Approximate Solution to Standing Water Waves of Finite Amplitude in Material Description

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Abstract

An analysis is given of standing water waves of finite amplitude growing in time. In the model considered, a piston-type wave maker generates the waves in a rectangular fluid domain. The frequency of the generation corresponds to water waves of lengths equal to double the length of the fluid domain. In this way the case of resonance was obtained and the amplitude of the generated standing wave grew in time. The analysis has been confined to the second order approximation expressed in the material variables. Theoretical results have been compared with experimental data.

1. Introduction

In analysis of water waves an important problem is the description of waves of finite amplitude propagating in fluid of constant depth. The importance of the problem results from its relative simplicity which is specially appreciated when performing laboratory experiments in a hydraulic flume. Results of such experiments are used to estimate the accuracy of theoretical models of description of the phenomenon. One of the problems of this kind is the generation of standing waves of finite amplitude increasing in time. In the paper, we investigate the initial value problem of the standing waves generated in fluid of constant depth. The fluid, initially at rest, starts to move at a certain moment in time. We focus our attention on the potential motion of the fluid with the potential function Φ depending on material variables and time. The results of the theoretical model considered are assumed to describe the main features of the phenomenon. Most of solutions of such problems encountered in the literature of the subject is described by means of space variables. With these variables and under the assumption that the fluid is inviscid and incompressible, and the flow is irrotational, the relevant differential equations of fluid dynamics are reduced to the Laplace equation for the velocity potential. The main difficulty in obtaining a solution of the problem is the solution to the initial and boundary conditions, especially on moving boundaries of the fluid

domain. There are also regions of our interest however, where it is more prefer-able to use the material (Lagrangian) description of the problem considered. An example of such a problem is the structure-fluid dynamic interaction, where the Lagrangian variables are more convenient in constructing a solution to the bound-ary condition on the wetted surface of the structure. As concerns the Lagrangian description of the phenomenon, an important contribution belongs to Fontanet (1961) who gave the complete second order solution to the harmonic generation of waves in fluid of constant depth. The solution was obtained by means of the method of successive approximations. Madsen (1970) discussed the problem of water waves generated by a piston-type wave maker starting from rest. In this case, a more familiar Eulerian description has been applied. The second component of the solution was constructed for a relatively large lapse of time from the starting point, it thus being justified to confine the analysis to the classical steady state wave maker theory. Madsen, Mei and Savage (1970) analysed the evolution of time-periodic long waves of finite amplitude. They found that the occurrence of the secondary crests of the waves is a dominant feature and that waves periodic in time do not remain simply periodic in space. Tadjbakhsh and Keller (1960) considered periodic in time and horizontal direction, wave motion of an inviscid incompressible fluid bounded by a rigid horizontal bottom and a free surface. It was assumed that the motion is symmetric about a vertical plane $x = 0$, where x is the horizontal Cartesian coordinate. In this way their solution describes the standing wave, which results from the reflection of a normally incident wave from the wall $x = 0$. Applying a perturbation procedure, the authors obtained a third order formula describing the fluid pressure on the vertical wall. Goda (1967) in-vestigated a similar problem of standing water waves. He extended the solution of Tadjbakhsh and Keller to the fourth order approximation. The pressure obtained in this way was in good conformity with experimental data. For wave steepness exceeding a critical value double humps may be observed in the wave pressure on its time history. Third order approximate solution to short crested waves corres-ponding to an oblique reflection of an incident wave from a vertical wall has been given by Hsu, Tsuchiya and Silvester (1979). The formulation reduces to stand-ing waves for the normal, to the boundary, incidence of an approaching wave. An interesting problem of small vibrations of a vertical plate immersed in water of finite depth and loaded with pressure resulting from a breaking standing wave has been discussed by Romańczyk (1994). In the theoretical model considered the breaking wave was obtained as a limiting case of a standing wave with amplitude growing in time. The latter was created with the help of a generation of the wa-ter wave with the frequency belonging to the resonance range of the rectangular fluid domain. The domain was bounded on one side by a rigid vertical plate of the generator and on the other – by the vertical plate suspended elastically within the fluid. Starting from rest, after a sufficiently long lapse of time, the growing standing wave reached the breaking point. The approximate analytical solution of

the problem was in good conformity with data obtained in experiments performed in a hydraulic flume. In the present paper a similar problem of standing waves generated in the rectangular fluid domain is considered. The problem is similar to that described by Romańczyk, but now, a material description of the problem on hand is investigated. In particular, we are looking for an approximate solution of the initial boundary value problem of the potential motion of fluid within the rectangular domain. The analysis is confined to the second order approximation of an equations describing water waves.

2. Fundamental Relations

In the following we confine our attention to the plane problem of fluid motion in Euclidean space. In order to describe the motion we introduce the Cartesian coordinate system z^r, $(r = 1, 2)$ in an actual configuration. In the reference configuration, the Cartesian coordinates, corresponding to names of the fluid particles, are denoted by Z^λ, $(\lambda = 1, 2)$. In the cases considered it is convenient to introduce a common Cartesian coordinate system. The motion of the fluid is described by the mapping of the names into the positions occupied by the material points at the time $t \geq 0$

$$z^i(Z^\alpha, t) = \delta^i_\lambda Z^\lambda + w^i(Z^\lambda, t), \tag{1}$$

where δ^i_λ is the Kronecker's delta and w^i are components of the displacement vector.

The Jacobian of the transformation is the determinant of the matrix of the transformation gradient

$$J = \det\left[z^i_{,\alpha}\right], \tag{2}$$

where the symbol $, \alpha$ denotes the partial derivative with respect to Z^α. Similarly, the symbol $, i$ denotes the partial derivative with respect to z^i, and the subscript $, t$ means the partial derivative with respect to time. For the assumed fluid incompressibility the Jacobian is equal to one. The inverse of the matrix of the transformation gradient reads

$$\left[Z^\alpha_{,i}\right] = \frac{1}{J}\begin{bmatrix} z^2_{,2} & -z^1_{,2} \\ -z^2_{,1} & z^1_{,1} \end{bmatrix}. \tag{3}$$

Knowing the above relations we may transform important formulae from the Eulerian variables into the Lagrangian variables and vice versa. Thus, let us consider a potential motion of the fluid with the potential function $\Phi(z^i, t)$ expressed in terms of the space variables. In these variables the potential should satisfy the Laplace equation

$$\nabla^2\Phi = 0 \tag{4}$$

and appropriate initial and boundary conditions.

With respect to the potential $\Phi(Z^\lambda, t)$, the velocity components are

$$\dot{w}_r(Z^\lambda, t) = \Phi_{,\beta} Z_r^\beta, \tag{5}$$

where the dot denotes the material time derivative.

