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# Empirical Method for Studying the Development of Plastic Strains before Failure of Sand

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## Abstract

An empirical method for studying the development of plastic strains before failure of sand is proposed. The starting point of the analysis constitutes experimental results of triaxial compression tests with local measurement of both vertical and lateral strains. The plastic strains are extracted from the total ones, assuming that the unloading is purely elastic. Experimental results are then approximated by analytical formulae which are very convenient for theoretical analyses. The results presented were obtained for specific loading paths which enable studying of the coupling between the volumetric and deviatoric effects.

## 1. Introduction

Determination and description of plastic strains developed during the loading of granular materials is one of the key issues in soil mechanics. There exist many contributions dealing with this issue, dispersed in various professional journals and conference proceedings, but it seems that the problem is far from being solved. Despite the large number of theoretical models attempting to describe the behaviour of soils, a general theory of granular materials has not yet emerged. Some of the existing models describe only chosen features of the soil behaviour, some being simply just "curve fitting" exercises. There seems to be a gap between theoretical and experimental soil mechanics, and the need exists to collate the most important findings of these specializations in order to elaborate a practical and useful theory of granular materials, which also includes a description of the pre-failure behaviour of the materials. The importance of this issue has recently been recognized, some conferences also having been devoted to the problem, see Jamiolkowski et al. (1999).

Useful models of soil behaviour should be based on empirical sound results, hence we have initiated an experimental programme aiming to determine the stress-strain characteristics of sands before failure. The experiments are performed in a computer controlled hydraulic triaxial testing system from GDS Instruments Ltd., see Menzies (1988) and Świdziński (2000). The system has been additionally equipped with special gauges, utilising the Hall effect, enabling the local measurement of both lateral and vertical strains. The importance of this type of measurement is increasingly appreciated in soil mechanics, as it affords more realistic results, reducing the influence of such experimental errors, as bedding errors or inaccuracies connected with the external measurement of strains, see Tatsuoka et al. (1995). The experimental technique is described in the paper by Sawicki and Świdziński (2002). It should always be remembered that accurate measurement of strains is necessary, in order to determine soil parameters sensible from the quantitative point of view.

Some previous analyses show that the elastic response of investigated sand specimens is anisotropic. Both qualitative and quantitative analyses of this question are presented in Sawicki and Świdziński (2001). Having known the elastic behaviour of sand, one can extract the plastic strains from the total, in order to obtain a picture illustrating the development of plastic deformations for various loading histories. In this paper, the results of two typical experiments performed on the loose and dense specimens, are discussed in detail. Experimental results are presented by simple analytical formulae with good accuracy. Such a presentation is very convenient for such theoretical analyses as, for example, studying the coupling of volumetric and deviatoric effects.

The practically important aspect of this paper is an empirical method which enables studying of the development of plastic strains in sand in stages preceding failure in a systematic manner. Recall that the strains were measured by local gauges, hence a fairly realistic picture of the soil deformation is presented. The results shown may be used for validation of existing models soils, and will serve as a basis for further experimental and theoretical studies. In practical applications of various complex elasto-plastic models of soils, their parameters are often collated from many different sources, e.g. Dłużewski and Hrabowski (1998). The aim of our research programme is to obtain fairly extensive sets of experimental data, from which realistic values of such parameters could have been obtained.

## 2. Notation and Some Assumptions

During the experiments, the following basic quantities were measured: $\sigma_1$ = vertical stress, $\sigma_3$ = horizontal stress, $\varepsilon_1$ = total vertical stress, $\varepsilon_3$ = total horizontal stress. Previous analyses show that soil behaviour during the unloading may be treated as linear elastic during the first stage of unloading. Therefore, it is assumed that the total strain can be decomposed into its elastic and plastic parts, i.e.

$$\varepsilon_1 = \varepsilon_1^{el} + \varepsilon_1^{pl} , \tag{1}$$

$$\varepsilon_3 = \varepsilon_3^{el} + \varepsilon_3^{pl}. \tag{2}$$

For future considerations, it is convenient to introduce the following invariants, after Schofield and Wroth (1968):

$$p = \frac{1}{3}\left(\sigma_1 + 2\sigma_3\right), \tag{3}$$

$$q = \sigma_1 - \sigma_3, \tag{4}$$

$$\varepsilon_v = \varepsilon_1 + 2\varepsilon_3, \tag{5}$$

$$\varepsilon_q = \frac{2}{3}\left(\varepsilon_1 - \varepsilon_3\right). \tag{6}$$

Eqs. (5) and (6) are also valid for the elastic and plastic parts of the strain tensor. The soil mechanics sign convention is used, which means that compression is considered as positive.

