boussinesq approach is formulated, and then solved by means of successive approximations with the help of a perturbation scheme. The detailed discussion is confined to the third order expansion of equations describing one-dimensional periodic waves of moderate length and height, propagating in shallow water.

2. The Governing Equations

In what follows we confine our attention to the plane problem of a fluid motion in Euclidean space. In order to describe the motion we introduce the Cartesian coordinate system (x, z) where x is the horizontal axis and z – the vertical one, respectively. With respect to the axes, the vertical velocity component is assumed in the form (Abbott 1979, Abbott et al. 1984)

$$w = w(z, t) = \frac{\partial h}{\partial t}\frac{z}{h}, \tag{1}$$

where $z = 0$ means the bed and $z = h$ denotes the free surface.

After Abbott (1979), the pressure distribution is described by the formula

$$\frac{p(z)}{\rho} = g\,(h - z) + \frac{\partial^2 h}{\partial t^2}\frac{h^2 - z^2}{2h}, \tag{2}$$

where p is the pressure and g – the gravitational acceleration.

With respect to the assumption of shallow water, it is reasonable to introduce the average horizontal velocity defined by the equation

$$u(x,t) = \frac{1}{h}\int_0^h u(x,z,t)dz. \tag{3}$$

On the basis of the relations, the continuity equation assumes the form

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0. \tag{4}$$

At the same time, the average horizontal momentum equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{h}\int_0^h \frac{1}{\rho}\frac{\partial p}{\partial x}dz. \tag{5}$$

From substitution of the equation (2) into the right hand side of the relation the following is obtained

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + \frac{1}{3}h\frac{\partial^3 h}{\partial x\,\partial t^2} + \frac{2}{3}\frac{\partial h}{\partial x}\frac{\partial^2 h}{\partial t^2} = 0. \tag{6}$$

In the Abbott monograph (1979) the last term of the relation describing the product of derivatives has been neglected. In our case however, in order to obtain a more compact form of the momentum equation, this term is retained. In relations (4) and (6) we have two unknown variables, namely h and u. For our purposes it is convenient to multiply equation (6) by h and then, by performing simple manipulations, to write the equation in the following conservative form

$$\frac{\partial (uh)}{\partial t} + \frac{\partial}{\partial x}\left(u^2 h + \frac{1}{2}gh^2 + \frac{1}{3}h^2\frac{\partial^2 h}{\partial t^2}\right) = 0. \tag{7}$$

Equations (4) and (7) form the fundamental system of equations of the problem considered. From a mathematical point of view we deal with the hyperbolic system of partial differential equations in which convective fluxes of the state variables play an important role in the description of flow phenomenon. In shallow water equations obtained from the Euler equations the last term in the round brackets in equation (7) does not exist (Toro 1997). In our case this term describes the influence of the vertical acceleration of the fluid on the momentum equation associated with the horizontal axis. Taking the time derivative of the equation (4) and the derivative of equation (7) with respect to the space variable, and then, subtracting the results, one obtains the additional equation

$$\frac{\partial^2 h}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(u^2 h + \frac{1}{2}gh^2 + \frac{1}{3}h^2\frac{\partial^2 h}{\partial t^2}\right) = 0. \tag{8}$$

which contains the two variables.

In order to save the place, in what follows, the subscripts x and t will be used to denote the partial derivatives with respect to x – co-ordinate and time, respectively. Simple manipulations on the equation give

$$\begin{aligned} h_{tt} - (gh + u^2)h_{xx} - \frac{1}{3}h^2 h_{ttxx} - g(h_x)^2 - 2\Big[h(u_x)^2 + 2uu_x h_x + uhu_{xx}\Big] + \\ -\frac{2}{3}\Big[(h_x)^2 h_{tt} + 2hh_x h_{ttx} + hh_{xx}h_{tt}\Big] = 0. \end{aligned} \tag{9}$$

In the further discussion we shall confine our attention to a series of approximate solutions of the system of equations (4) and (9) by means of a perturbation method.

3. 3 Perturbation Approach to the Non-linear Problem

In order to find an approximate solution of the problem mentioned, the variables h and u are expressed in the form of the following series

$$\begin{aligned} h &= h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \varepsilon^3 h_3 + \cdots, \\ u &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots. \end{aligned} \tag{10}$$

where:

h_0	–	the fluid depth at rest,
$h_1, h_2, h_3,\ \ u_1, u_2, u_3$	–	"components" of the solution,
ε	–	the small parameter $(0 \le \varepsilon < 1)$.

