Archives of Hydro-Engineering and Environmental Mechanics
Vol. 49 (2002), No. 4, pp. 77–88

Groundwater Flow Through an Observation Well

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(Received November 12, 2001; revised April 18, 2003)

Abstract

The paper presents the analytical solution to the problem of groundwater flow through an observation well. Assumptions enabling the formulation of the problem using the classical theory of seepage are specified. The solution was derived from the variable separation method. Obtained results confirm the outcomes of investigations, conducted on physical models, published by other researchers.

Numerical modelling may be useful to obtain the solution to the poblem for more complicated cases e.g. complex geometry, other boundary conditions, anisotropy of medium. An example of calculation applying the finite element method is given.

Notation

h_1, h_2 – piezometric heads,
r, φ – polar co-ordinates,
x, y – Cartesian co-ordinates,
n, s – local coordinates (n is unit outward normal to surface Γ),
r_0 – radius of a well,
k_1, k_2 – seepage coefficient (hydraulic permeability),
$\mathbf{q}$ – seepage velocity vector,
q_x, q_y – components of vector $\mathbf{q}$,
q_n, q_s – components of vector $\mathbf{q}$ in local co-ordinates (n, s),
i – slope of piezometric surface,
q – discharge, $q = k_1 i$,
Q_n, Q_m – flow through an observation well.

1. Introduction

Basic information concerning groundwater flow can usually be obtained using piezometric measurements. The direction and the intensity of water flow in aquifer are traditionally determined on the basis of measurements conducted in observation well systems forming a hydrogeologic node (Wieczysty 1982).

Flow rate can be also measured in a single well by introducing appropriate sensors. Obviously, it must be remembered that the velocities of water particles inside an observation well, as in any other porous medium, are very small i.e. within the range of fractions of millimeters per second. Thus, measuring methods ensuring adequate accuracy must be applied. Majewski (1980) was one of those who investigated this problem from the metrological point of view.

Determination of the quantitative relation between the measured rate inside a well and seepage rate in aquifer is an important problem. An attempt to present an analytical description of flow through an observation well was considered by Chirek (1995). Unfortunately, the solution presented there turned out to be incorrect. Sroka and Wosiewicz (1998) undertook a study giving an outline of an analytical solution to the problem. For comparison, an approximate numerical solution with finite element method was obtained. The analysis of a similar problem, but for measuring sensor placed directly in porous medium (spherical cap), not in an observation well was presented by Majewski (1980). However, Majewski's solution cannot be transformed from a sphere to a cylinder as in the case of a well.

Seepage rate in natural aquifers can also be measured by radiometric method. A tracer (isotope) is introduced into a testing well and the variation in time, of water activity in a well is tested. The diffusion phenomenon is usually neglected at small tracer concentrations (Kowalski 1998). Variations in tracer concentrations are involved in this case only by the dilution caused by water flowing from the aquifer to the well. In this case it is important to determine the relation between the measured flow through a well and the natural flow in an aquifer (undisturbed flow). This relation is usually written in the form of: $Q_m = \beta Q_n$, where Q_m is measured flow and Q_n the natural flow. Kowalski (1998) referring to Piętka (1962) research (sand-box model) states that β coefficient assumes values within the range of $0 \div 2$, depending on the ratio between the seepage coefficient of the medium and the filter. When the filter conductivity is one rank (or more) higher than the conductivity of the porous medium, it is advised, for practical purposes, to assume the value of $\beta = 2.0$.

Rethati (1983) quotes publications in which a value of the β coefficient equal to 1.0 is recommended, however, some others, give a value of 2.0. He also reports Krätschmar and Luckner investigations on the basis of which an empiric formula was prepared. The value of β coefficient in the formula depends, among other parameters, on filter porosity. At a higher filter porosity, of above 60%, the β coefficient value is about 2.0.

The main purpose of this paper is to present a detailed description of groundwater flow through an observation well, beginning from its formulation, specification of simplifying assumptions and boundary condition up to the solution and analysis of obtained results. The comparison of results obtained with the results of investigations conducted on physical models confirms the correctness of the solution presented.

The problem discused could be solved by applying appropriate numerical methods. Numerical modelling will be particularly useful in the case of more complex area geometry, other boundary conditions or when the anisotropy of the medium has occurred.

The analytical solution formulated below could be obtained as the particular case analyzed by Dagan (1989), i.e. the problem of determination of effective conductivity for heterogeneous porous medium. The solution presented in this paper is less general than that obtained by Dagan (1989). However, it was obtained using a simpler mathematical method, hence the analysis of results is decisively easier.

