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Numerical Solution of 2-D Advection Equation Using the Modified Finite Element Method and Directional Splitting Technique

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Abstract

A method of the numerical solution of a 2-D linear advection equation is proposed. In the solution, the modified finite element method and directional splitting technique is used. For the time integration the two-step differential scheme is used. Two weighting parameters introduced into the scheme determine the accuracy and stability of the solution. Numerical diffusion and dispersion tensors are derived for the pure advection equation solved using the proposed method. They enable the explanation of numerical properties and applicability of this scheme. The proposed scheme is of third-order accuracy and is adaptive, allowing for simultaneous elimination of numerical diffusion and dispersion from the solution of a 2-D advection equation.

1. Introduction

Problems with numerical diffusion and dispersion may be avoided if a more accurate scheme is used, for instance, of third order accuracy. In this paper such a scheme will be proposed.

An advection-diffusion equation may be solved using the splitting technique. However, this is not a numerical method of solution, but rather a technique that enables significant simplification of the solving procedure. The numerical solution is usually obtained applying well-known finite differences or finite element schemes.

In scientific literature, many versions of the splitting technique are described. They are applied to different problems, including hydrodynamics as well as mass and energy transport. A basis to all the later techniques was the fractional step method developed by Chorin (1967) for incompressible flow, consequently improved and used by other authors including: Cunge et al. (1980), Donea et al. (1982), Gresho et al. (1984), Kim and Moin (1985), Kawahara and Ohmiya (1985), Ramaswamy (1988), Shimura and Kawahara (1988), Zienkiewicz et al. (1990), Laval and Quartapelle (1990), Peeters et al. (1991), Yeh and Chang (1992), Jiang and Kawahara (1993), Pinelli and Vacca (1994) and many others. Unfortunately not all attempts at implementing the splitting technique were satisfactory. This was possibly the result of insufficient quality of methods used for solving equations obtained due to the application of the splitting procedure. For instance, in the case of the ADI method, the finite differences method is used most often. The method has limitations influencing the quality of the solution. Szymkiewicz (1993) presented an interesting proposition for solving shallow water equations that combined directional splitting technique with the splitting technique referring to physical processes. The method proved effective for relatively high Courant numbers. The splitting technique significantly simplifies the solution of the algorithm in which sets of algebraic equation systems of tridiagonal matrixes are solved. In this paper it will be proved that in the case of the advection-diffusion equation, a combination of the directional splitting technique with the finite element method gives very good results.

2. Formulation of a Problem

Let us write a 2D advection equation (1)

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} = 0, \tag{1}$$

where:

- x, y – space coordinates,
- u, v – velocity vector components in x and y directions respectively,
- f – any scalar value,
- t – time,

in the following form:

$$\frac{\partial f}{\partial t} = F, \tag{2}$$

where F includes all terms of Eq. (1) excluding the derivative in relation to time, i.e.

$$F = -u\frac{\partial f}{\partial x} - v\frac{\partial f}{\partial y}. \tag{2a}$$

The Eq. (2) may be integrated in time within the limits from t to $t + \Delta t$ giving the general formula

$$f_{t+\Delta t} = f_t + \int_t^{t+\Delta t} F dt, \tag{3}$$

where f_t describes the initial condition.

All known methods of solving equation (2) differ depending on the method of approximation of differential operators included in F and the numerical calculating of the integral in (3). Since function F depicts the process of advection of quantity f in directions x and y, it may be presented as the sum of two terms:

$$F = F_x + F_y, \tag{4}$$

where

$$F_x = -u\frac{\partial f}{\partial x}, \; F_y = -v\frac{\partial f}{\partial y}. \tag{4a}$$

Thus the equation (2) may be written as follows:

$$\frac{\partial f}{\partial t} = F_x + F_y, \tag{5}$$

and its general solution (3) assumes the form:

$$f_{t+\Delta t} = f_t + \int_t^{t+\Delta t} (F_x + F_y)\, dt = f_t + \int_t^{t+\Delta t} F_x dt + \int_t^{t+\Delta t} F_y dt. \tag{6}$$

Implementation of auxiliary denotation

$$f^{(1)}_{t+\Delta t} = f_t + \int_t^{t+\Delta t} F_x dt \tag{7}$$

results in

$$f_{t+\Delta t} = f^{(1)}_{t+\Delta t} + \int_t^{t+\Delta t} F_y dt. \tag{8}$$

As can be noticed, the advection equation is solved in two stages in each time step Δt: first 1D equation is solved for direction x (Eq. 7), then the result obtained is used to solve similar equation (8) in direction y.