It is assumed that the elastic response of sand is known and can be described by the following relation:

$$\left\{ \begin{array}{c} \varepsilon_v^{el} \\ \varepsilon_q^{el} \end{array} \right\} = \left[ \begin{array}{cc} M_{vv} & M_{vq} \\ M_{qv} & M_{qq} \end{array} \right] \left\{ \begin{array}{c} p \\ q \end{array} \right\}. \tag{7}$$

The elastic compliances $M_{ij}$, $i, j = v, q$, were determined for loose and dense sands, see Sawicki and Świdziński (2002). Eq. (7) means that, in general, there is a coupling of elastic volumetric and deviatoric effects. In the case of elastic isotropy, we have $M_{vq} = M_{qv} = 0$. The principal strains can be determined from Eq. (7) using the following formula:

$$\left\{ \begin{array}{c} \varepsilon_1^{el} \\ \varepsilon_3^{el} \end{array} \right\} = \left[ \begin{array}{cc} \frac{1}{3} & 1 \\ \frac{1}{3} & -\frac{1}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_v^{el} \\ \varepsilon_q^{el} \end{array} \right\}. \tag{8}$$

For the sake of convenience, the following units will be applied in this paper: stress unit: $10^5$ N/m$^2$, strain unit: $10^{-3}$. Therefore, the elastic compliance unit is $10^{-8}$ m$^2$/N, etc. Fig. 1 shows a specific loading path applied in the majority of experiments performed. The specimens were initially pre-consolidated under hydrostatic pressure, up to point A, then two types of loading path were applied. Along path pq, the sample was hydrostatically loaded to point B, then unloaded to point A, and subsequently loaded by the stress deviator to point C, then unloaded to point A. The second loading programme (qp) corresponds to path OACABA. Note that along path AB only the mean stress p changes and the stress deviator q = 0. Along path AC, the mean stress p = const, and the stress deviator increases from zero at point A to its maximum value at C. Such loading paths enable the studying of the deformation effects caused by p and q separately.

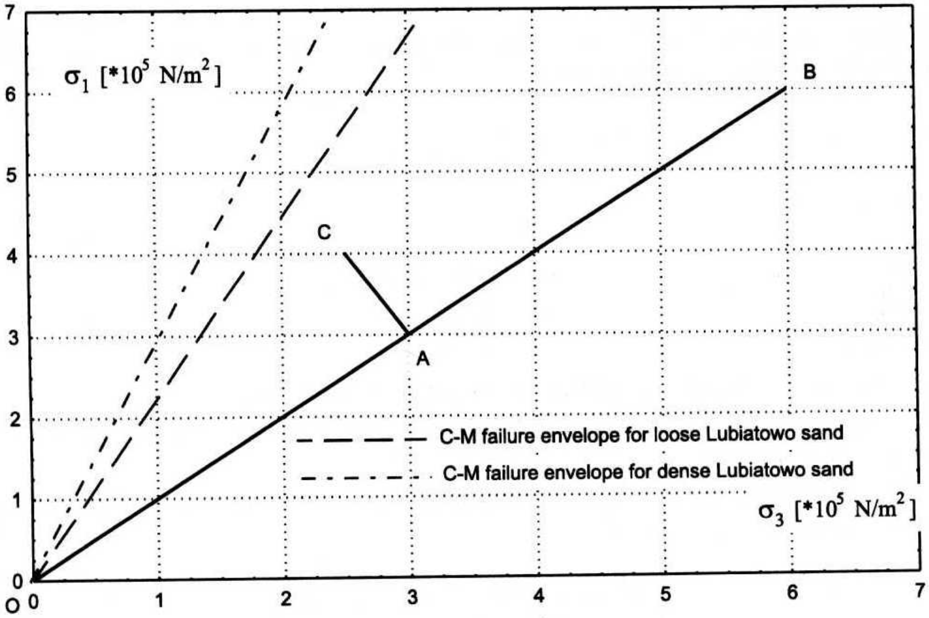

**Fig. 1.** Stress paths

Some comments as to loading and unloading are necessary. In this paper, the process of loading is understood intuitively. When the mean stress p increases along path AB, it means loading. When it decreases (path BA), it is unloading. Similarly, when q increases along path AC it is deviatoric loading, and unloading takes place along CA when q decreases. Note that the vertical stress $\sigma_1$ increases along path AC, but the horizontal stress $\sigma_3$ simultaneously decreases. In the plasticity theory, the yield surface should be defined first, in order to decide whether subsequent stress increments cause loading or unloading. In the present stage of analysis, a simple intuitive definition of loading and unloading is sufficient.