At the same time, the Laplace equation (4) in the material coordinates assumes the form

$$\delta^{rs} \left[\Phi_{,\beta} Z_{,s}^\beta \right]_{,\alpha} Z_{,r}^\alpha = 0. \tag{6}$$

The fluid pressure in the material variables describes the formula

$$p(Z^\lambda, t) = \rho \left[-h - \dot{\Phi} + \frac{1}{2} \delta^{rs} \Phi_{,\beta} Z_{,r}^\beta \Phi_{,\gamma} Z_{,s}^\gamma + C(t) \right], \tag{7}$$

where $C(t)$ is a "constant" of the solution, and h is the potential of the mass force due to the gravitational field. In the spatial description the potential is given by the relation

$$h(z^r) = -g_i z^i. \tag{8}$$

When the coordinates are chosen in such a way that z^3 acts vertically upwards the coefficients g_i are: $g_1 = g_2 = 0$, $g_3 = -g$, where g is the gravitational acceleration. In the case discussed the fluid density is constant and in what follows, it is convenient to introduce the "pressure" function

$$P(Z^\lambda, t) = \frac{p}{\rho} = -h - \dot{\Phi} + \frac{1}{2} \delta^{rs} \Phi_{,\beta} Z_{,r}^\beta \Phi_{,\gamma} Z_{,s}^\gamma + C(t). \tag{9}$$

For the two-dimensional problem considered one may introduce the classic notation $z^1 = x, z^2 = y$ for the current configuration and $Z^1 = X, Z^2 = Y$ for the reference configuration. With respect to this notation Eq. (5) may be written in the form as follows

$$\begin{aligned} \dot{w}_1 &= \dot{u} = \Phi_{,X}(1 + v_{,Y}) - \Phi_{,Y} v_{,X}, \\ \dot{w}_2 &= \dot{v} = -\Phi_{,X} u_{,Y} + \Phi_{,Y}(1 + u_{,X}). \end{aligned} \tag{10}$$

Having the velocity we may calculate the displacement components

$$\begin{aligned} u(Z^\lambda, t) &= \int_0^t \dot{u}(Z^\alpha, \xi) d\xi + u(Z^\lambda, t = 0), \\ v(Z^\lambda, t) &= \int_0^t \dot{v}(Z^\alpha, \xi) d\xi + v(Z^\lambda, t = 0). \end{aligned} \tag{11}$$

In order to describe the initial and boundary conditions, let us consider the case shown in Fig. 1. The motion of the fluid is induced by the piston-type wave-maker (the rigid wall AB in the figure) starting to move at a certain moment of time. For the case shown in the figure the boundary conditions are:
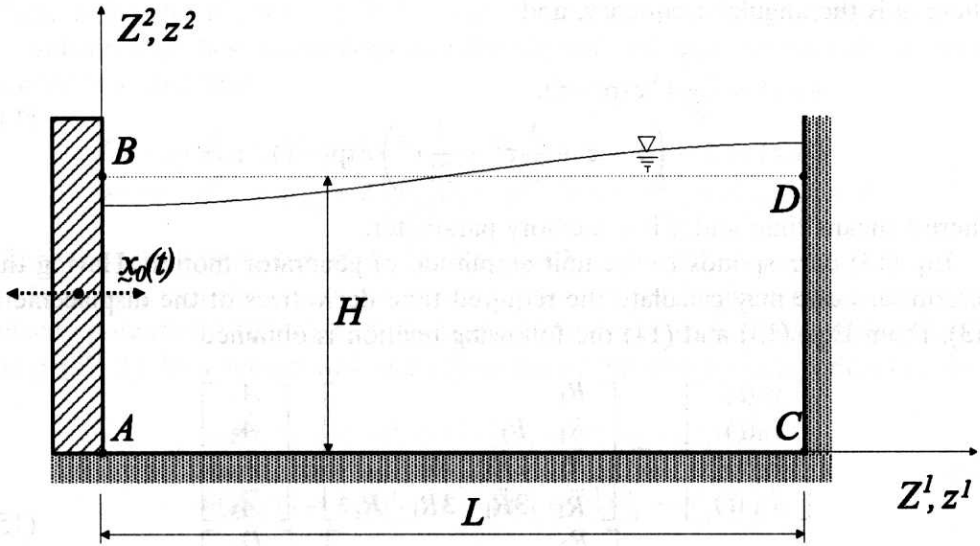
Fig. 1. Fluid domain with piston-type wave-maker

$$\text{a)} \quad u(Z^1 = 0, t) = x_0(t),$$

$$\text{b)} \quad u(Z^1 = L, t) = 0,$$

$$\text{c)} \quad v(Z^2 = 0, t) = 0, \tag{12}$$

$$\text{d)} \quad P(Z^2 = H, t) = \text{const.},$$

where $x_0(t)$ describes the horizontal displacement of the wall AB and the constant in Eq. (12d) will be assumed equal to zero.

3. Generation of Standing Waves

Let us consider the piston type generator starting to move at a certain moment in time. The simplest case is the harmonic generation of the fluid motion with smooth beginning for which not only velocities, but also the acceleration field disappears at the initial moment of time. Now, we aim to construct a harmonic generation of the waves by the piston-type generator whose displacement, velocity and acceleration are equal to zero at the beginning and thus no abrupt fluid loading exists. In describing the generation we follow the method developed by Wilde and Wilde (2001). We outline here some important results. The motion of the generator is assumed in the form

$$x_0(t) = A_3(\tau) \cos \omega t + D_3(\tau) \sin \omega t, \tag{13}$$

where ω is the angular frequency, and

$$A_3(\tau) = \frac{s^{-3}}{3!}t^3 \exp(-\tau),$$
$$D_3(\tau) = 1 - \left(1 + \tau + \frac{1}{2!}\tau^2 + \frac{1}{3!}\tau^3\right)\exp(-\tau), \quad \tau = \eta t, \tag{14}$$

where t means time and η is a memory parameter.

Eq. (13) corresponds to the unit amplitude of generator motion. Having the descriptions one may calculate the required time derivatives of the displacement (13). From Eqs. (13) and (14) the following relation is obtained

$$
\begin{bmatrix} x_0(t) \\ \dot{x}_0(t) \\ \ddot{x}_0(t) \\ \dddot{x}_0(t) \end{bmatrix} =
\begin{bmatrix} R_1 & & & \\ \dot{R}_1 & R_1 & \cdot & \\ \ddot{R}_1 & 2\dot{R}_1 & R_1 & \\ \dddot{R}_1 & 3\ddot{R}_1 & 3\dot{R}_1 & R_1 \end{bmatrix} \cdot
\begin{bmatrix} A_3 \\ \dot{A}_3 \\ \ddot{A}_3 \\ \dddot{A}_3 \end{bmatrix} +
$$
$$
+ \begin{bmatrix} R_2 & & & \\ \dot{R}_2 & R_2 & & \\ \ddot{R}_2 & 2\dot{R}_2 & R_2 & \\ \dddot{R}_2 & 3\ddot{R}_2 & 3\dot{R}_2 & R_2 \end{bmatrix} \cdot
\begin{bmatrix} D_3 \\ \dot{D}_3 \\ \ddot{D}_3 \\ \dddot{D}_3 \end{bmatrix}, \tag{15}
$$

where $R_1 = \cos\omega t$ and $R_2 = \sin\omega t$, and the dots denote derivatives with respect to time.