Substituting the expansion into equations (4) and (9) and collecting terms with the same powers in ε, a series of linear differential equations of the problem is obtained. In the further analysis we confine our attention to the third order terms in the relations (10). The first order power in ε gives the linearized version of the shallow water equations

$$\begin{aligned} &h_{1_{tt}} - gh_0 h_{1_{xx}} - \frac{1}{3}(h_0)^2 h_{1_{xxtt}} = 0, \\ &h_{1_t} + h_0 u_{1_x} = 0. \end{aligned} \tag{11}$$

Substituting $h = h_0 + \eta(x,t)$ into the first equation (11) and performing simple manipulations one obtains equation (B) as the linearized version of the Boussinesq system of shallow water equations (Whitham 1974, p. 462). From the second relation in (11) the horizontal velocity can be calculated

$$u_1(x,t) = -\frac{1}{h_0}\int_x h_{1_t}(x,t)dx + \text{const.} \tag{12}$$

Collecting terms with the second power expansion in terms of the small parameter, the following system of equations is derived

$$\begin{aligned} &h_{2_{tt}} - gh_0 h_{2_{xx}} - \frac{1}{3}(h_0)^2 h_{2_{xxtt}} + RA = 0, \\ &h_{2_t} + h_0 u_{2_x} + h_{1_x} u_1 + h_1 u_{1_x} = 0. \end{aligned} \tag{13}$$

where

$$\begin{aligned} RA = &-gh_1 h_{1_{xx}} - \frac{2}{3}h_0 h_1 h_{1_{xxtt}} - g(h_{1_x})^2 - 2h_0(u_{1_x})^2 + \\ &-2h_0 u_1 u_{1_{xx}} - \frac{4}{3}h_0 h_{1_x} h_{1_{xtt}} - \frac{2}{3}h_0 h_{1_{xx}} h_{1_{tt}}. \end{aligned} \tag{14}$$

Similarly way the third power in ε yields

$$\begin{aligned} &h_{3_{tt}} - gh_0 h_{3_{xx}} - \frac{1}{3}(h_0)^2 h_{3_{xxtt}} + RB = 0, \\ &h_{3_t} + h_0 u_{3_x} + h_{1_x} u_2 + h_{2_x} u_1 + h_1 u_{2_x} + h_2 u_{1_x} = 0. \end{aligned} \tag{15}$$

where

$$\begin{aligned} RB \;=\; &-\left[gh_1 h_{2_{xx}} + gh_2 h_{1_{xx}} + (u_1)^2 h_{1_{xx}}\right] + \\ &-\frac{1}{3}\left[2h_0 h_1 h_{2_{xxtt}} + (h_1)^2 h_{1_{xxtt}} + 2h_0 h_2 h_{1_{xxtt}}\right] - 2gh_{1_x} h_{2_x} + \\ &-2\left[2h_0 u_{1_x} u_{2_x} + h_1(u_{1_x})^2\right] - 4u_1 u_{1_x} h_{1_x} + \\ &-2\left[h_1 u_1 u_{1_{xx}} + h_0(u_2 u_{1_{xx}} + u_1 u_{2_{xx}})\right] - \frac{2}{3}(h_{1_x})^2 h_{1_{tt}} + \\ &-\frac{4}{3}\left[h_0(h_{1_x} h_{2_{xtt}} + h_{2_x} h_{1_{xtt}}) + h_1 h_{1_x} h_{1_{xtt}}\right] + \\ &-\frac{2}{3}\left[h_0(h_{1_{xx}} h_{2_{tt}} + h_{1_{tt}} h_{2_{xx}}) + h_1 h_{1_{xx}} h_{1_{tt}}\right]. \end{aligned} \tag{16}$$

The result of the procedure on hand is the system of linear differential equations, which enable us to calculate successively the coefficients of the series (10), assuming that such series exist. The system describes an arbitrary flow of the fluid, for example, transient flow and motions starting from rest. In the further analysis we confine our discussion to solutions of the equations for a simpler case of periodic waves of rigid form propagating in fluid of constant depth.

4. Periodic Waves Propagating in Shallow Water

Let us consider now the case of waves of rigid form propagating in the direction of positive values of the space coordinate. For this case it is reasonable to assume that our variables h and u are functions of the parameter

$$\theta = kx - \omega t \tag{17}$$

where k is the wave number and $\omega = \omega(k)$ is the angular frequency of the propagating wave.