Point C was chosen in such a way, that it is distant from the Coulomb-Mohr failure envelope, defined as:

$$\sigma_1 = \frac{1+\sin\phi}{1-\sin\phi}\sigma_3, \tag{9}$$

where $\phi$ = angle of internal friction. $\phi = 22.3°$ for loose "Lubiatowo" sand and $\phi = 29.02°$ for dense one.

## 3. Behaviour of Loose Sand

In this Section, the behaviour of loose "Lubiatowo" sand ($\rho$ = 1.549 g/cm$^3$, $I_D$ = 0.12) will be described in detail. The loading path was of the pq type (path OABACA). Figs 2 and 3 show the experimental records of the specimen's strains. Linear approximation of the data from Fig. 2 is presented in Fig. 4.

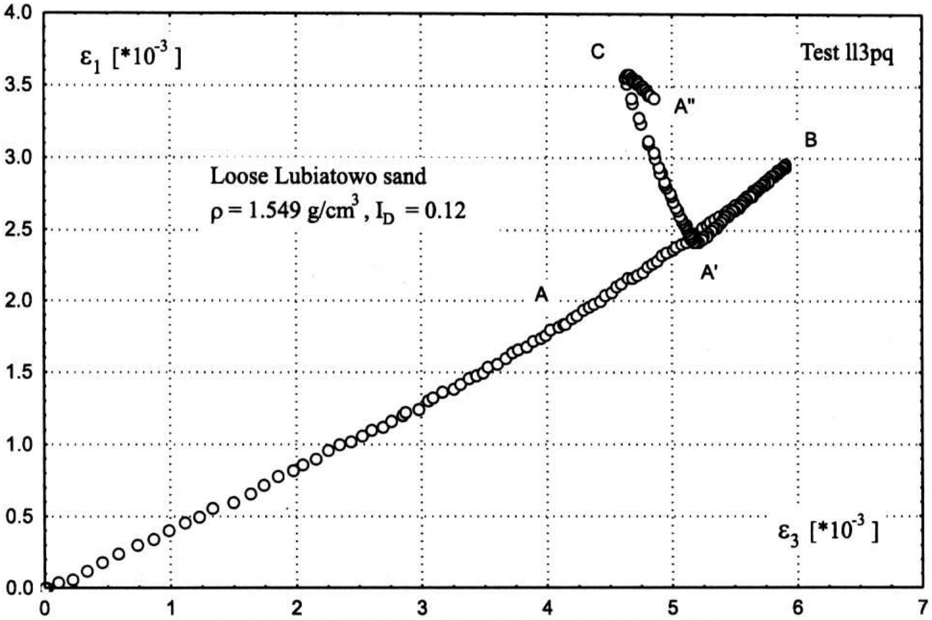

**Fig. 2.** Vertical against horizontal total strains. Loose sand – experimental record

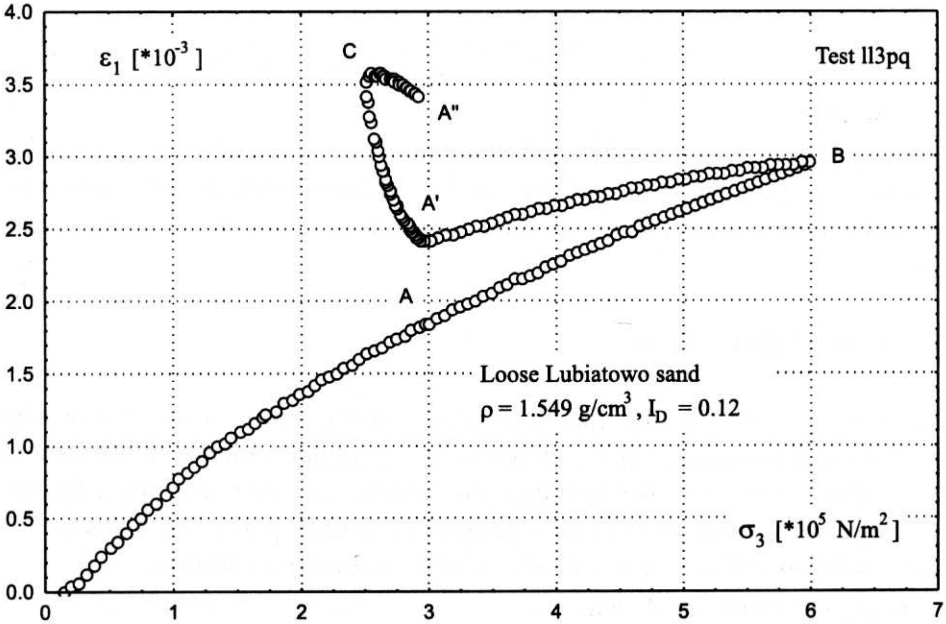

**Fig. 3.** Vertical strain against horizontal stress. Loose sand – experimental record

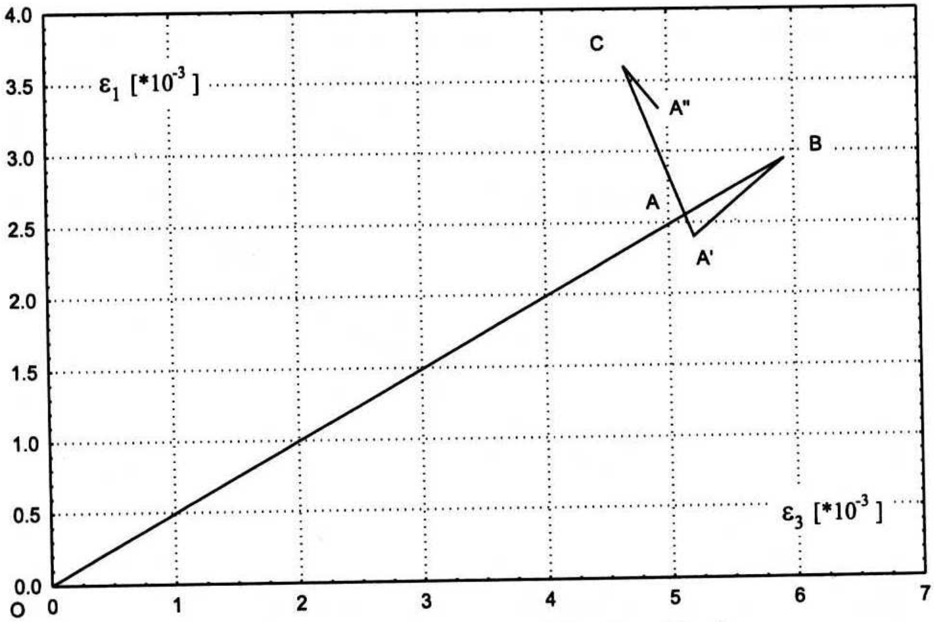

**Fig. 4.** Linear approximation of data from Fig. 2

The matrix of elastic compliances, determined from unloading paths BA and CA, has the following form:

$$\mathrm{M} = \begin{bmatrix} 0.67 & -0.173 \\ -0.04 & 0.258 \end{bmatrix} \times 10^{-8} \quad \mathrm{m}^2/\mathrm{N}, \tag{10}$$