One may check that the displacement together with its first and second derivatives are equal to zero at the starting point. Moreover, with the passage of time the motion of the generator goes asymptotically to the case of harmonic displacement with constant amplitude. In the further discussion we confine our attention to generation described by the formulae.

4. Small Parameter Representation of the Fundamental Relations

The non-linear problem considered has no closed analytical solution and therefore, in order to find a solution to this, we have to approximate the fundamental equations by ones which are more tractable. For waves of small amplitude one of the methods of approximation is based on the assumption that the potential function and the surface elevation possess power series expansions with respect to a small parameter ε (Stoker 1957). Although the method applies to the infinitesimal wave approximation it fits into a general scheme for the approximating non-linear equations (Wehausen and Laitone 1960) and is therefore applied to the problem discussed in this paper. Thus, let us consider the following expansions in the small parameter ε

$$\Phi(Z^\lambda, t) = \varepsilon\phi^1 + \varepsilon^2\phi^2 + \varepsilon^3\phi^3 + \cdots,$$
$$u(Z^\lambda, t) = \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 + \cdots, \tag{16}$$
$$v(Z^\lambda, t) = \varepsilon v^1 + \varepsilon^2 v^2 + \varepsilon^3 v^3 + \cdots.$$

where ϕ^i, u^i and v^i for $i = 1, 2, 3, \cdots$ are "components" of the solutions.

Substituting these expansions into Eq. (6) and collecting terms with the same powers in ε, one finds

$$
\begin{aligned}
\varepsilon \to \quad & \phi^1_{,11} + \phi^1_{,22} = 0, \\
\varepsilon^2 \to \quad & \phi^2_{,11} + \phi^2_{,22} + 2\left[\phi^1_{,11}v^1_{,2} - \phi^1_{,12}\left(u^1_{,2} + v^1_{,1}\right) + \phi^1_{,22}u^1_{,1}\right] = 0.
\end{aligned}
\tag{17}
$$

where the terms up to the second order have been displayed.

It is seen that the linear component results in the Laplace equation for the velocity potential $\phi^1(Z^\lambda, t)$, and the higher component leads to Poisson's equation for $\phi^2(Z^\lambda, t)$. In a similar way, the expansion of the velocity components reads

$$
\begin{aligned}
\dot{u}(Z^\lambda, t) &= \varepsilon \cdot \phi^1_{,1} + \varepsilon^2 \cdot \left[\phi^2_{,1} + \phi^1_{,1}v^1_{,2} - \phi^1_{,2}v^1_{,1}\right], \\
\dot{v}(Z^\lambda, t) &= \varepsilon \cdot \phi^1_{,2} + \varepsilon^2 \cdot \left[\phi^2_{,2} + \phi^1_{,2}u^1_{,1} - \phi^1_{,1}u^1_{,2}\right].
\end{aligned}
\tag{18}
$$

Knowing that

$$
h(Z^\lambda, t) = gz^3(Z^\lambda, t) = g\left[Z^3 + v(Z^\lambda, t)\right],
\tag{19}
$$

we may assume $C = gH$ in equation (9) and write

$$
P(Z^\lambda, t) = g(H - Z^3) - gv(Z^\lambda, t) - \dot{\Phi}(Z^\lambda, t) + \frac{1}{2}\delta^{rs}\Phi_{,\beta}Z^\beta_{,r}\Phi_{,\gamma}Z^\gamma_{,s}.
\tag{20}
$$

The first term on the right hand side of the equation means the hydrostatic pressure $P^0 = g(H - Z^3)$. The pressure function may also be written in the following form

$$
P = P^0 + \varepsilon P^1 + \varepsilon^2 P^2,
\tag{21}
$$

where

$$
\begin{aligned}
P^1 &= -gv^1 - \dot{\phi}^1, \\
P^2 &= -gv^2 - \dot{\phi}^2 + \frac{1}{2}\left[(\phi^1_{,1})^2 + (\phi^1_{,2})^2\right].
\end{aligned}
\tag{22}
$$

With respect to the expansion (21) we can write the sequence of the dynamic boundary conditions on the upper boundary. From the first of equations (22) it follows

$$
gv^1(Z^1, H, t) + \dot{\phi}^1(Z^1, H, t) = 0.
\tag{23}
$$

Calculating the partial time derivative of the equation one obtains

$$
\ddot{\phi}^1 + g\phi^1_{,2}\Big|_{Z^2 = H} = 0.
\tag{24}
$$

A similar procedure for the square term (the second equation in relations 22) gives

$$\ddot{\phi}^2 + g\phi_{,2}^2 + g\left(\phi_{,2}^1 u_{,1}^1 - \phi_{,1}^1 u_{,2}^1\right) - \left(\phi_{,1}^1 \dot{\phi}_{,1}^1 + \phi_{,2}^1 \dot{\phi}_{,2}^1\right)\Big|_{Z^2=H} = 0. \quad (25)$$

The boundary condition on the bottom $Z^2 = 0$ results in

$$\dot{v}^1\big|_{Z^2=0} = 0, \quad \rightarrow \quad \phi_{,2}^1\big|_{Z^2=0} = 0,$$
$$\dot{v}^2\big|_{Z^2=0} = 0, \quad \rightarrow \quad \phi_{,2}^2 + \phi_{,2}^1 u_{,1}^1 - \phi_{,1}^1 u_{,2}^1\big|_{Z^2=0} = 0. \quad (26)$$

On the rigid vertical wall $Z^1 = 0$ we have the prescribed velocity $\dot{x}_0(t)$, and thus

$$\phi_{,1}^1\big|_{Z^1=0} = \dot{x}_0(t),$$
$$\phi_{,1}^2 + \phi_{,1}^1 v_{,2}^1 - \phi_{,2}^1 v_{,1}^1\big|_{Z^1=0} = 0. \quad (27)$$

Similarly, on the right boundary $Z^1 = L$ we have

$$\phi_{,1}^1\big|_{Z^1=L} = 0,$$
$$\phi_{,1}^2 + \phi_{,1}^1 v_{,2}^1 - \phi_{,2}^1 v_{,1}^1\big|_{Z^1=L} = 0. \quad (28)$$

5. First Order Solution of the Potential Motion

The first order solution of the problem in the Lagrangian variables is similar to the linear solution of the potential flow in the Eulerian variables. In both cases we have to solve the Laplace equation for the velocity potential satisfying given boundary and initial conditions. The only difference between the formulations is the system of coordinates. In the first description we have functions dependent on the material coordinates while in the second one, the relevant functions depend on space coordinates with the obvious relation that at $t = 0^+$, $z^r = \delta_\lambda^r Z^\lambda$. Therefore, in constructing the first order solution for the velocity potential $\phi^1(Z^\lambda, t)$ we do not need to distinguish the coordinate systems.