In view of the relation (17) the partial derivatives of the variable h with respect to x and t are

$$\frac{\partial h}{\partial x} = k\frac{\partial h}{\partial \theta} = kh', \quad \frac{\partial h}{\partial t} = -\omega\frac{\partial h}{\partial \theta} = -\omega h'. \tag{18}$$

Similar formulae hold for the second variable. Hereinafter the primes denote the derivatives with respect to θ. From substitution of the relations (17) and (18) into equation (9) the following equation results

$$\begin{aligned}&\left[\omega^2 - k^2\left(gh + u^2\right)\right]h'' - \frac{1}{3}k^2\omega^2h^2h'''' - gk^2(h')^2 + \\ -2k^2&\left[h(u')^2 + 2uu'h' + uhu''\right] - \frac{2}{3}k^2\omega^2\left[(h')^2h'' + 2hh'h''' + h(h'')^2\right] = 0.\end{aligned} \tag{19}$$

In a similar way, the continuity equation gives

$$-\omega h' + k\left(u'h + uh'\right) = 0. \tag{20}$$

For the case considered the expansions (10) assume the form

$$\begin{aligned}h &= h_0 + \varepsilon h_1(\theta) + \varepsilon^2 h_2(\theta) + \varepsilon^3 h_3(\theta) + \cdots, \\ u &= \varepsilon u_1(\theta) + \varepsilon^2 u_2(\theta) + \varepsilon^3 u_3(\theta) + \cdots.\end{aligned} \tag{21}$$

At the same time, the frequency ω is also expanded into power series in ε (Whitham 1974)

$$\omega = \omega(k) = \omega_0(k) + \varepsilon\omega_1(k) + \varepsilon^2\omega_2(k) + \varepsilon^3\omega_3(k) + \cdots. \tag{22}$$

Substituting the expansions (21) and (22) into equations (19) and (20) and collecting terms with the same power in ε, a set of ordinary differential equations is obtained. The linear component of the equation (19) is

$$\left[(\omega_0)^2 - gk^2h_0\right]h_1'' - \frac{1}{3}k^2(\omega_0)^2(h_0)^2h_1'''' = 0. \tag{23}$$

The quadratic term of the equation assumes the form

$$\left[(\omega_0)^2 - gk^2 h_0\right] h_2'' - \frac{1}{3}k^2(\omega_0)^2(h_0)^2 h_2'''' + RA = 0, \tag{24}$$

where

$$\begin{aligned} RA = {} & (2\omega_0\omega_1 - gk^2 h_1)h_1'' - \frac{1}{3}k^2\left[2(\omega_0)^2 h_0 h_1 + 2(h_0)^2\omega_0\omega_1\right]h_1'''' + \\ & -gk^2(h_1')^2 - 2k^2\left[h_0(u_1')^2 + h_0 u_1 u_1''\right] - \frac{2}{3}k^2(\omega_0)^2\left[2h_0 h_1' h_1''' + h_0(h_1'')^2\right]. \end{aligned} \tag{25}$$

The third power component of the expansion leads to the equation

$$\left[(\omega_0)^2 - gk^2 h_0\right] h_3'' - \frac{1}{3}k^2(\omega_0)^2(h_0)^2 h_3'''' + RB = 0, \tag{26}$$

where

$$\begin{aligned} RB = {} & \left(2\omega_0\omega_1 - gk^2 h_1\right) h_2'' + \left[(\omega_1)^2 + 2\omega_0\omega_2 - gk^2 h_2 - k^2(u_1)^2\right] h_1'' + \\ & -\frac{1}{3}k^2\left[2(\omega_0)^2 h_0 h_1 + 2(h_0)^2\omega_0\omega_1\right] h_2'''' + \\ & -\frac{1}{3}k^2\left[(\omega_0)^2\left[(h_1)^2 + 2h_0 h_2\right] + (h_0)^2[(\omega_1)^2 + 2\omega_0\omega_2 + 4\omega_0\omega_1 h_0 h_1\right] h_1'''' + \\ & -2gk^2 h_1' h_2' - 2k^2\left[2h_0 u_1' u_2' + h_1(u_1')^2 + 2u_1 u_1' h_1' + h_0(u_1 u_2'' + u_2 u_1'') + \right. \\ & \left. + h_1 u_1 u_1''\right] - \frac{2}{3}k^2\left\{(\omega_0)^2\left[(h_1')^2\, h_1'' + 2h_1 h_1' h_1''' + 2h_0\left(h_1' h_2''' + h_2' h_1'''\right) + \right.\right. \\ & \left.\left. + h_1(h_1'')^2 + 2h_0 h_1'' h_2''\right] + 2\omega_0\omega_1\left[2h_0 h_1' h_1''' + h_0(h_1'')^2\right]\right\}. \end{aligned} \tag{27}$$

In order to solve the homogeneous equation (23) it is convenient to introduce the notation

$$r^2 = \frac{3}{(\omega_0 k h_0)^2}\left[gk^2 h_0 - (\omega_0)^2\right] \tag{28}$$

and to express the equation in the form

$$h_1'''' + r^2 h_1'' = 0. \tag{29}$$

We are looking for a solution describing a wave of constant shape, therefore the following inequality should be satisfied

$$r^2 > 0, \rightarrow \left(\frac{\omega_0}{k}\right)^2 < gh_0. \tag{30}$$

The general solution of the equation (29) reads

$$h_1 = A_1 \cos r\theta + A_2 \sin r\theta + A_3\theta + A_4, \tag{31}$$

where $A_1, \cdots A_4$ are constants.