This matrix is obviously slightly different from the matrix of average compliances of loose sand presented in Sawicki and Świdziński (2002), as it corresponds to a particular experiment. Let us consider subsequent stages of the stress-strain history.

### Virgin consolidation OAB

The behaviour of granular media depends on the stress-strain history which is generally unknown in natural conditions. It is common practice in soil mechanics, that during laboratory investigations, the sample is subjected to some hydrostatic pre-consolidation and then to other prescribed loading paths, see Atkinson (1993). Such virgin consolidation is a kind of simple stress-strain history.

It follows from Fig. 4 that:

$$\varepsilon_1 = a\varepsilon_3 = 0.4973\varepsilon_3 \cong 0.5\varepsilon_3. \tag{11}$$

The virgin loading $\varepsilon_1 - \sigma_3$ curve from Fig. 3, where $\sigma_3 = p$ in this case, can be approximated by the following relation:

$$p = b\varepsilon_3^2 = 0.17\varepsilon_3^2, \tag{12}$$

or

$$\varepsilon_3 = 2.43\sqrt{p}. \tag{13}$$

The volumetric and deviatoric strains from Eqs. (5) and (6) are the following:

$$\varepsilon_v = 6.08\sqrt{p} \quad \text{and} \quad \varepsilon_q = -0.81\sqrt{p}. \tag{14}$$

Elastic strains can be determined from Eqs. (7) and (10):

$$\varepsilon_v^{el} = 0.67p \quad \text{and} \quad \varepsilon_q^{el} = -0.04p. \tag{15}$$