Thus, let us consider the first order solution. With respect to the boundary conditions on hand, the solution for the velocity potential $\phi^1(Z^\beta, t)$ may be expressed in the form

$$\phi^1(Z^\lambda, t) = \dot{x}_0(t)\left[Z^1 - \frac{(Z^1)^2 - (Z^2)^2}{2L}\right] + \sum_{n=0}^{\infty} \dot{B}_n(t)\cosh k_n Z^2 \cos k_n Z^1, \quad (29)$$

where

$$k_n = \frac{n\pi}{L}, \quad n = 0, 1, 2, \cdots \quad (30)$$

From substitution of the equation into condition (24) the following differential equations are obtained

$$\ddot{B}_0 + \frac{1}{3}L\left[1 + \frac{3}{2}\left(\frac{H}{L}\right)^2\right]x_0 + \frac{gH}{L}\dot{x}_0 = 0, \tag{31}$$

$$\ddot{B}_n + r_n^2 B_n = RA_n, \quad n = 1, 2, \cdots,$$

where

$$r_n^2 = gk_n \tanh k_n H, \tag{32}$$

$$RA_n(t) = \frac{2\ddot{x}_0(t)}{k_n^2 L \cosh k_n H}, \quad n = 1, 2, \cdots.$$

The solutions of the equations are

$$\ddot{B}_0(t) = -\frac{1}{3}L\left[1 + \frac{3}{2}\left(\frac{H}{L}\right)^2\right]\ddot{x}_0(t) - \frac{gH}{L}x_0(t) + C_0,$$

$$\dot{B}_n(t) = C_n^1 \cos r_n t + C_n^2 \sin r_n t + \frac{1}{r_n}\int_0^t RA_n(\xi)\sin r_n(t - \xi)\,d\xi \tag{33}$$

$$n = 1, 2, \cdots.$$

The constants C_0, C_n^1 and C_n^2 ($n = 1, 2, \cdots$) in the last relations are obtained from initial conditions.

Knowing the potential function (29) one may calculate the first order hydrodynamic part of the pressure on the free surface. The first relation (22) leads to the expression

$$-P^1(Z^1, H, t = 0) = \dot{\phi}^1(Z^1, H, t = 0) =$$

$$= \ddot{x}_0\left[Z^1 - \frac{(Z^1)^2 - (H)^2}{2L}\right] + \ddot{B}_0 + \sum_{n=1}^{\infty}\ddot{B}_n \cosh k_n H \cos k_n Z^1\bigg|_{t=0} = 0. \tag{34}$$

From the first formula in (33) and equation (34) one obtains

$$C_0 = \frac{gH}{L}\ddot{x}_0(t = 0^+). \tag{35}$$

At the same time from the second formula in (33) and relation (34) it follows that

$$\ddot{B}_n = \frac{2\ddot{x}_0(t = 0^+)}{k_n^2 L \cosh k_n H}, \quad \rightarrow \quad C_n^2 = \frac{2\ddot{x}_0(t = 0^+)}{k_n^2 r_n L \cosh k_n H}, \quad n = 1, 2, \cdots. \tag{36}$$

In order to find the remaining constants we evaluate the components of the velocity field

$$\dot{u}^1 = \phi^1_{,1} = \dot{x}_0(t)\left(1 - \frac{Z^1}{L}\right) - \sum_{n=1}^{\infty} \dot{B}_n(t)k_n \cosh k_n Z^2 \sin k_n Z^1,$$

$$\dot{v}_1 = \phi^1_{,2} = \dot{x}_0(t)\frac{Z^2}{L} + \sum_{n=1}^{\infty} \dot{B}_n(t)k_n \sinh k_n Z^2 \cos k_n Z^1. \tag{37}$$

Assuming that at $t = 0^+$ we have a zero velocity field the following is obtained

$$\dot{B}_n(t = 0^+) = 0, \quad \rightarrow \quad C^1_n = 0, \; n = 1, 2, \cdots \tag{38}$$

On the basis of the above results we get

$$\ddot{B}_0(t) = -\frac{1}{3}L\left[1 + \frac{3}{2}\left(\frac{H}{L}\right)^2\right]\ddot{x}_0(t) + \frac{gH}{L}\left[x_0(0^+) - x_0(t)\right],$$

$$\dot{B}_n(t) = \frac{2\ddot{x}_0(0^+)\sin r_n t}{k_n^2 r_n \cosh k_n H} + \frac{1}{r_n}\int_0^t RA_n(\xi)\sin r_n(t - \xi)\, d\xi, \; n = 1, 2, \cdots, \tag{39}$$

where $RA_n(\xi)$ is described by the second of (32).

In further calculations we will also need values of $\dot{B}_0(t)$ and $B_n(t)$ for chosen moments of time $t_k = k\Delta t$, where Δt is the time step. In order to obtain the latter quantities we have to integrate the formulae (39) in the time domain, i.e.

$$\dot{B}_0(t) = -\frac{1}{3}L\left[1 + \frac{3}{2}\left(\frac{H}{L}\right)^2\right]\dot{x}_0(t) + \frac{gH}{L}\left[x_0(0^+)\cdot t - \int_t x_0(t)dt\right],$$

$$B_n(t) = \frac{2\ddot{x}_0(0^+)}{k_n^2 r_n \cosh k_n H}\int_t \sin r_n t\, dt + \frac{1}{r_n}\int_t \int_0^t RA_n(\xi)\sin r_n(t - \xi)\, d\xi dt, \tag{40}$$

$$n = 1, 2, \cdots,$$

The integrals entering the last relations may be evaluated numerically, according to the formulae

$$J_1(t_k) = J_1(t_{k-1}) + \int_{t_{k-1}}^{t_k} x_0(t)dt,$$

$$J_2(t_k) = J_2(t_{k-1}) + \int_{t_{k-1}}^{t_k} \ddot{x}_0(\xi)\cos r_n\xi\, d\xi,$$

$$J_3(t_k) = J_3(t_{k-1}) + \int_{t_{k-1}}^{t_k} \ddot{x}_0(\xi)\sin r_n\xi\, d\xi, \tag{41}$$

$$J_4(t_k) = J_2(t_k)\sin r_n t_k - J_3(t_k)\cos r_n t_k,$$

$$J_5(t_k) \cong J_5(t_{k-1}) + \frac{1}{2}[J_4(t_k) + J_4(t_{k-1})]\cdot \Delta t\,.$$

With the last relations all constants of the solution (33) are determined. Knowing the solution it is a simple task to calculate the first order velocity field and the first order pressure field as functions of the material variables and time.