The technique presented above enables the solving of Eq. (1) by splitting it in two directions in relation to independent variables x and y, and may be written as:

$$\frac{\partial f^{(1)}}{\partial t} = F_x, \tag{9a}$$

with the initial condition $f_t^{(1)} = f_t$ and

$$\frac{\partial f^{(2)}}{\partial t} = F_y, \tag{9b}$$

with the initial condition $f_t^{(2)} = f_{t+\Delta t}^{(1)}$.

Calculations on the level $t + \Delta t$ are conducted in two stages. In the first an 1D differential equation is solved for direction x (9a) giving as a result $f_{t+\Delta t}^{(1)}$. This is an intermediate value being an initial condition for calculations in direction y (Eq. 2b). The value in question on the level $t + \Delta t$ is $f_{t+\Delta t} = f_{t+\Delta t}^{(2)}$.

The modified finite element method is used for solving both 1D equations (Szymkiewicz 1995). The solution domain S is covered by the rectangular grid of M columns and N rows, and dimensions of a grid cell are $\Delta x \cdot \Delta y$ (Fig. 1).

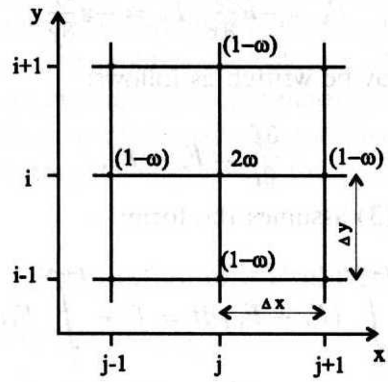

Fig. 1. Grid of nodes applied in the proposed scheme

Each row and column is treated as a separate structure of the length L, consisting of linear elements of elementary lengths Δx and Δy respectively. Let us solve equation (9a) in a selected row i first. Let us neglect index „i" in the notation for simplicity. The Galerkin method requires the following condition to be satisfied (Zienkiewicz 1972):

$$\int_0^L N\Omega(f_a)dx = \sum_{j=1}^{M-1} \int_{x_j}^{x_{j+1}} N\Omega(f_a)dx = 0, \tag{10}$$

where:

Ω	–	symbolic notation of equation (9a),
$N = (N_1(x), \dots N_M(x))^T$	–	vector of linear basis functions,
L	–	distance between the first and last nodes in row i.

Substituting equation (9a) into condition (10) results in:

$$\sum_{j=1}^{M-1} \int_{x_j}^{x_{j+1}} \mathbf{N} \left(\frac{\partial f_a^{(1)}}{\partial t} + u \frac{\partial f_a^{(1)}}{\partial x} \right) dx = 0, \tag{11}$$

where:

$$f_a^{(1)}(x,t) = \sum_{j=1}^{M} N_j(x) f_j^{(1)}(t)_j = \mathbf{N}^T \mathbf{f}^{(1)} \tag{12}$$

is an approximation of function $\mathbf{f}^{(1)}$,

$$\mathbf{f}^{(1)} = (f_1, f_2, \dots f_M)^T.$$

In this case also modification of the finite element method's algorithm is proposed, analogical to the case of the 2D equation. Szymkiewicz (1995) has presented a detailed description of the modified method proposing a following alternation of the condition (11):

$$\sum_{j=1}^{M-1} \int_{x_j}^{x_{j+1}} \mathbf{N} \left(\frac{\partial f_c^{(1)}}{\partial t} + u_c \frac{\partial f_a^{(1)}}{\partial x} \right) dx = 0 \tag{13}$$

where:

- f_a – depicts approximation according to the formula applied in (11),
- f_c – depicts approximation based on the weighting mean of nodal values in the element.