Therefore, the plastic strains developed along path OAB can be expressed by the following expressions:

$$\begin{aligned} \varepsilon_v^{pl} &= 6.08\sqrt{p} - 0.67p, \\ \varepsilon_q^{pl} &= -0.81\sqrt{p} + 0.04p. \end{aligned} \tag{16}$$

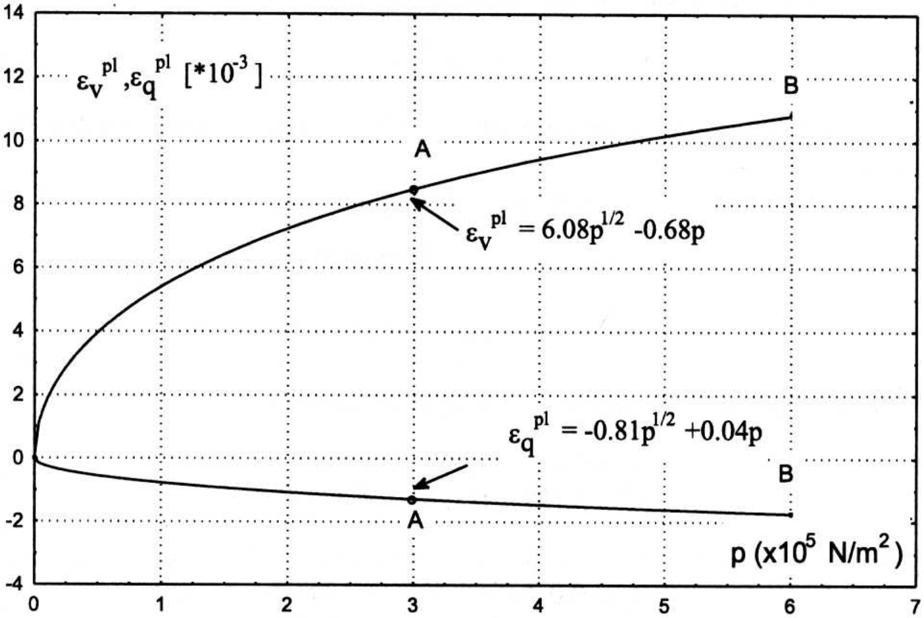

**Fig. 5.** Development of plastic strains during virgin compression of loose sand

The above result is illustrated in Fig. 5. An interesting feature of the behaviour observed is that deviatoric strains develop during hydrostatic consolidation. There is $\varepsilon_q^{pl}/\varepsilon_v^{pl} = -0.16$ for $p = 6$. These strains are probably caused by slight anisotropy of sand, or perhaps they are due to some experimental errors. The resolution of this problem is beyond the scope of this paper, as we do not have better experimental facilities. It should be mentioned that we have been perfecting our experimental work on the GDS apparatus for more than two years, and therefore believe that the results obtained are correct, see Świdziński and Mierczyński (2002).

An other interesting observation is that plastic volumetric strains dominate during virgin compression. For example, $\varepsilon_v^{pl}$ is 73% of the total volumetric strain for $p = 6$, whilst the elastic one is 27%.

### First unloading BA

During the first unloading only some elastic strains are recovered according to Eqs. (15). The plastic strains remain unchanged along path BA.

### Deviatoric loading AC

During the deviatoric loading along path AC, the stress deviator increases from 0 to 1.5, whilst the mean stress $p = 3 =$ const. The respective strain path from Fig. 4 can be approximated by the following formula:

$$\varepsilon_3 = -0.458\varepsilon_1 + 6.3. \tag{17}$$

The $\varepsilon_1 - \sigma_3$ relationship along path AC from Fig. 3 can be approximated by the following equation:

$$\varepsilon_1 = 6.54\sigma_3^2 - 38.35\sigma_3 + 58.62, \tag{18}$$

where $\sigma_3 \in < 2.5,\ 3 >$, and $\sigma_1 = 9 - 2\sigma_3$. There is also:

$$\sigma_3 = 3 - \frac{1}{3}q. \tag{19}$$

The above equations lead to the following expressions:

$$\begin{aligned} \varepsilon_1 &= 2.43 - 0.297q + 0.727q^2, \\ \varepsilon_3 &= 5.187 + 0.136q - 0.333q^2, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \varepsilon_v &= 0.061q^2 - 0.025q + 12.804, \\ \varepsilon_q &= 0.707q^2 - 0.289q - 1.838. \end{aligned} \tag{21}$$

The elastic strains are the following along this path:

$$\begin{aligned} \varepsilon_v^{el} &= -0.173q + 2.01, \\ \varepsilon_q^{el} &= -0.258q - 0.12. \end{aligned} \tag{22}$$