6. Second Order Solution of the Motion

In order to find a second order solution we have to solve Poisson's equation for the second component of the velocity potential together with proper boundary and initial conditions. The equation contains terms resulting from the first order solution. In order to calculate the terms we have to know the first and second order derivatives of the first order potential function with respect to the material variables. As compared with the first order solution the problem becomes more complicated. Poissons's equation for the second order velocity potential (the second of equation 17) is written in the form

$$\nabla^2 \phi^2(Z^1, Z^2, t) = -RA(Z^1, Z^2, t), \tag{42}$$

where

$$RA(Z^\alpha, t) = 2\left[\phi^1_{,11} v^1_{,2} - \phi^1_{,12}\left(u^1_{,2} + v^1_{,1}\right) + \phi^1_{,22} u^1_{,1}\right]. \tag{43}$$

The solution to the Poisson's equation (42) may be obtained by means of the Green theorem (Sokolnikoff and Redheffer 1966)

$$\phi^2(P) = \frac{1}{2\pi}\oint_C\left(\ln r\frac{\partial\phi^2}{\partial n} - \phi^2\frac{\partial}{\partial n}\ln r\right)ds + \frac{1}{2\pi}\iint_S RA(Z^1, Z^2)\ln r\, dS, \tag{44}$$

where S is the fluid region, C is the fluid contour in the reference configuration and $r = r(P, C)$ the vector connecting the domain point P with the boundary point C.

The right hand side term of equation (42) reads

$$\begin{aligned}
RA &= -4\left[\phi^1_{,11}\int_t \phi^1_{,11}dt + \phi^1_{,12}\int_t \phi^1_{,12}dt\right] = \\
&= -4\left[\frac{\dot{x}_0}{L} + \sum_{n=1}^{\infty}\dot{B}_n k_n^2 \cosh k_n Z^2 \cos k_n Z^1\right] \times \\
&\quad \times\left[\frac{x_0}{L} + \sum_{n=1}^{\infty}B_n k_n^2 \cosh k_n Z^2 \cos k_n Z^1\right] + \\
&\quad -4\left[\sum_{n=1}^{\infty}\dot{B}_n k_n^2 \sinh k_n Z^2 \sin k_n Z^1\right] \times \\
&\quad \times\left[\sum_{n=1}^{\infty}B_n k_n^2 \sinh k_n Z^2 \sin k_n Z^1\right].
\end{aligned} \tag{45}$$

At the same time, for $Z^1 = 0$ we have the boundary condition

$$\phi_{,1}^2 \Big|_{Z^1=0} = -\dot{x}_0(t) \left[\frac{x_0(t)}{L} + \sum_{n=1}^{\infty} B_n(t) k_n^2 \cosh k_n Z^2 \right]. \tag{46}$$

The boundary conditions on the bottom and on the right vertical wall are

$$\phi_{,2}^2 \Big|_{Z^2=0} = 0, \quad \phi_{,1}^2 \Big|_{Z^1=L} = 0. \tag{47}$$

The dynamic boundary condition on the upper surface of the fluid assumes the form

$$\ddot{\phi}^2 + g\phi_{,2}^2 + g \ RB - RC = 0, \tag{48}$$

where

$$
\begin{aligned}
RB &= -\left[\dot{x}_0 \frac{H}{L} + \sum_{n=1}^{\infty} \dot{B}_n k_n \sinh k_n H \cos k_n Z^1 \right] \times \\
&\times \left[\frac{x_0}{L} + \sum_{n=1}^{\infty} B_n k_n^2 \cosh k_n H \cos k_n Z^1 \right] + \\
&+ \left[\dot{x}_0 \left(1 - \frac{Z^1}{L} \right) - \sum_{n=1}^{\infty} \dot{B}_n k_n \cosh k_n H \sin k_n Z^1 \right] \times \\
&\times \left[\sum_{n=1}^{\infty} B_n k_n^2 \sinh k_n H \sin k_n Z^1 \right]
\end{aligned} \tag{49}
$$

and

$$
\begin{aligned}
RC &= \left[\dot{x}_0 \left(1 - \frac{Z^1}{L} \right) - \sum_{n=1}^{\infty} \dot{B}_n k_n \cosh k_n H \sin k_n Z^1 \right] \times \\
&\times \cdot \left[\ddot{x}_0 \left(1 - \frac{Z^1}{L} \right) - \sum_{n=1}^{\infty} \ddot{B}_n k_n \cosh k_n H \sin k_n Z^1 \right] + \\
&+ \left[\dot{x}_0 \frac{H}{L} + \sum_{n=1}^{\infty} \dot{B}_n k_n \sinh k_n H \cos k_n Z^1 \right] \times \\
&\times \left[\ddot{x}_0 \frac{H}{L} + \sum_{n=1}^{\infty} \ddot{B}_n k_n \sinh k_n H \cos k_n Z^1 \right].
\end{aligned} \tag{50}
$$

To find a solution to the problem at each level of time it is necessary to calculate the integrals entering equation (44). In order to avoid tedious calculations we resort to a discrete solution of the Poisson's equation by means of the finite difference method (FDM). With this method only a finite number of nodal points of an assumed net is considered. Thus, let us consider now the discrete formulation of the problem within the rectangular domain $(0 \le Z^1 \le L, \ 0 \le Z^2 \le H)$. Let the spacing of vertical lines of the assumed net be equal to a and of the horizontal lines – to b, respectively. Since the main part of the Poisson's equation is formed

by the same operator as in the case of the Laplace equation, it is reasonable to consider first the finite difference representation of the Laplace equation for the potential function. To simplify the notation we shall use the notation ϕ for the velocity potential $\phi^2(Z^\lambda, t)$. For a typical nodal point (i, j) within the fluid, where i means the number of a vertical line and j denotes the number of a horizontal line, the finite difference representation of the Laplace equation reads

$$- \gamma\phi_{i-1,j} - \phi_{i,j-1} + K\phi_{i,j} - \phi_{i,j+1} - \gamma\phi_{i+1,j} = 0, \qquad (51)$$

where

$$\gamma = \left(\frac{b}{a}\right)^2 \text{ and } K = 2(1 + \gamma). \qquad (52)$$

The equations (51) are written for all nodal points of the assumed net, including boundary points. In order to write the equations at the boundary points (at $Z^1 = 0$, $Z^1 = L$, $Z^2 = 0$ and $Z^2 = H$) we have to extend the discrete net in such a way that together with each of the boundary points there is a neighbouring nodal point placed on the outward normal to the boundary at the considered point. The unknown value of the potential function at these external points is expressed, by means of the boundary conditions, in terms of values of the function at internal and boundary nodal points. As compared with description (51), in the finite difference representation of the Poisson's equation we have non-zero right hand side terms. Knowing the first order solution (relations 45, 49 and 50) we can calculate the right hand side term of equation (42) at each point of the assumed net including boundary points, and finally, we can establish the relevant system of algebraic equations of the problem mentioned. It is worth adding that the boundary conditions of the problem on hand involve not only the values of the potential function but also the first and second time derivatives of it. For example, the second time derivative of the potential function enters the boundary condition on the upper surface of the fluid. In order to solve the problem in the time domain we resort to discrete description of the time, i.e. instead of continuous time we introduce a sequence of time steps with the increment $\Delta t > 0$. With the discrete time, the FD equations written for the assumed level of time, say at $t_k = k \cdot \Delta t$, contain an unknown value of the potential function corresponding to the next moment of time. In order to overcome this difficulty and construct an approximate solution to the initial value problem considered, it is convenient to use the Wilson θ method. The method enables us to build the discrete system of FD equations for the unknown values of the potential function corresponding to a single level of time. It is based on an assumed linear approximation to the second order time derivative of a function at every point of the discrete time. In the problem discussed, the second time derivative of the potential function is approximated by a linear function within the vicinity of the considered moment of time. To make the discussion clear, we attached here some of the fundamental