With respect to our demands the last two terms in the solution should be cancelled, and finally, the solution can be written as

$$h_1 = C_1 \cos(r\theta + \beta_1), \tag{32}$$

where C_1 and β_1 are new constants.

Having the solution we can find the relevant velocity component

$$u_1 = \gamma h_1 + \text{const.}, \tag{33}$$

where $\gamma = \omega_0/kh_0$.

In order to find the second order solution, the differential equation (24) is written in the following form

$$F_2'' + r^2 F_2 = \frac{3}{(kh_0\omega_0)^2} RA = RA^*, \tag{34}$$

where $F_2 = h_2''$.

The right hand side of the equation depends solely on the linear solution. Substituting the solution (32) into equation (25) and performing the prescribed differentiation one obtains

$$RA^* = -\alpha_1 C_1 \cos(r\theta + \beta_1) + \alpha_2 (C_1)^2 \cos 2(r\theta + \beta_1), \tag{35}$$

where

$$\begin{aligned} \alpha_1 &= \frac{6}{(kh_0)^2} \frac{\omega_1}{\omega_0} r^2 \left[1 + \frac{1}{3}(kh_0 r)^2\right], \\ \alpha_2 &= 9\frac{r^2}{(kh_0)^2} \frac{1}{h_0} \left[1 - \frac{1}{3}(kh_0 r)^2\right]. \end{aligned} \tag{36}$$

The general solution of the non-homogeneous equation (34) is

$$\begin{aligned} &F_2 = A_2 \cos r\theta + B_2 \sin r\theta + \\ &-\frac{\alpha_1 C_1}{2r}\left[\theta \sin(r\theta + \beta_1) + \frac{1}{2r}\cos(r\theta + \beta_1)\right] - \frac{\alpha_2 (C_1)^2}{3r^2} \cos 2(r\theta + \beta_1), \end{aligned} \tag{37}$$

were A_2 and B_2 are constants.

The term in the square brackets does not satisfy the assumed condition of periodic waves and therefore the multiplier of the term has to be set equal to zero. From the last condition it follows that $\omega_1 = 0$ in the expansion (22). Having the solution (37) it is a simple task to find the function $h_2(\theta)$. Integration of the equation (37) with respect to θ leads to the solution

$$h_2 = D_1 \cos(r\theta + \beta_2) + D_2 \cos 2(r\theta + \beta_1), \tag{38}$$

where $D_2 = \dfrac{\alpha_2}{12r^4}(C_1)^2$ and, D_1 and β_2 are new constants.

From substitution of the relations (21), (22) and (38) into equation (20), the second component of the velocity is obtained

$$u_2 = \gamma \left[h_2 - \frac{1}{h_0}(h_1)^2 \right] + \text{const.} \tag{39}$$

Substituting the solutions (32) and (38) into relation (27) and performing simple, but tedious manipulation, we arrive at the formula

$$\begin{aligned} RB = {} & \cos(r\theta + \beta_1) r^2 \left\{ -2\omega_0\omega_2 \left[1 + \frac{1}{3}(kh_0 r)^2 \right] C_1 + \right. \\ & \left. + k^2 \left[\frac{1}{2}g - \frac{5}{3}(\omega_0)^2 h_0 r^2 + \gamma^2 h_0 \right] C_1 D_2 - \frac{1}{4}k^2 \left[(\omega_0)^2 r^2 + 3\gamma^2 \right] (C_1)^3 \right\} + \\ & + \cos(2r\theta + \beta_1 + \beta_2) k^2 r^2 \left[2g - \frac{8}{3}(\omega_0)^2 h_0 r^2 + 4\gamma^2 h_0 \right] C_1 D_1 + \\ & + \cos 3(r\theta + \beta_1) k^2 r^2 \left\{ \left[\frac{9}{2}g - 15(\omega_0)^2 h_0 r^2 + 9\gamma^2 h_0 \right] C_1 D_2 + \right. \\ & \left. - \frac{3}{4} \left[(\omega_0)^2 r^2 + 3\gamma^2 \right] (C_1)^3 \right\}. \end{aligned} \tag{40}$$

From the equation (26) describing the third order solution it follows that

$$F_3'' + r^2 F_3 = \frac{3}{(kh_0\omega_0)^2} RB = RB^*, \tag{41}$$

where $F_3 = h_3''$.