In a selected element limited by nodes j and $j+1$ the value f_c is defined as follows:

$$f_c(t) = \omega f_j(t) + (1-\omega) f_{j+1}(t) \quad \text{for a product of type } N_j f_c \tag{14a}$$

$$f_c(t) = (1-\omega) f_j(t) + \omega f_{j+1}(t) \quad \text{for a product of type } N_{j+1} f_c \tag{14b}$$

where ω is a weighting parameter from the range $< 0, 1 >$. According to this method a differential equation is obtained for node j in which $f^{(1)}$ is denoted by f for simplification of the notation:

$$\begin{aligned} &\left[(1-\omega) - \theta \frac{u \Delta t}{\Delta x}\right] f_{j-1}^{k+1} + 2\omega f_j^{k+1} + \left[(1-\omega) + \theta \frac{u \Delta t}{\Delta x}\right] f_{j+1}^{k+1} = \\ &= \left[(1-\omega) + (1-\theta) \frac{u \Delta t}{\Delta x}\right] f_{j-1}^{k} + 2\omega f_j^{k} + \left[(1-\omega) - (1-\theta) \frac{u \Delta t}{\Delta x}\right] f_{j+1}^{k}. \end{aligned} \tag{15a}$$

Notation of the above equation for all nodes $j = 2,3,4,\dots M-1$ of row i leads to a system of linear algebraic equations with tridiagonal matrix of the following form (Szymkiewicz 1995):

$$(\mathbf{A} + \Delta t \theta \mathbf{B}_{t+\Delta t}) \mathbf{f}_{t+\Delta t} = (\mathbf{A} - \Delta t (1-\theta) \mathbf{B}_t) \mathbf{f}_t, \tag{15b}$$

where:

A – constant matrix, banded and symmetrical,

B – variable matrix, banded and asymmetrical,

θ – weighting parameter from the range $< 0, 1 >$.

The system of equations must be completed with given boundary conditions. The system may be solved using Thomas method (Fortuna et al. 1982).

The same method is applied to solve equation (9b) for direction y. The following differential equation is then obtained :

$$\left[(1-\omega)-\theta\frac{v\Delta t}{\Delta y}\right]f_{i-1}^{k+1}+2\omega f_i^{k+1}+\left[(1-\omega)+\theta\frac{v\Delta t}{\Delta y}\right]f_{i+1}^{k+1}= \\ =\left[(1-\omega)+(1-\theta)\frac{v\Delta t}{\Delta y}\right]f_{i-1}^{k}+2\omega f_i^{k}+\left[(1-\omega)-(1-\theta)\frac{v\Delta t}{\Delta y}\right]f_{i+1}^{k}, \tag{15c}$$

which, for successive nodes, denoted $i = 2,3,4,...N-1$ of the column j also leads to a system of linear algebraic equations with tridiagonal matrix.

As a result of the applied technique the presented algorithm is faster and requires less computational memory as compared with a 2D problem, since two sets of 1D equations with tridiagonal matrixes are solved.

The described solution method has two weighting parameters. The parameter θ defines how the spatial derivatives are centered in time, whereas parameter ω determines the averaging of the variables in space. An analysis of accuracy and stability for the assumed finite element grid enables estimation of the role of both parameters.

The numerical stability analysis has been performed applying the Neumann method (Fletcher 1991). It has been proved that the scheme is always stable when $\theta \geq 1/2$ and $\omega \geq 1/2$ (Bielecka-Kieloch 1998).

3. Accuracy Analysis

Application of the splitting technique leads to the solution of the following sequence of 1D differential equations:

$$\frac{\partial f^{(1)}}{\partial t}+u\frac{\partial f^{(1)}}{\partial x}=0, \tag{16a}$$

with the initial condition $f_t^{(1)} = f_t$ and

$$\frac{\partial f^{(2)}}{\partial t}+v\frac{\partial f^{(2)}}{\partial y}=0, \tag{16b}$$

with the initial condition $f_{t+\Delta t}^{(2)} = f_{t+\Delta t}^{(1)}$.