equations of the method (for details see Bathe 1982). Assuming that we know the solution of the problem at time t (ϕ, $\dot{\phi}$ and $\ddot{\phi}$) the standard equations of the method are

$$\ddot{\phi}(\tau) = \ddot{\phi}_{(1)} + \frac{\tau}{DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}),\tag{53}$$

and

$$\dot{\phi}(\tau) = \dot{\phi}_{(1)} + \ddot{\phi}_{(1)}\tau + \frac{\tau^2}{2DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}),$$

$$\phi(\tau) = \phi_{(1)} + \dot{\phi}_{(1)}\tau + \frac{1}{2}\ddot{\phi}_{(1)}\tau^2 + \frac{\tau^3}{6DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}),\tag{54}$$

where $\phi_{(1)} = \phi(t)$, $\phi_{(3)} = \phi(t + DT)$ and $DT = \theta\Delta t$, $\theta = 1.47$, and τ is measured from the first point, i.e. $\phi(\tau = 0) = \phi(t)$.

From the relations one obtains

$$\dot{\phi}_{(3)} = \frac{3}{DT}(\phi_{(3)} - \phi_{(1)}) - 2\dot{\phi}_{(1)} - \frac{DT}{2}\ddot{\phi}_{(1)},$$

$$\ddot{\phi}_{(3)} = \frac{6}{DT^2}(\phi_{(3)} - \phi_{(1)}) - \frac{6}{DT}\dot{\phi}_{(1)} - 2\ddot{\phi}_{(1)}.\tag{55}$$

In the Wilson θ method all equations of the problem together with boundary conditions are written for the time $(t + DT)$. Having the solution at this time we can calculate the solution at the level $(t + \Delta t)$

$$\ddot{\phi}(t + \Delta t) = \ddot{\phi}_{(1)} + \frac{\Delta t}{DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}),$$

$$\dot{\phi}(t + \Delta t) = \dot{\phi}_{(1)} + \ddot{\phi}_{(1)}\Delta t + \frac{(\Delta t)^2}{2DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}),\tag{56}$$

$$\phi(t + \Delta t) = \phi_{(1)} + \dot{\phi}_{(1)}\Delta t + \frac{1}{2}\ddot{\phi}_{(1)}(\Delta t)^2 + \frac{(\Delta t)^3}{6DT}(\ddot{\phi}_{(3)} - \ddot{\phi}_{(1)}).$$

The procedure is repeated for subsequent levels of discrete time.

7. Numerical Examples

In order to illustrate the consideration and to investigate accuracy of the approximate solution obtained above, some numerical examples are presented below. Numerical computations have been performed for a set of different lengths of the fluid domain, different amplitudes and frequencies of the generator motion. In all cases the amplitudes of the standing waves grow in time, and therefore, for each case we have to assume a range of time, measured from the starting point, within which the approximate solution may be considered as being sufficiently accurate. This range of time depends on the approximate formulation of the original non-linear problem which is based on the fundamental assumption of

small waves. Although the assumption does not form a precise limit, it is understood that the amplitude of the standing wave should be small (small compared to wave length and water depth) and the second order solution should be smaller (much smaller) than that of first order. The range of time corresponds to a continuous solution far from the point where the phenomenon of breaking of the waves may occur. It should be stressed however that, because of the approximate formulation, the computational model may formally lead to smooth solutions also in the range corresponding to the breaking point. Some of the results obtained in numerical calculations are presented in the subsequent graphs. Fig. 2 shows the elevation of the free surface on the right boundary of the fluid domain (point D in the figure). The plots correspond to the fluid domain $L \times H = 1.2$ m $\times$ 0.6 m and the generation frequency $\omega = 4.8533$ s^{-1}. In Fig. 3 the resultant of pressure force and its moment relative to the bottom as functions of time are presented. It is seen that the hydrodynamic resultant exceeds the hydrostatic one significantly. Subsequently, Fig. 4 shows the evolution in time of the pressure distribution on the right boundary for the same fluid domain and frequency generation as in Fig 2. The amplitude of the generator motion is equal to 0.04 m. From the plots it can be seen how the motion of the fluid influences the pressure distribution on the vertical wall. In order to evaluate the accuracy of the numerical model and the range of its reasonable application, the results of the discrete formulations should be compared with results of experiments in a hydraulic flume. Such comparison is presented in Fig. 5 where the results of calculations are compared with results of experiments described by Wilde et al. (1998). In experiments, the standing waves were generated in the rectangular fluid domain of dimensions $H \times L = 0.8$ m $\times 2.325$ m. The graphs in the figure correspond to water waves of a length equal twice the length of the domain. From the plots it may be seen that the computational model presented above gives surprisingly good results.