We are interested in periodic solution of the problem considered and therefore the multiplier of the function $\cos(r\theta + \beta_1)$ entering the equation (40) should be equal to zero. The last condition provides the formula describing the second component of the wave frequency

$$\omega_2 = \omega_0 \left(\frac{3}{4r} \frac{C_1}{h_0} \right)^2 \frac{1 - 2(kh_0 r)^2 + (kh_0 r)^4/9}{(kh_0 r)^2 \left[1 + (kh_0 r)^2/3 \right]}. \tag{42}$$

With respect to the result, the right hand side of the equation (41) becomes

$$RB^* = \alpha_3 C_1 D_1 \cos(2r\theta + \beta_1 + \beta_2) + \alpha_4 (C_1)^3 \cos 3(r\theta + \beta_1), \tag{43}$$

where

$$\alpha_3 = \frac{18}{h_0}\left(\frac{r}{kh_0}\right)^2\left[1-\frac{1}{3}(kh_0r)^2\right],$$
$$\alpha_4 = \frac{27}{4(kh_0)^2}\frac{1}{(h_0)^2}\left\{\frac{9}{2}\frac{[1-(kh_0)^2][1-(kh_0)^2/3]}{(kh_0)^2} - r^2\left[1+(kh_0)^2/3\right]\right\}. \tag{44}$$

With the notations introduced, the general solution of the equation (41) is written as

$$F_3 = A_3\cos r\theta + B_3\sin r\theta +$$
$$-\frac{1}{3r^2}C_1D_1\alpha_3\cos(2r\theta+\beta_1+\beta_2) - \frac{1}{8r^2}(C_1)^3\alpha_4\cos 3(r\theta+\beta_1), \tag{45}$$

where A_3 and B_3 are constants.

From integration of the equation in the θ domain, the following is obtained

$$h_3 = E_1\cos(r\theta+\beta_3) + E_2\cos(2r\theta+\beta_1+\beta_2) + E_3\cos 3(r\theta+\beta_1). \tag{46}$$

In the solution, E_1 and β_3 are new constants and

$$E_2 = \frac{1}{12r^4}\alpha_3 C_1 D_1, \quad E_3 = \frac{1}{32r^4}\alpha_4(C_1)^3. \tag{47}$$

Substitution of the equations (21), (22) and (46) into expression (20) yields

$$u_3 = \gamma\left\{h_3 + h_1\left[\frac{\omega_2}{\omega_0} - 2\frac{h_2}{h_0} + \left(\frac{h_1}{h_0}\right)^2\right]\right\} + \text{const.} \tag{48}$$

The solutions of the set of three differential equations for the components h_1, h_2 and h_3 contain six arbitrary constants, i.e. C_1, D_1, E_1 and β_1, β_2, β_3. From the practical point of view the most interesting is the case of equal phase shifts of the components, and thus, let us consider the case $\beta_1 = \beta_2 = \beta_3 = \beta$ for which the solution of the problem mentioned assumes the form

$$h = h_0 + \varepsilon\left(C_1 + \varepsilon D_1 + \varepsilon^2 E_1\right)\cos(r\theta+\beta) +$$
$$+\varepsilon^2\frac{C_1}{12r^4}\left(\alpha_2 C_1 + \varepsilon\alpha_3 D_1\right)\cos 2(r\theta+\beta) + \varepsilon^3\frac{\alpha_4}{32r^4}(C_1)^3\cos 3(r\theta+\beta), \tag{49}$$

which may be rewritten as

$$h = h_0 + \varepsilon A_1\cos(r\theta+\beta) + \varepsilon^2 A_2\cos 2(r\theta+\beta) + \varepsilon^3 A_3\cos 3(r\theta+\beta), \tag{50}$$

where A_1, A_2 and A_3 are new constants.

From the descriptions (49) and (50) it may be seen that, with higher order expansions the constants $A_1, A_2, \cdots$, being the amplitudes of the subsequent components, depend on the higher order terms in the expansion procedure. In other words, with the solution procedure of the small parameter expansion, the components of the solutions are not defined in an absolute unique way. Knowing, however, that the description (50) is admissible in constructing an approximate solution to the problem of periodic waves, described by the equations (4) and (7), we may substitute the expression (50) directly to these equations.