2. Assumptions and Formulation of the Problem

Water flow in a complex system comprising porous medium of aquifer, filter around test well and the flow in the well (lack of porous medium) should be analyzed. The filter resistance is so low compared with the seepage resistances in aquifer that it can be neglected without introducing significant errors. For calculation purposes, it is better to assume that the porous medium is also placed inside the well. The whole phenomenon can than be described as a problem of seepage flow. Assuming such a model, the formulation of the problem and searching for the solution, as well as the analysis of results, is much easier. All considerations are conducted within the classical theory of seepage. On the other hand, when seepage coefficient inside the observation well is assumed to be several ranks higher than in the aquifer and the porosity equals one, then the true situation will be properly described in such an approximation. An additional argument for the application of such a scheme is the fact that when the medium conductivity inside the piezometer is properly matched, filter action can be taken into consideration.

The problem of water flow through the observation well, including the searched relation between flow rate measured in the well and seepage rate in the aquifer, was obtained assuming as follows:

a) the problem is flat in the plane (the flow is averaged in the vertical axis),
b) the depth of water seepage strata is constant – confined flow in aquifer covered with impermeable formations,

c) seepage area is infinite,
d) the flow is steady in time (stationary),
e) the field of seepage velocity (neglecting local disturbance near the well) is uniform i.e. seepage rate vector is parallel to selected direction (in the work to x-axis),
f) the Darcy seepage law is valid in the porous medium,
g) porous medium is isotropic with the k_1 seepage coefficient for aquifer, and the k_2 in the well.

From the mathematical point of view the solution of the formulated problem can be put down to find two functions, the h_1 and the h_2, which describe the distribution of piezometric heads in the Ω_1 – aquifer, and in the Ω_2 – observation well (Fig. 1).

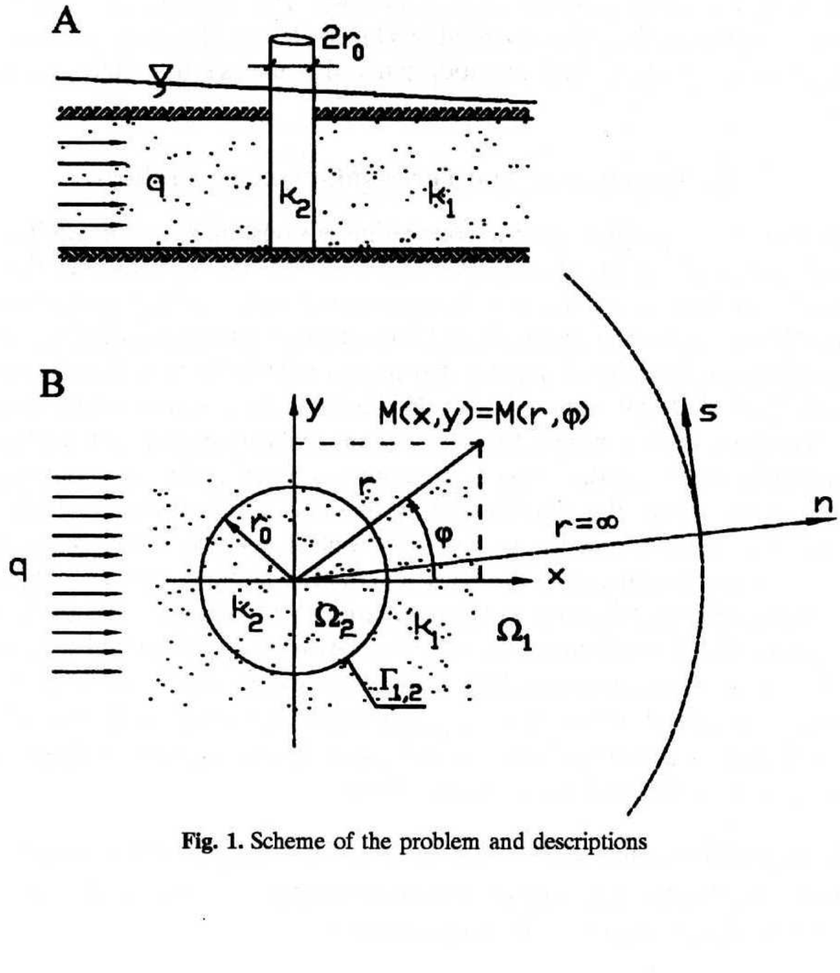

Fig. 1. Scheme of the problem and descriptions

These functions must satisfy an adequate flow equation inside their flow area. Due to isotropy and homogenity of seepage properties it will be a Laplace equation (Bear, Verruijt 1987) thus:

$$\text{in} \quad \Omega_1 \quad \Delta(k_1 h_1) = 0, \qquad k_1 = \text{const}, \tag{1}$$

$$\text{in} \quad \Omega_2 \quad \Delta(k_2 h_2) = 0, \qquad k_2 = \text{const}. \tag{2}$$