There are some round-off errors in the above expression, which however, do not essentially influence the results. The plastic strains are:

$$\begin{aligned} \varepsilon_v^{pl} &= 0.061q^2 + 0.148q + 10.8, \\ \varepsilon_q^{pl} &= 0.707q^2 - 0.547q - 1.72. \end{aligned} \tag{23}$$

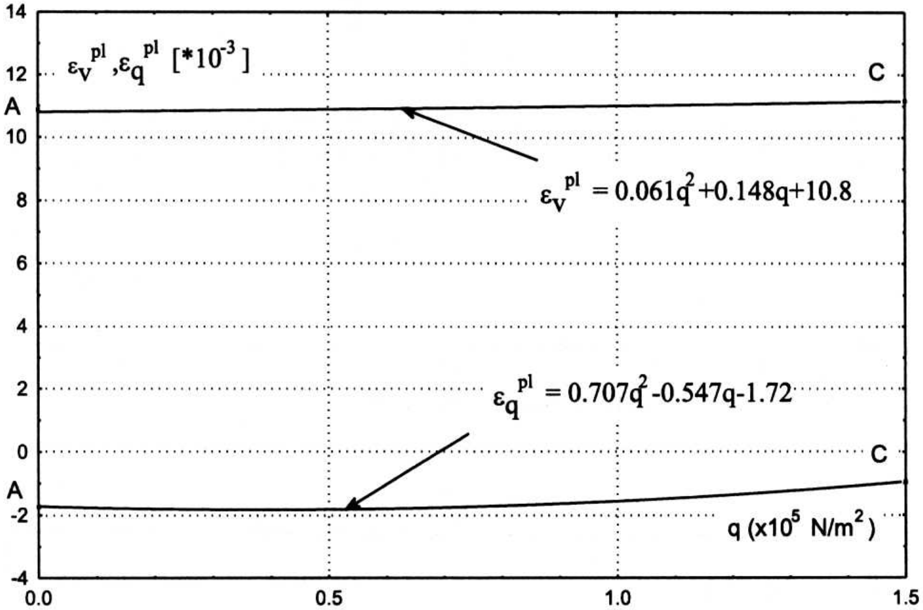

**Fig. 6.** Development of plastic strains during deviatoric loading. Loose sand

Development of plastic strains during the deviatoric loading at constant $p$ is illustrated in Fig. 6. An increase of both the volumetric and deviatoric strains is visible. The absolute values of these increases of strains are much smaller than those developed during the virgin compression of the specimen.

## 4. Behaviour of Dense Sand

The second example deals with behaviour of the dense "Lubiatowo" sand specimen ($\rho = 1.692$ g/cm$^3$, $I_D = 0.70$). The loading path was of the qp type (OACABA). Fig. 7 shows the linearized strain path followed during this loading programme drawn from the experimental record with good accuracy. Fig. 8 shows the respective $\varepsilon_1 - \sigma_3$ plot also drawn from the experimental record. The behaviours shown in these figures are linear, in contrast to the previously described non-linear response of loose sand.