5. Direct Solution to Harmonic Waves

From the solution obtained it may be seen that the parameter r may be incorporated into the variable θ, namely

$$\Theta = r\theta = r(kx - \omega t) = \kappa x - \sigma t. \tag{51}$$

At the same time

$$\sigma(k) = r\omega(k) = r(\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots) = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots. \tag{52}$$

Obviously, the solution expressed in the new variables may be obtained by formal substitution of $r = 1$ into the above formulae, and therefore, in what follows we also use the old symbols (for instance k, ω and θ) to denote the new parameters. With respect to the explanations, formula (28) leads to the dispersion relation

$$\omega_0^2 = \frac{gk^2h_0}{1 + (1/3)k^2h_0^2}, \tag{53}$$

which is equal to that given by Whitham (1974) for the linearized Boussinesq equation.

Now we seek a solution of the problem by means of the direct substitution of the description (50) into equations (4) and (7). Moreover, without loss of generality it is assumed that $\beta = 0$ in the expression, i.e.

$$h_1 = A_1 \cos\theta, h_2 = A_2 \cos 2\theta, h_3 = A_3 \cos 3\theta. \tag{54}$$

Substitution of the first component of the equations into the first of equations (11) gives the dispersion formula (53). From substitution of the expansion (52) and the components h_1 and h_2 into equations (13) and (14) it follows that $\omega_1 = 0$, and

$$\frac{A_2}{h_0} = \frac{3}{4}\frac{1}{(kh_0)^2}\left[1 - \frac{1}{3}(kh_0)^2\right]\left(\frac{A_1}{h_0}\right)^2. \tag{55}$$

In a similar way, from substitution of the expressions (52) and (54) into relations (15) and (16) the relevant components of the dispersion relation and water depth may be derived. Simple manipulations give

$$\omega_2 = \omega_0 \frac{1 - 2(kh_0)^2 + (kh_0)^4/9}{(kh_0)^2[1 + (kh_0)^2/3]}\left(\frac{3}{4}\frac{A_1}{h_0}\right)^2 \tag{56}$$

and

$$\frac{A_3}{h_0} = \frac{1}{(kh_0)^4}\left[27 - 42(kh_0)^2 + 7(kh_0)^4\right]\left(\frac{A_1}{4h_0}\right)^3. \tag{57}$$

Having the components we may write the formula describing the water depth

$$h = h_0 + \varepsilon A_1 \cos\theta + \varphi_1(\varepsilon A_1)^2 \cos 2\theta + \varphi_2(\varepsilon A_1)^3 \cos 3\theta, \tag{58}$$

where

$$\begin{aligned}\varphi_1 &= \frac{1}{h_0}\frac{1}{(kh_0)^2}\frac{3}{4}\left[1 - \frac{1}{3}(kh_0)^2\right],\\ \varphi_2 &= \frac{1}{(h_0)^2}\frac{1}{(kh_0)^4}\frac{1}{64}\left[27 - 42(kh_0)^2 + 7(kh_0)^4\right].\end{aligned} \tag{59}$$

In the discussion presented so far the small parameter ε has not been specified. From the solutions obtained it follows, that the parameter can be incorporated into the constant A_1 and thus, the solution (58) may be written in the following form

$$h = h_0 + A\cos\theta + \varphi_1 A^2 \cos 2\theta + \varphi_2 A^3 \cos 3\theta, \tag{60}$$

where $A = \varepsilon A_1$ is the amplitude of the first component of the solution.

Knowing the water depth we can calculate the average horizontal velocity

$$\begin{aligned}u = \gamma A\bigg[&\left(1 + \frac{\omega_2}{\omega_0}\right)\cos\theta + \varphi_1 A \cos 2\theta + \varphi_2 A^2 \cos 3\theta +\\ &-\frac{A}{h_0}\cos^2\theta - \frac{2}{h_0}\varphi_1 A^2 \cos\theta\cos 2\theta + \frac{1}{h_0^2}A^2\cos^3\theta\bigg].\end{aligned} \tag{61}$$

At the same time, following the notation introduced, the expansion (22) gives

$$\begin{aligned}\omega(k) &= \omega_0(k) + \varepsilon^2\omega_2(k) =\\ &= \omega_0\left[1 + \frac{9}{16}\frac{1 - 2(kh_0)^2 + (kh_0)^4/9}{(kh_0)^2[1 + (kh_0)^2/3]}\left(\frac{A}{h_0}\right)^2\right].\end{aligned} \tag{62}$$

In derivation of the solutions presented above it has been assumed that A/h_0 and (kh_0) are small numbers. In accordance with the assumption, the last formula may be written as

$$\frac{\omega}{c_0 k} \cong 1 - \frac{1}{6}(kh_0)^2 + \frac{9}{16}\frac{1}{(kh_0)^2}\left[\frac{1 - 2(kh_0)^2 + (kh_0)^4/9}{1 + (kh_0)^2/3}\right]\left(\frac{A}{h_0}\right)^2 + \cdots, \tag{63}$$

where $c_0 = \sqrt{gh_0}$.