From the results of computations it follows that for the considered resonance case the second component is responsible for the proper description of the bottom influence on the final shape of the elevation wave. In the discussed case of the standing wave the second component forces the mean level of the free surface on the right vertical wall up over the fluid level at rest. At the same time the height of the second order components grows non-linearly with time (the height of the first order component grows linearly in time). From the solution it follows that in order to get a more convincing results the higher order terms of the solution procedure should be taken into account, especially for higher standing waves. This however, may cause serious difficulties in a discrete formulation because higher order terms need higher order derivatives of the functions entering the fundamental equations of the problem which cannot be calculated with acceptable accuracy in the discrete space.
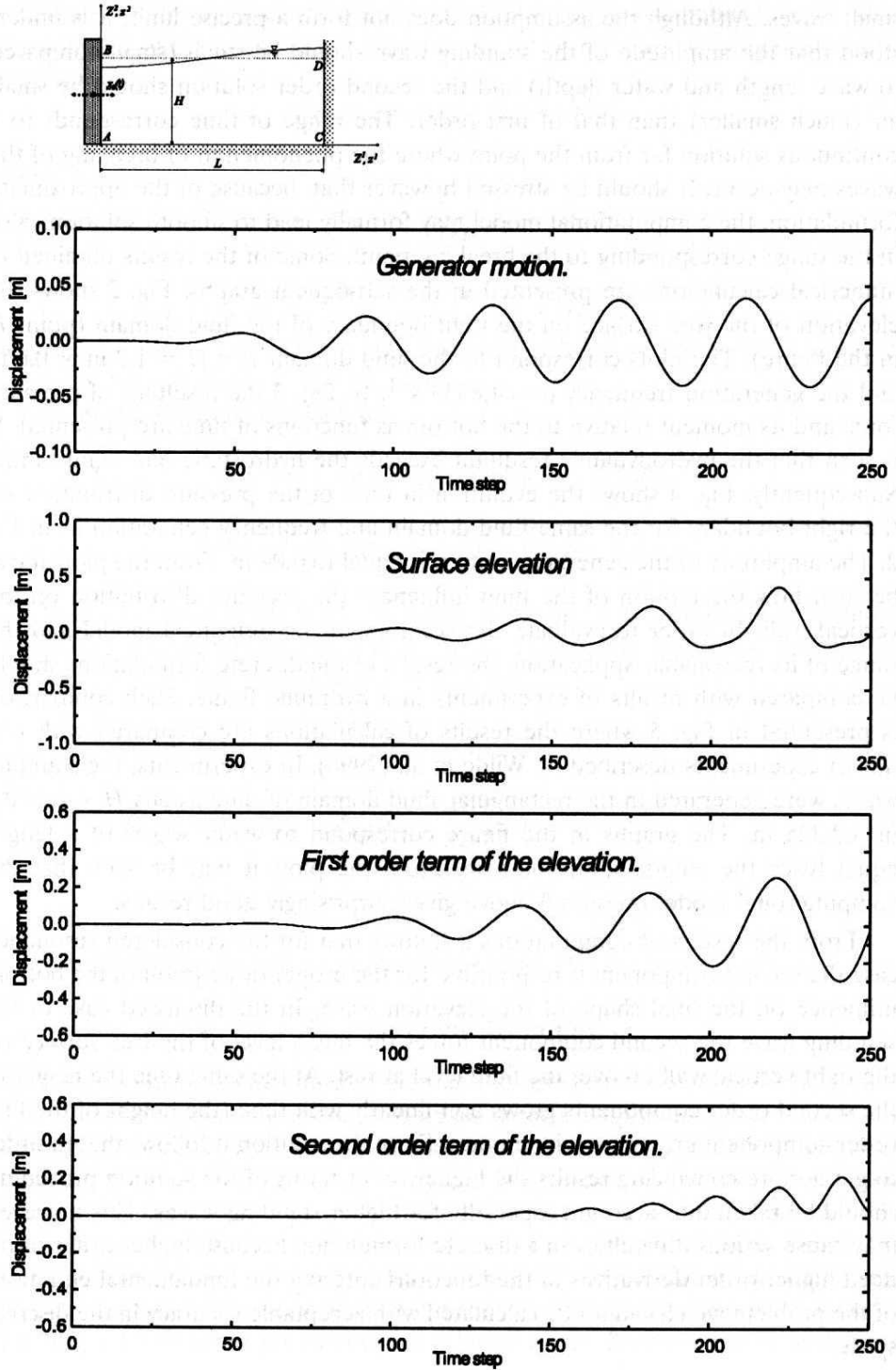
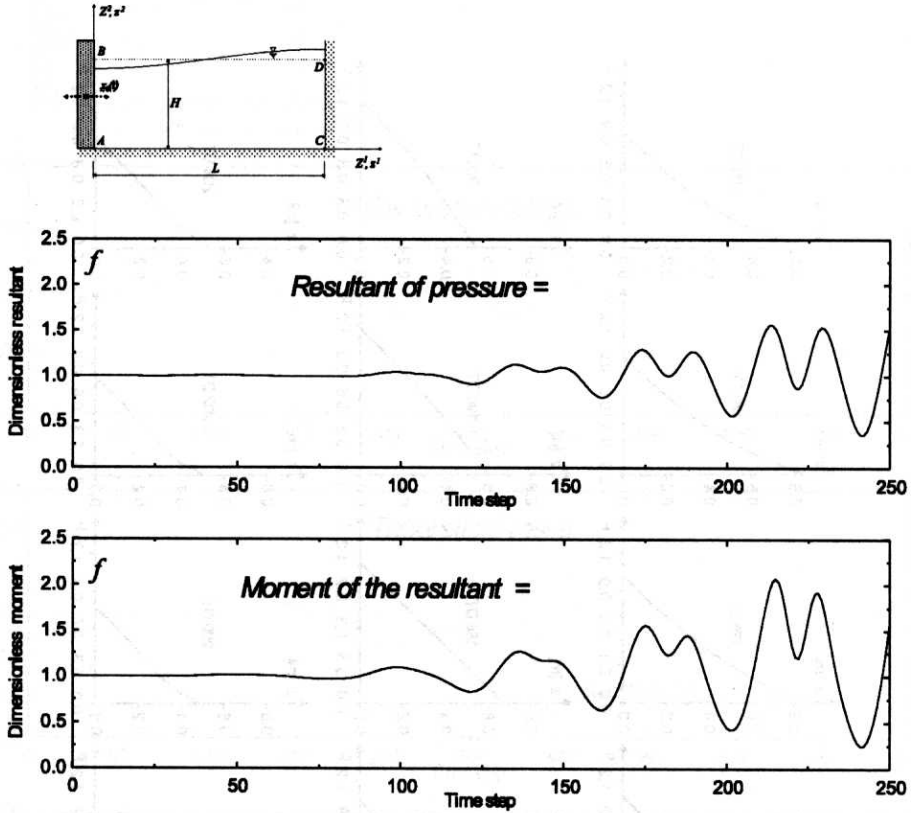Fig. 2. Free surface elevation on the right boundary ($H = 0.60$ m, $L = 2H$, $\lambda = 2L$)

Fig. 3. Resultant of pressure on the right boundary and its moment relative to the fluid bottom
($H = 0.60$ m, $L = 2H$, $\lambda = 2L$)

8. Concluding Remarks

The approximate analysis of the non-linear problem of generation of standing water waves allows us to formulate some important conclusions as follows