Equations (1) and (1) in a cartesian coordinates system take the following form:

$$\frac{\partial^2(k_i h_i)}{\partial x^2} + \frac{\partial^2(k_i h_i)}{\partial y^2} = 0 \quad i = 1, 2. \tag{3}$$

Adequate boundary condition must also be met. Prescribed flow must be set on the boundary at infinity:

$$\begin{aligned} &-k_1 \frac{\partial h_1}{\partial n} = -q \quad \text{for } x \to -\infty, \quad -k_1 \frac{\partial h_1}{\partial n} = q \quad \text{for } x \to \infty, \\ &-k_1 \frac{\partial h_1}{\partial n} = 0 \qquad \text{for } y \to \pm\infty. \end{aligned} \tag{4}$$

At the $\Gamma_{1,2}$ boundary between the areas, continuity conditions must be satisfied:

$$h_1 = h_2 \qquad (x, y) \in \Gamma_{1,2}, \tag{5}$$

$$k_1 \frac{\partial h_1}{\partial n_1} = -k_2 \frac{\partial h_2}{\partial n_2} \qquad (x, y) \in \Gamma_{1,2}. \tag{6}$$

Minus sign on the right side of formula (6) results from the opposite sense of normal vector to $\Gamma_{1,2}$ boundary for Ω_1 and Ω_2.

It is worth noting that, due of the symmetry of the problem, searched functions h_1 and h_2 should also be symmetric with regard to the x axis.

In a polar coordinates system ($x = r\cos\varphi$, $y = r\sin\varphi$) equations (1) and (2) may be written as follow:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r}(k_i h_i)\right] + \frac{1}{r^2}\frac{\partial^2(k_i h_i)}{\partial \varphi^2} = 0 \quad i = 1, 2 \tag{7}$$

and boundary condition at infinity takes the simple form:

$$-k_1 \frac{\partial h_1}{\partial n} = q\cos\varphi \quad \text{for } r \to \infty. \tag{8}$$

3. Solution

An analytical solution of the formulated problem was sought in the (r, φ) polar coordinates system (Fig. 1) with variables separation method. This solution was assumed in the following form:

$$h = f(r)\cos\varphi. \tag{9}$$

It is worth noting, that such a form of solution meets the conditions of flow symmetry related to the x axis. Introducing the function (9) into the equation (7) and solving this characteristic equation, the following function in external area (Ω_1) was assumed:

$$h_1 = \left(Ar + Br^{-1}\right)\cos\varphi \tag{10}$$

and similarly inside the observation well (Ω_2)

$$h_2 = \left(Cr + Dr^{-1}\right)\cos\varphi. \tag{11}$$

From the above mentioned conditions four integration constants occurring in the solution were determined:

- on the boundary at infinity ($r \to \infty$) condition (9) can be written in the following form:

$$q_n = -k_1 \frac{\partial h_1}{\partial n}\bigg|_{r\to\infty} = -k_1 \frac{\partial h_1}{\partial r}\bigg|_{r\to\infty} = q\cos\varphi. \tag{12}$$

  It is obtained from (10):

$$\frac{\partial h_1}{\partial r}\bigg|_{r\to\infty} = A\cos\varphi. \tag{13}$$

  Taking the above (12) and (13) into account:

$$A = -q/k_1. \tag{14}$$

  It is easy to prove that (14) automatically satisfied the conditions formulated in (4);
- the h_2 function in Ω_2 area, i.e. inside the well must be a bounded function. In the face of this fact the second constant from formula (11) is equal to zero

$$D = 0, \tag{15}$$

- the condition of compatibility of piezometric heads at the $\Gamma_{1,2}$ boundary must be ensured, i.e. condition (5) must be met. Considering (14) and (15) the following equation is given

$$-\frac{q}{k_1}r_0 + \frac{B}{r_0} = Cr_0, \tag{16}$$

- continuity at the $\Gamma_{1,2}$ boundary shall be ensured, i.e. condition (6) shall be met by the following equation