In a similar way, the free surface elevation may be expressed in the form

$$\begin{aligned} \frac{\eta}{h_0} \cong \frac{A}{h_0}\cos\theta + \frac{3}{4}\frac{1}{(kh_0)^2}\left(\frac{A}{h_0}\right)^2 \cos 2\theta + \\ + \frac{27}{64}\frac{1}{(kh_0)^4}\left[1 - \frac{42}{27}(kh_0)^2 + \frac{7}{27}(kh_0)^4\right]\left(\frac{A}{h_0}\right)^3 \cos 3\theta + \cdots. \end{aligned} \tag{64}$$

The formulae (63) and (64) are similar to those given in Whitham (1974, p. 473) for the case of the Stokes waves propagating in fluid of constant depth. As compared with the latter results the differences occur in the third order terms. A remark is needed. The formulae have been derived under the assumption that (kh_0) is a moderately small number. At first glance, from the relations (59) and (64) it can be seen that the ratio A/h_0 plays the role of the small parameter ε. On the other hand, solutions (55) and (57) strongly depend on the magnitude (kh_0). This means,that in fact, the expansion parameter is really $\varepsilon/(kh_0)^2$. Therefore, in order to get reliable results, we have to formulate an additional restriction imposed on the latter parameter. From the condition that the parameter should be less than one it follows that

$$\frac{A}{h_0}\frac{1}{(kh_0)^2} < 1, \rightarrow \ (kh_0) > \sqrt{A/h_0}. \tag{65}$$

Knowing, that the expansion procedure has been justified only for waves of small amplitude, say $A < 0.2h_0$, we obtain the condition

$$kh_0 > \sqrt{0.2} \cong 0.4472, \tag{66}$$

which may be considered as a strong restriction imposed on the approximate formulation. It means that the perturbation scheme may be applied only for waves of moderate length. Otherwise, for very long waves, the higher order components may exceed the lower order ones which contradicts the fundamental assumption used in derivation of the solutions.

In order to illustrate the considerations, some numerical computations have been performed for a chosen set of parameters describing the length and leading

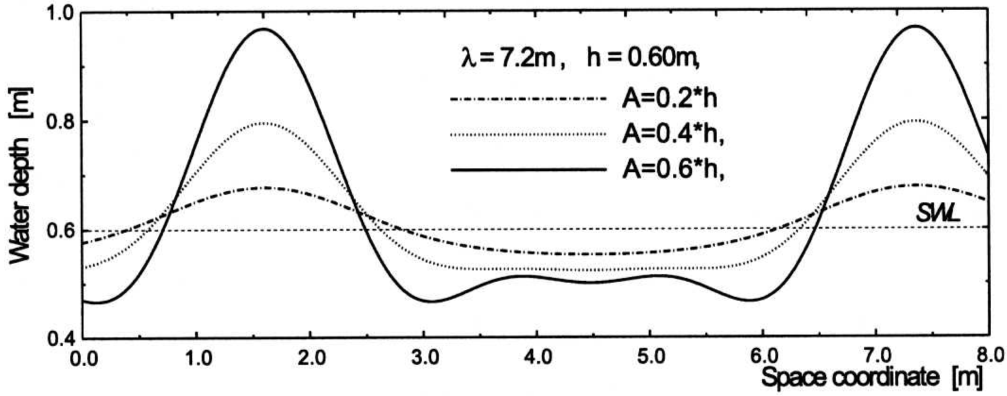
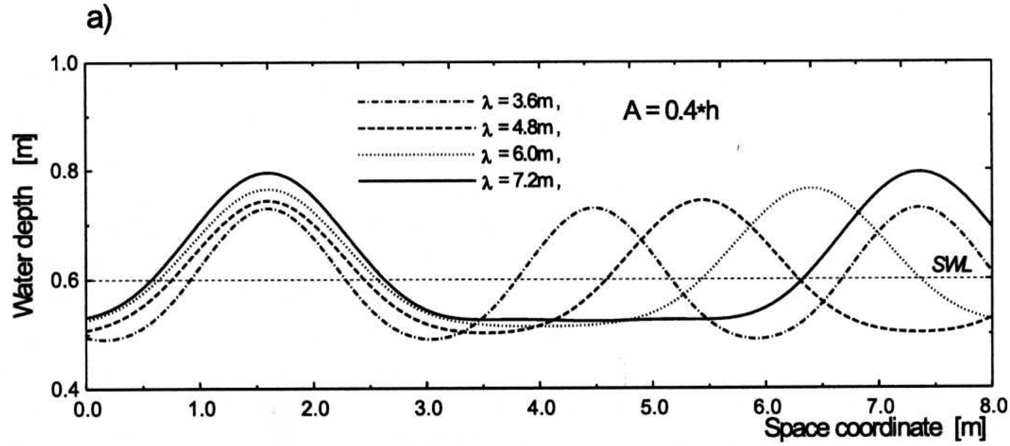