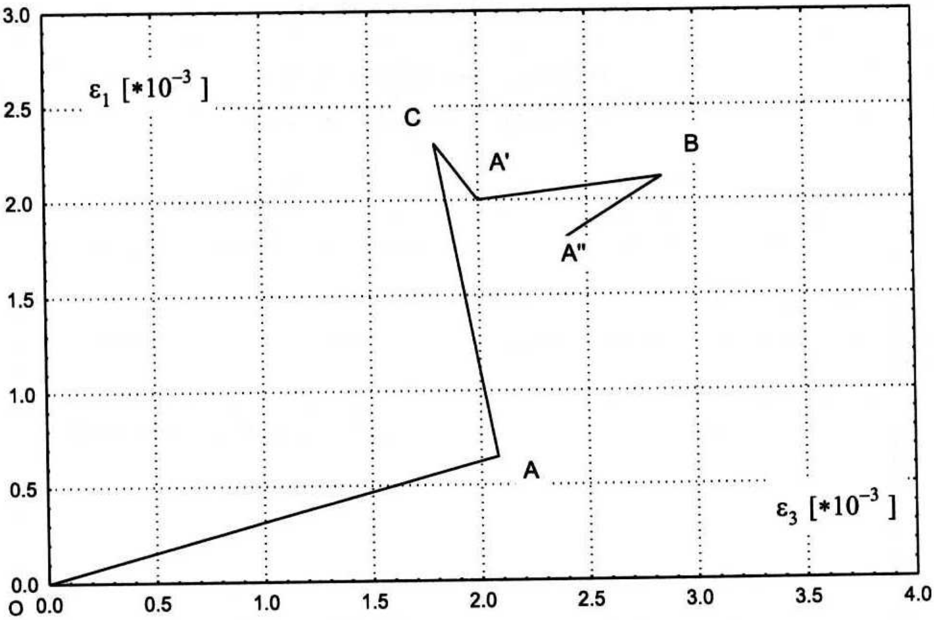

**Fig. 7.** Sstrain path for dense sand

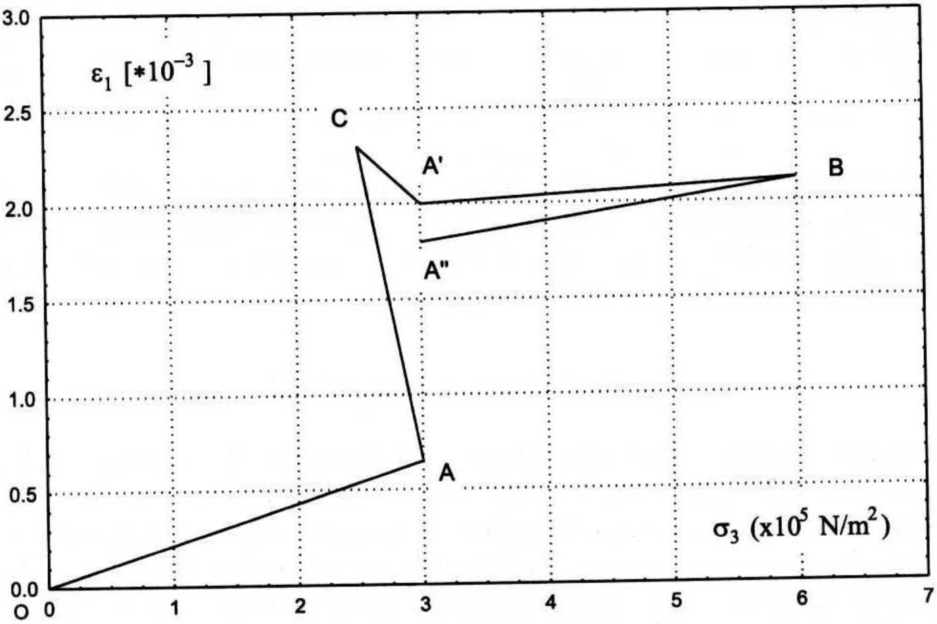

**Fig. 8.** $\varepsilon_1 - \sigma_3$ relation. Dense sand

The matrix of elastic compliances is the following in this case:

$$\mathrm{M} = \begin{bmatrix} 0.407 & -0.067 \\ -0.029 & 0.222 \end{bmatrix} \times 10^{-8} \ \mathrm{m}^2/\mathrm{N}, \tag{24}$$

It also differs slightly from the matrix of average elastic compliances presented in Sawicki and Świdziński (2002). Let us consider the subsequent stages of loading path as in the previous case.

**Virgin compression OA**

The virgin hydrostatic compression OA is linear in this case, in contrast to the behaviour of loose sand, described in the previous Section. The relation between the global strains is the following in this case:

$$\varepsilon_1 = 0.3125\varepsilon_3 \ \text{ or } \ \varepsilon_3 = 3.2\varepsilon_1. \tag{25}$$

There is also:

$$\varepsilon_1 = 0.217p \ \text{ and } \ \varepsilon_3 = 0.693p, \tag{26}$$

$$\varepsilon_v = 1.603p \ \text{ and } \ \varepsilon_q = -0.317p. \tag{27}$$

The elastic strains can be determined from Eqs. (7) and (22):

$$\varepsilon_v^{el} = 0.407p \ \text{ and } \ \varepsilon_q^{el} = -0.029p, \tag{28}$$

Therefore, the plastic strains are:

$$\varepsilon_v^{pl} = 1.196p \ \text{ and } \ \varepsilon_q^{pl} = -0.288p. \tag{29}$$

**Deviatoric loading AC**

The strain path AC from Fig. 7 can be approximated by the following formula:

$$\varepsilon_1 = -5.893\varepsilon_3 + 12.907. \tag{30}$$

Simple manipulations lead to the following empirical expressions:

$$\varepsilon_v = 0.726q + 4.81, \ \ \varepsilon_q = 0.858q - 0.953, \tag{31}$$

$$\varepsilon_v^{el} = -0.067q + 1.221, \ \ \varepsilon_q^{el} = 0.222q - 0.087, \tag{32}$$

$$\varepsilon_v^{pl} = 0.793q + 3.59, \ \ \varepsilon_q^{pl} = 0.636q - 0.866. \tag{33}$$

## Elastic unloading CA

Some elastic strains are recovered along this path, and the following plastic strains remain in the specimen: $\varepsilon_v^{pl} = 4.78$ and $\varepsilon_q^{pl} = 0.088$.