As is known, the solution of the advection equation using the finite element method modifies the initial equation (1) leading to the following formula (Bielecka-Kieloch 1998):

$$\frac{\partial f}{\partial t}+u_i\frac{\partial f}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\mathbf{D^n}\frac{\partial f}{\partial x_j}\right)+\frac{\partial}{\partial x_i}\left(\mathbf{T^n}\frac{\partial^2 f}{\partial x_j^2}\right)+\ldots=0 \quad i=1,2. \tag{17}$$

Further it has been proved that tensors $\mathbf{D^n}$ and $\mathbf{T^n}$ in equation (17) assume the following form (Bielecka-Kieloch 1998):

$$\mathbf{D}^n=\begin{bmatrix} D_x^n & 0 \\ 0 & D_y^n \end{bmatrix} \tag{17a}$$

and

$$\mathbf{T^n}=\begin{bmatrix} T_x^n & 0 \\ 0 & T_y^n \end{bmatrix}, \tag{17b}$$

where:

D_x^n, D_y^n – numerical diffusion tensors in the x and y directions respectively and

T_x^n, T_y^n – numerical dispersion tensors in the x and y directions respectively.

Let us perform an accuracy analysis for equation (16a). Its discrete form is represented by equation (15a). In this equation the nodal values are replaced by their estimates resulting from Taylor-series expansion around the node „j" at time level $t+\theta\Delta t$. As a result of some transformations the modified initial equation (16a) is obtained (Bielecka-Kieloch 1998):

$$\begin{aligned}\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}&=\left(\theta-\frac{1}{2}\right)\Delta t u^2\frac{\partial^2 f}{\partial x^2}+\\ &+u\Delta x^2\left[\frac{1}{3}+(1-6\theta+6\theta^2)\frac{C_x^2}{6}-\frac{\omega}{2}\right]\frac{\partial^3 f}{\partial x^3}\end{aligned} \tag{18a}$$

The same procedure applies to the initial equation (16b), the discrete form of which is represented by equation (15c), giving as a result the modified equation similar to equation (18a)

$$\begin{aligned}\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial y}&=\left(\theta-\frac{1}{2}\right)\Delta t v^2\frac{\partial^2 f}{\partial y^2}+\\ &+v\Delta y^2\left[\frac{1}{3}+\left(1-6\theta+6\theta^2\right)\frac{C_y^2}{6}-\frac{\omega}{2}\right]\frac{\partial^3 f}{\partial y^3}.\end{aligned} \tag{18b}$$

The above equations allow for formulation of relations:

$$D_x^n = \left(\theta - \frac{1}{2}\right)\Delta t u^2, \qquad D_y^n = \left(\theta - \frac{1}{2}\right)\Delta t v^2,$$
$$T_x^n = u\Delta x^2\left[-\frac{1}{3} - r\frac{C_x^2}{6} + \frac{\omega}{2}\right], \quad T_y^n = v\Delta y^2\left[-\frac{1}{3} - r\frac{C_y^2}{6} + \frac{\omega}{2}\right]. \tag{19}$$

The tensors in equation (17): $\mathbf{T}^n$ and $\mathbf{D}^n$ therefore assume the following form:

$$\mathbf{D}^n = \begin{bmatrix} \left(\theta - \frac{1}{2}\right)\Delta t u^2 & 0 \\ 0 & \left(\theta - \frac{1}{2}\right)\Delta t v^2 \end{bmatrix}, \tag{19a}$$

$$\mathbf{T}^n = \begin{bmatrix} u\Delta x^2\left(\frac{1}{12} + r\frac{C_x^2}{6} - \frac{\alpha}{4}\right) & 0 \\ 0 & v\Delta y^2\left(-\frac{1}{3} - r\frac{C_y^2}{6} + \frac{\omega}{2}\right) \end{bmatrix} \tag{19b}$$

$$r = 1 - 6\theta + 6\theta^2. \tag{19c}$$

In this case, both numerical diffusion, as well as numerical dispersion tensors have elements different from zero only on the main diagonal. This is a result of the application of the directional splitting technique. The method enables avoiding the introduction of mixed derivatives, reducing in this way, the influence of numerical diffusion and dispersion. At the same time it can be seen that positive elements in the numerical diffusion tensor are identical with the corresponding elements of the tensor derived for the modified finite element method (Szymkiewicz, Bielecka-Kieloch 1995). The tensor disappears when $\theta = 1/2$, i.e. when the approximation of the derivatives in time is of the second order. It can also be noticed that the stability condition $\theta \geq 1/2$ assures positive values of tensor's $\mathbf{D}^n$ elements.