$$k_1\left(A - Br_0^{-2}\right) = k_2 C. \tag{17}$$

Equations (16) and (17) enable the finding of constants B and C, thus determination of the final form of the h_1 function in the area beyond the well and the h_2 inside is possible. Solving this system we obtain:

$$B = \frac{q}{k_1}\frac{k_2 - k_1}{k_2 + k_1} r_0^2, \tag{18}$$

$$C = -\frac{2q}{k_2 + k_1}. \tag{19}$$

The following dimensionless quantities are introduced:

$$R = \frac{r}{r_0}, \quad H_1 = \frac{h_1}{r_0}, \quad H_2 = \frac{h_2}{r_0}, \quad \lambda = \frac{k_2 - k_1}{k_2 + k_1}. \tag{20}$$

Finally, one can obtain a solution with the following form:

$$H_1 = \frac{q}{k_1}\left(\lambda R^{-1} - R\right)\cos\varphi, \tag{21}$$

$$H_2 = \frac{q}{k_1}(\lambda - 1) R \cos\varphi. \tag{22}$$

4. Results and Discussions

The obtained solution (22) determines the uniform field rate inside the observation well described by the following equations:

$$q_x = 2q\frac{k_2}{k_2 + k_1}, \quad q_y = 0 \ \text{ in } \ \Omega_2. \tag{23}$$

Thus, the flow rate in the well is constant, irrespective of its position but depending on the value of k_1 and k_2 seepage coefficients. Assuming that k_2 is much greater than k_1, the obtained description can be related to the physical situation when there is no porous medium in the well. Flow rate inside the well is then in practice twice as great as the seepage rate in the surrounding medium.

Distributions of dimensionless piezometric heads $H_1/|i|$, $H_2/|i|$ obtained for three selected values of the parameter λ, i.e. -1.0, 0.6 and 0.98 are presented in Fig. 2 (note that $|i| = q/k_1$).

The following particular cases can be selected when the solution obtained is analysed:

1) seepage coefficients (permeability) of medium in a layer and in a well are equal, then λ parameter is equal to zero and the uniform field rate is described by the H_1 and the H_2 functions ($q_x = q$, $q_y = 0$),