Fig. 1. Comparison between profiles of waves of the same length and different amplitudes



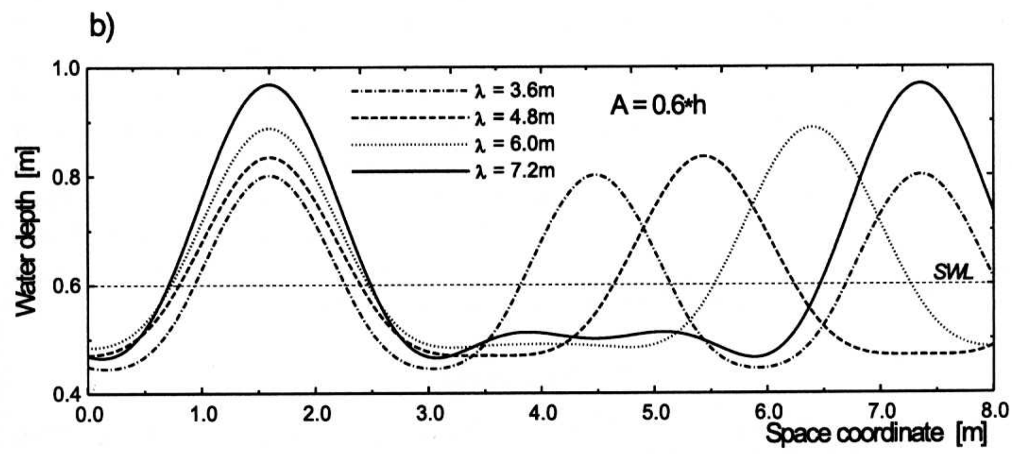


Fig. 2. Comparison of profiles for waves of different length and the leading term amplitude: a) $A = 0.4h$ and b) $A = 0.6h$, where h is the still water depth

amplitude of the waves. The results of computations are presented in the subsequent figures 1 and 2. Fig. 1 shows the free surface profiles corresponding to single length and a chosen set of amplitudes of the first, leading component of the waves. The plots in Fig. 2 represent the free surface elevations of waves of different lengths, but with the same leading amplitude. From the plots it is seen how the wave length influences the final height of the wave. With increasing length of the waves having the same leading amplitude, the final height of the waves increases. From the above analysis and results of computations it follows that the procedure of the solution of the problem considered leads to sufficiently good results only for moderately long waves of finite, but relatively small, amplitudes.

6. Concluding Remarks

The paper deals with a new form of the Boussinesq equations describing waves in shallow water. The governing differential equations are formulated in a conservative form which seems to be very convenient in approximate description of the phenomenon. An approximate solution of the equations is obtained with the help of a perturbation procedure in which expansions of variables of the problem in powers of a small parameter are introduced directly to the differential equations. The analysis has improved the dispersion characteristic of waves in shallow waters, which in the case discussed depends on the amplitude of the propagating wave. A similar conclusion holds for the phase celerity of the wave. An improvement of the formulation is possible in which an unevenness of the water depth may be taken into account. The latter problem would be more complicated and is not within the present scope of work. The analysis has revealed that the lower order components of the perturbation procedure depend on the higher order terms and thus the amplitudes of subsequent components are not uniquely defined. On the other hand, such ambiguity does not exist in the case of direct substitution of the trigonometric components into equations of the problem corresponding to harmonic waves. The solution obtained strongly depends on the magnitude of the wave amplitude and the wave length. In practice, the solution is valid only for waves satisfying the condition $kh_0 \geq 1/2$. At the same time, it is perhaps of importance to emphasize here that analytical solutions contain components which do not correspond to progressive waves. Therefore, one should be careful in a discrete integration of the non-linear equations in the time domain, because a formal approach to the problem may lead to incorrect results.

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