The tensor of numerical dispersion also differs from the one derived for the modified finite element method (Szymkiewicz, Bielecka-Kieloch 1995). Also in this case only the tensor's elements on the main diagonal differ from zero, what is the result of the application of the directional splitting technique.

Rotation of the co-ordinate system $x - y$ by an angle ϕ results in transformation of the numerical diffusion and dispersion tensors $\mathbf{D}^n$ and $\mathbf{T}^n$ respectively. In the rotated $l - n$ co-ordinate system for $u = v = w/\sqrt{2}$, $\phi = \pi/4$, $\Delta x = \Delta y = \Delta$, $C_x = C_y = C$ the $\mathbf{D}^n$ tensor assumes the following form:

$$\vec{\mathbf{D}}^n = \begin{bmatrix} \left(\theta - \frac{1}{2}\right)\Delta t \frac{w^2}{2} & 0 \\ 0 & \left(\theta - \frac{1}{2}\right)\Delta t \frac{w^2}{2} \end{bmatrix}. \tag{20a}$$

Since the tensor's components for $\theta \geq 1/2$ have positive values for any Δt, it proves that the scheme is stable, which is consistent with the previous stability analysis.

Assuming additionally $\theta = 1/2$, i.e. the situation without numerical diffusion, the numerical dispersion tensor form in the rotated $l - n$ co-ordinate system is as follows:

$$\vec{\mathbf{T}}^n = \begin{bmatrix} \frac{w\Delta^2}{\sqrt{2}}\left(-\frac{1}{3} + \frac{1}{12}C^2 + \frac{1}{2}\omega\right) & 0 \\ 0 & \frac{w\Delta^2}{\sqrt{2}}\left(-\frac{1}{3} + \frac{1}{12}C^2 + \frac{1}{2}\omega\right) \end{bmatrix}. \tag{20b}$$

Both numerical diffusion and dispersion tensors are symmetrical. The form of $\vec{\mathbf{T}}^n$ tensor proves that it is possible to find such value of the parameter ω, for which each all of the tensor's elements are equal to zero. The described situation takes place when

$$\omega = \frac{2}{3} - \frac{C^2}{6}. \tag{21}$$

In this case the scheme does not introduce any numerical diffusion ($\theta = 1/2$) as well as dispersion. Therefore this is the third order approximation being a great advantage of the presented scheme.

For $\theta > 1/2$ numerical diffusion occurs, but it is symmetrical in both directions. Its magnitude depends on parameter θ, time step and flow velocity. However, it is still possible to determine such value of parameter ω for which numerical dispersion does not occur. In this case it is the following relation:

$$\omega = \frac{2}{3} + \frac{1 - 6\theta + 6\theta^2}{3}C^2. \tag{22}$$

Both condition (21) as well as (22) can be applied for such Courant numbers for which parameter ω must be not less than 1/2, due to stability of the solution. In case of relation (21) this is $C \leq 1$. If $C > 1$ occurs in calculations then $\omega = 1/2$ will not delete elements of tensor $\mathbf{T}^n$, however, their reduction will be significant.

4. Numerical Tests

The presented solution method was applied for calculations of advection of initial concentration in the form of Gaussian distribution of the maximum value $f_{max} = 1.0$, along the basin's diagonal (Fig. 2).

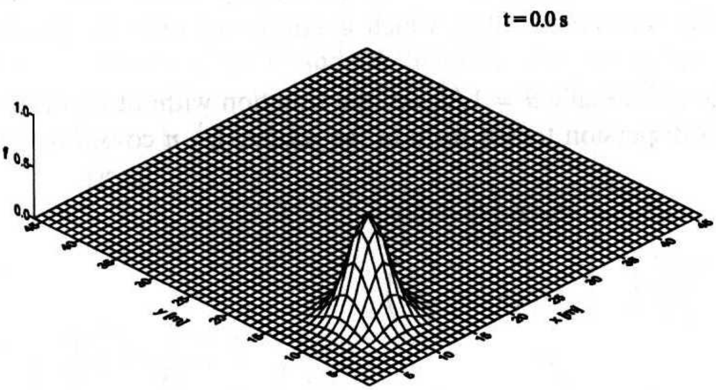

Fig. 2. The initial concentration distribution in case of flow along the basin's diagonal