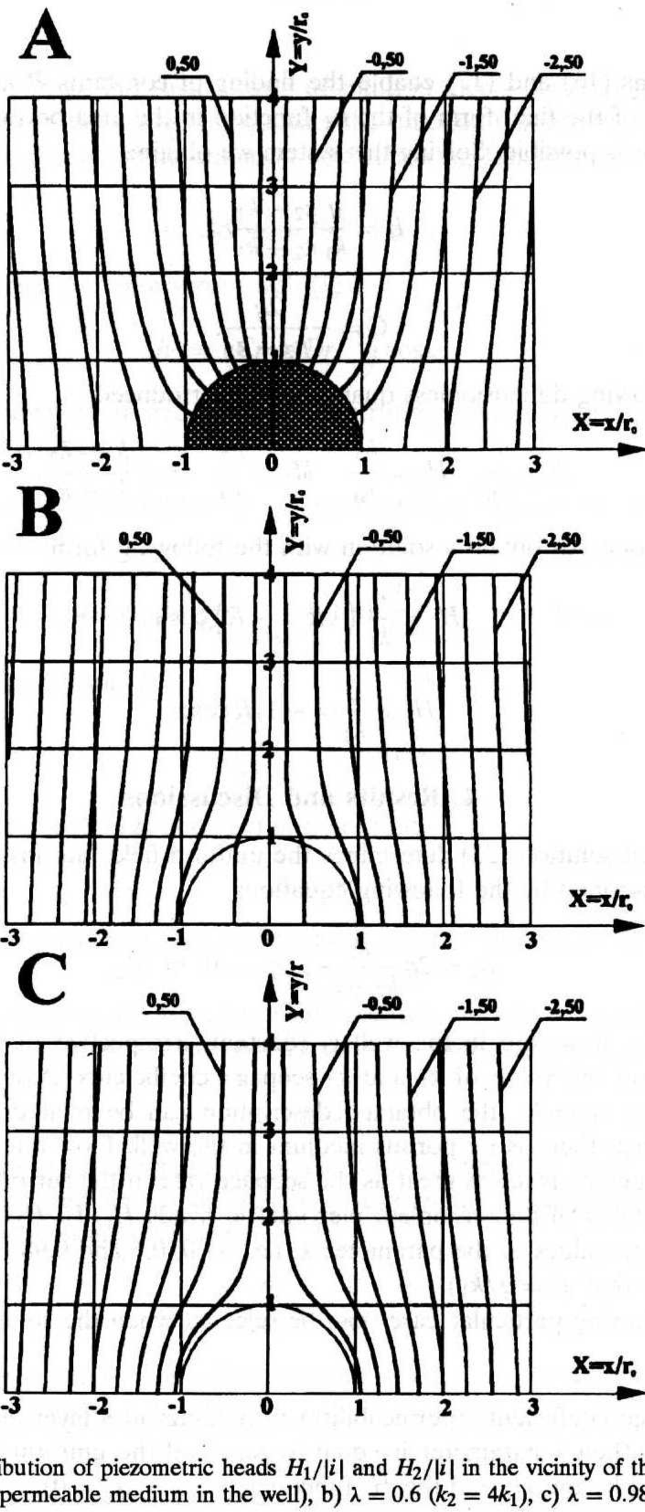

Fig. 2. Distribution of piezometric heads $H_1/|i|$ and $H_2/|i|$ in the vicinity of the well: a) for $\lambda = -1$ (impermeable medium in the well), b) $\lambda = 0.6$ ($k_2 = 4k_1$), c) $\lambda = 0.98$ ($k_2 = 99k_1$)

2) seepage coefficient in a well is equal to zero, $\lambda = -1$ and the solution obtained is analogous to flow around a cylinder, which is well known in hydromechanics (compare e.g. Puzyrewski, Sawicki 1987). On the surface of the cylinder ($r = r_0$) solution (21) gives, $q_s = -k_1 \frac{\partial h_1}{\partial s} = -k_1 \frac{1}{r_0} \frac{\partial h_1}{\partial \varphi} = -2q$ $\sin\varphi$. For $\varphi = \frac{\pi}{2}$ and $\varphi = \frac{3}{2}\pi$, the value of $q_x = |q_s| = 2q$,
3) the layer is impermeable, we have $k_1 = 0$, $\lambda = 1$ and the above-mentioned description is without physical sense which is equal to lack of flow.

A range of well influence on the uniform flow field is easy to assess on the basis of formula (21). The difference in the H_1 function value for the flow with a well ($\lambda = -1$) and without it ($\lambda = 0$) at a distance of $10r_0$ is no greater than 1% relative error. In practice, this means that flow disturbance caused by the introduction of a well vanishes several metres (depending upon well diameter) from its center.

It is worth explaining how boundary conditions are set. Constant potential or piezometric head at the $\Gamma_{1,2}$, boundary between a well and aquifer, as Chirek assumed in his work (Chirek 1995), is incorrect in fact. In this instance the potential inside the well should be constant. The value of harmonic function inside the area is limited by values at its boundary (Tichonow, Samarski 1963) and as a result any flow is possible.

Solution of the problem discused could be easy to assess by applying numerical methods. The finite element method (e.g. Wosiewicz, Sroka 1982, Zienkiewicz 1977) is a very useful tool for analysis seepage problems and is very often applied to create a numerical model. The following example demonstrates how the appropiate numerical model should be formulated.

Since the influence of observation well on uniform flow field at a distance of $10r_0$ is practicly negligable, the problem could be analyzed in the finite area. The scheme of the problem together with assumed boundary conditions are shown in Fig. 3a. Due to symmetry with regard to x axis only the upper part of the flow area was taken into consideration. The seepage flow problem in $\Omega = \Omega_1 \cup \Omega_2$ is described by the following equation:

$$\mathrm{div}(k\,\mathrm{grad}h) = 0, \quad k(x,y) = \mathrm{const} = k_1 \text{ in } \Omega_1, \quad k(x,y) = \mathrm{const} = k_2 \text{ in } \Omega_2. \quad (24)$$

Assumed boundary conditions are shown in Fig. 3a. The boundary condition is of the Neumann type and the solution is unique only to within an arbitrary additive constant (Brebbia et al. 1984). Because of this, a piezometric head equal to zero was assumed for one node placed at the origin of coordinates system (i.e. for $x = 0$, $y = 0$).