- The formulation of the problem in material coordinates leads to Poisson's equations for higher components of the velocity potential.
- The discrete formulation with the help of the FDM may be successfully applied to solution of the Poisson's equations only for the lowest orders of the small parameter method. In the case of higher order terms, we have to calculate higher order derivatives of the potential functions which may introduce serious difficulties and errors because of an approximate description of the functions in the discrete formulation.
- The amplitude of the first, linear term of the solution grows linearly in time as it should be for the considered case of the resonance motion of the generator-fluid system.
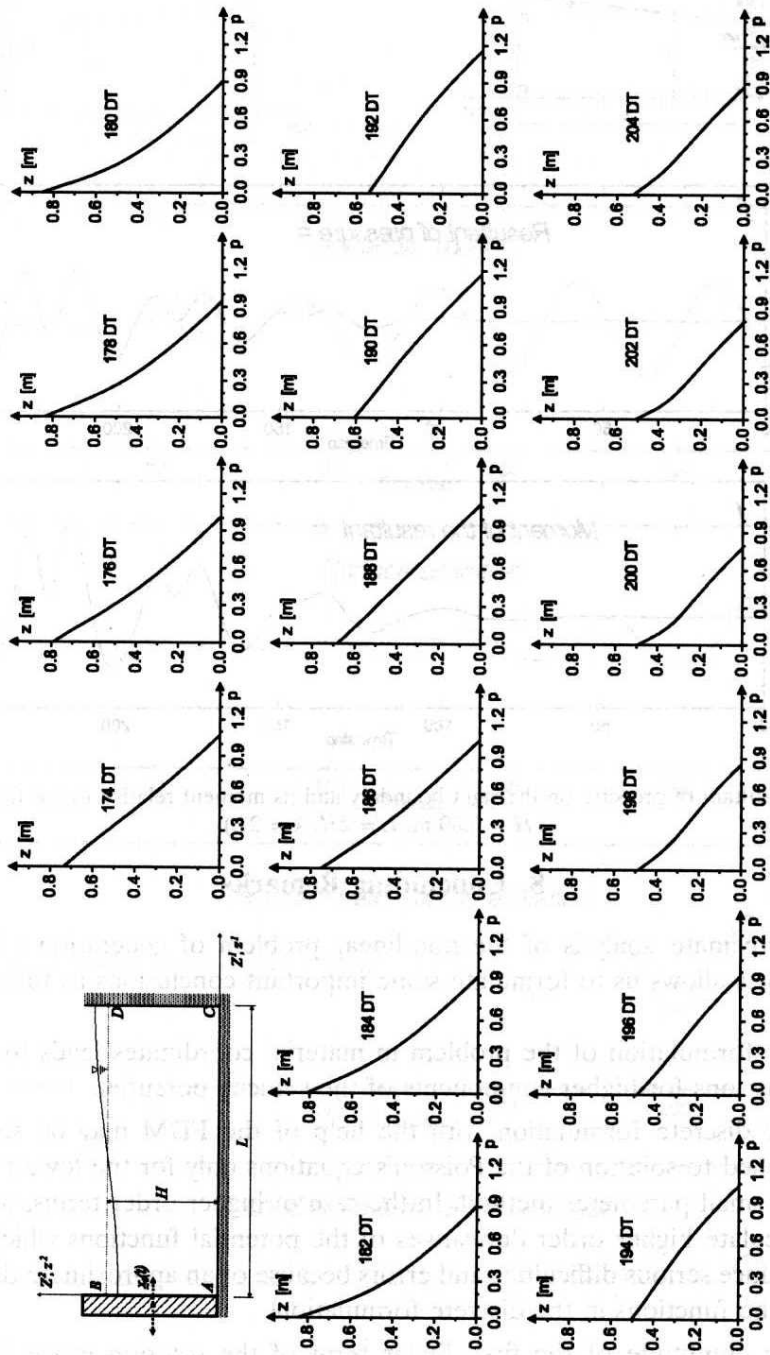
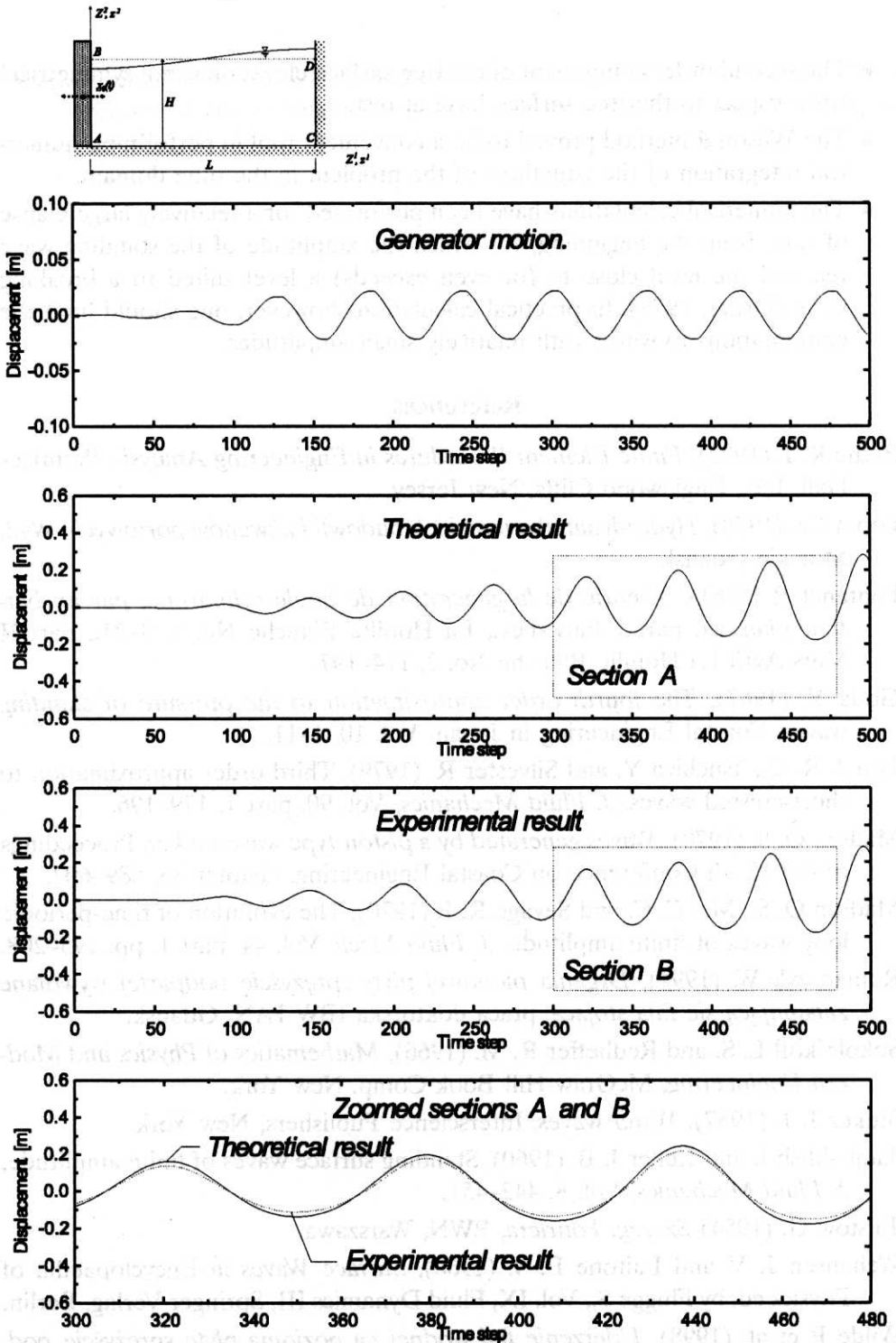Fig. 4. Pressure distribution on the right boundary at chosen moments of time

Fig. 5. Comparison of the theoretical and experimental free surface elevation on the right boundary ($L = 2.325$ m, $H = 0.80$ m, $\lambda = 2L$)

- The second order component of the free surface elevation is not symmetrical with respect to the free surface level at rest.
- The Wilson θ method proved to be a convenient tool in performing numerical integration of the equations of the problem in the time domain.
- The numerical calculations have been performed for a relatively large elapse of time from the beginning, for which the amplitude of the standing wave reached the level close to (or even exceeds) a level suited to a breaking wave (Druet, 1978). In practical calculations however, one should limit the computations to waves with relatively small amplitudes.

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