All calculations were performed for $\theta = 1/2$, assuring elimination of the numerical diffusion. Simultaneously, variability in both directions of the parameter ω, resulting from the dispersion tensor's form (19b), were taken into account:

$$\omega_x = \frac{2}{3} - \frac{C_x^2}{6}, \tag{23a}$$

$$\omega_y = \frac{2}{3} - \frac{C_y^2}{6}. \tag{23b}$$

The calculations proved that the quality of results is exceptionally good and depends on both parameter θ, as well as the Courant number. Solution using the directional splitting technique for $\theta = 1/2$ and $C = 1$ is accurate. After time $t = 200$ s the maximum concentration was equal to the initial one. For greater Courant numbers results are worse and oscillations occur. Decreasing C for $\theta = 1/2$ does not practically influence the quality of the results and the obtained solution is very accurate, without any oscillations or deformations (Tab. 1).

For instance for $C = 0.1$ the solution is very accurate with unnoticeable oscillations (Fig. 3).

Values of θ less than 1/2 make the scheme unstable. However, for $\theta > 1/2$ the quality of the results significantly decreases with the increasing Courant number

Table 1. Extreme concentrations calculated after 200 s of advection of the initial concentration distribution along basin's diagonal

	$\theta = 0.5$		$\theta = 0.75$			$\theta = 1.0$	
C	0.1	0.5	0.1	0.5	1.0	0.1	1.0
$f_{\min}$	−0.001	0.000	0.000	0.000	0.000	0.000	0.000
$f_{\max}$	0.988	0.993	0.797	0.446	0.290	0.667	0.170

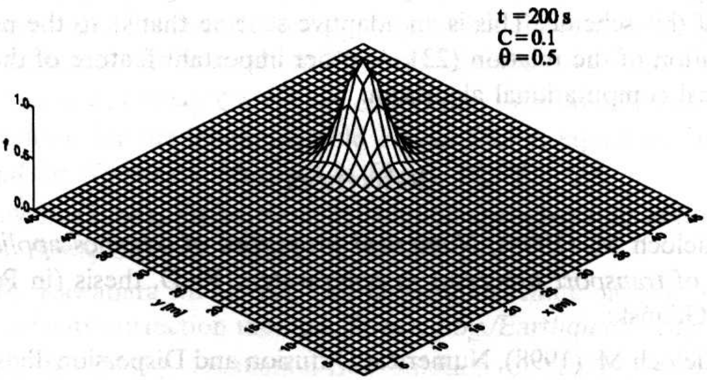

Fig. 3. Advection of the initial concentration distribution along the basin's diagonal for $\theta = 0.5$ and $C = 0.1$

(Tab. 1). As can be noticed for greater Courant numbers, strong numerical diffusion is generated, but oscillations resulting from numerical dispersion do not occur. Simultaneously, it should be mentioned that in the analyzed cases, distributions are axially symmetrical.

The presented scheme affords a highly accurate solution of pure advection or advection-diffusion equation with dominant advection transport when $\theta = 1/2$ is assumed, Courant numbers are less than one and ω is determined according to (23a, b). In other cases either an unstable solution is obtained or very strong numerical diffusion occurs. The presented numerical tests fully confirmed the conclusions resulting from stability and accuracy analysis of the proposed method.

5. Conclusions

In this paper a new method of solving linear advection equation has been presented. It was based on the modified linear finite element method where the integration procedure was generalized by introduction of weighting parameters and the directional splitting technique was implemented to the previously modified finite element solution. The method was analyzed by means of the derived numerical diffusion and dispersion tensors.

Both accuracy analysis, as well as numerical tests showed that the proposed method offers excellent accuracy of the solution. It is possible here not only to eliminate numerical diffusion, but also numerical dispersion by proper selection of two weighting parameters θ and ω. It was proved that for $\theta = 1/2$, ω determined according to the formula (23) and for Courant number not greater than one, the scheme does not generate any numerical diffusion or dispersion. It assures an approximation of the IIIrd order and produces a numerical solution very close to the analytical one. Ability of continuos adjustment of the weighting parameter ω depending on variations of Courant number during calculations, is a great advantage of this scheme. This is an adaptive scheme thanks to the possibility of local application of the relation (23). Another important feature of the scheme is its economical computational algorithm.

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