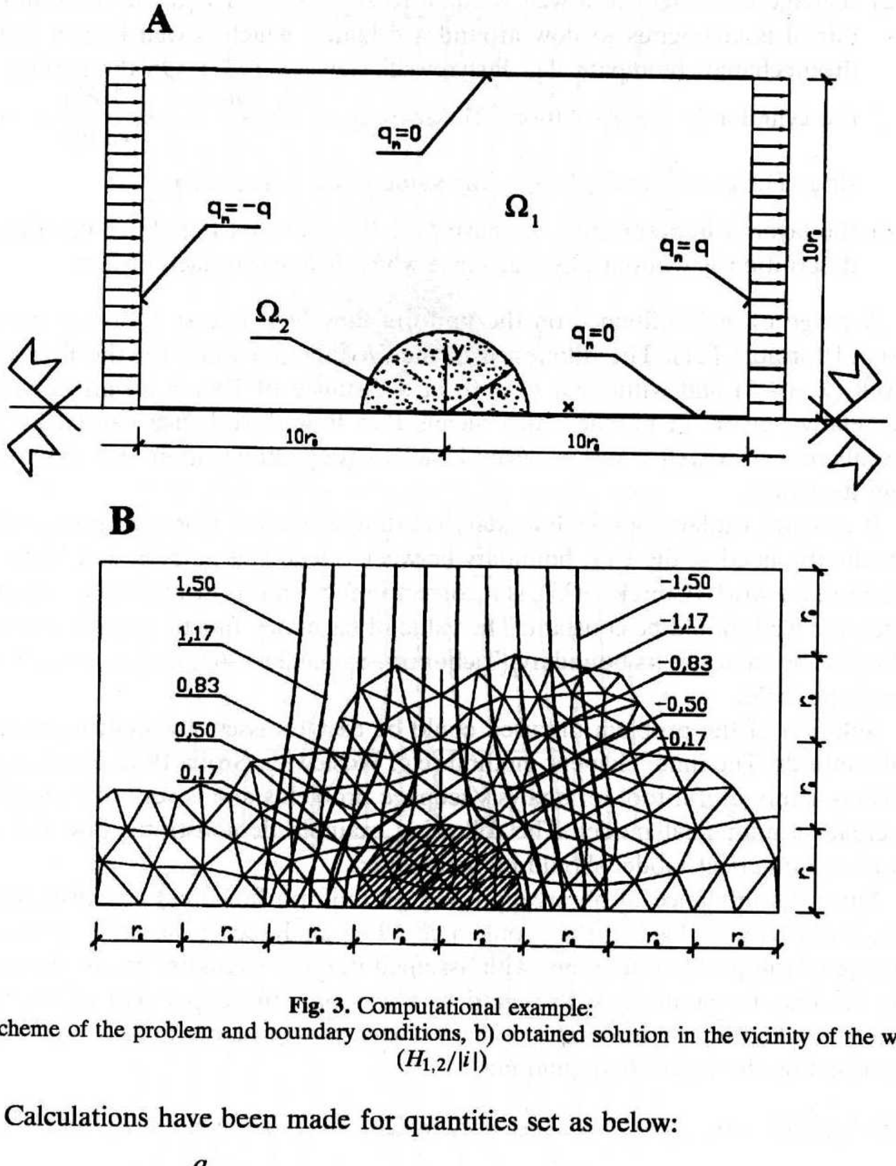

Fig. 3. Computational example:
a) scheme of the problem and boundary conditions, b) obtained solution in the vicinity of the well ($H_{1,2}/|i|$)

Calculations have been made for quantities set as below:

$$\frac{q}{k_1} = i = 0.01 \quad k_2 = 99k_1 \quad r_0 = 1 \quad \lambda = 0.98. \tag{25}$$

The simplest triangular elements with linear shape functions were used for discretization in space. The applied mesh of elements consists of 912 elements and 493 nodes. Part of it, situated near the well, is shown in Fig. 3b. Obtained results, piezometric heads for some nodes along the x axis, are given in Table 1. Distribiuton of piezometric head in the vicinity of the well is shown in Fig. 3b.

The results were compared with the analytical solution (equations (21) and (22)). Piezometric heads were determinated with good accuracy. Outside the well relative error value is less than 2%.

Table 1. Piezometric heads

$X = x/r_0$ ($Y = y/r_0 = 0$)	$H_1/\lvert i\rvert$		
	FEM	analytical solution	relative error [%]
1	−0.02027	−0.02000	1.35
4/3	−0.59580	−0.59833	0.42
5/3	−1.07400	−1.07867	0.43
2	−1.50200	−1.51000	0.53
2.5	−2.09400	−2.10800	0.66
3	−2.65400	−2.67333	0.72

Acknowledgement

The work was partially supported by the State Committee for Scientific Research, Poland under grant No. 5P0H 01918.

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