## Hydrostatic consolidation AB

The already known procedure affords the following results:

$$\varepsilon_v = 0.608p + 4.186, \quad \varepsilon_q = -0.163p + 0.485, \tag{34}$$

$$\varepsilon_v^{el} = 0.407p, \quad \varepsilon_q^{el} = -0.029p, \tag{35}$$

$$\varepsilon_v^{pl} = 0.201p + 4.186, \quad \varepsilon_q^{pl} = -0.134p + 0.485. \tag{36}$$

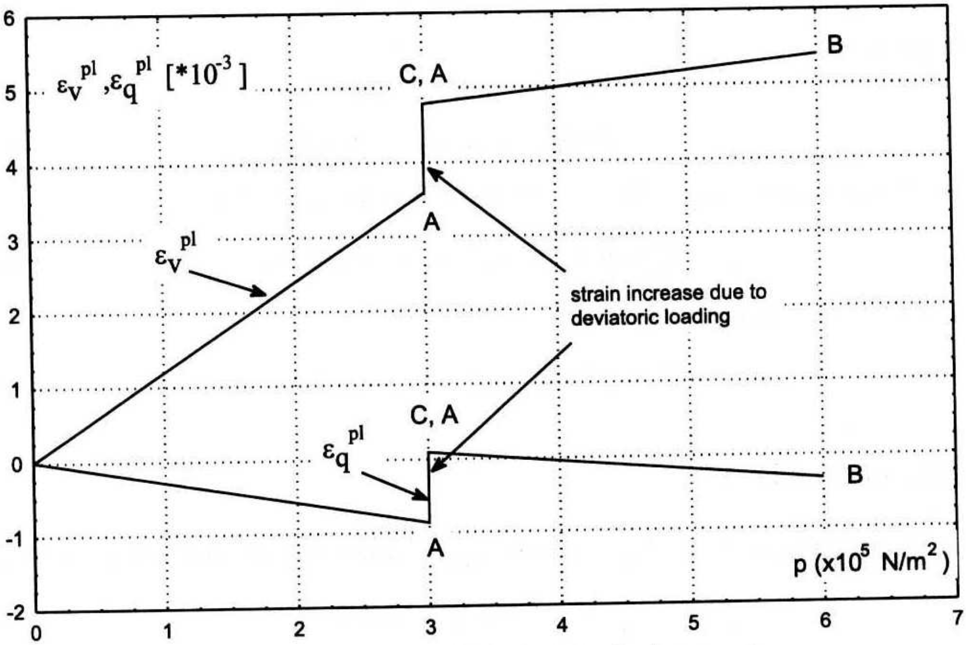

**Fig. 9.** Development of plastic strains in dense sand

The development of plastic strains during the loading history OACAB is shown in Fig. 9. During the virgin compression OA the volumetric plastic strain increases up to the value of $3.59 \times 10^{-3}$. Note that in the previous case of loose sand the respective value was $\varepsilon_v^{pl} = 8.5 \times 10^{-3}$ (see Fig. 5 or Eq. 16). This is a correct result from the qualitative point of view. The behaviour of loose sand was non-linear, whilst there was practically linear behaviour of the dense sand. The deviatoric loading (path AC) causes a further decrease in volume. Subsequent hydrostatic compression (AB) gives additional compaction, but the sand has become stiffer due to the preceding loading history.

The deviatoric plastic strains $\varepsilon_q^{pl}$ are much smaller than the volumetric ones during hydrostatic compression, but they are similar during deviatoric loading at $p$ = const. Note that in deviatoric loading the sign of these strains changes.

## 5. Discussion

a) The experimental results presented in this paper deal with two particular experiments performed on loose and dense specimens of "Lubiatowo" sand. A method of dealing with experimental data is described in detail. From the quantitative point of view, the results discussed are representative for a greater number of experiments, but still we have insufficient data to perform any statistical analysis which would lead to more general quantitative conclusions.

b) The plastic strains were determined under the assumption as to the elastic response of sand during unloading. This problem was discussed extensively in the companion paper of Sawicki and Świdziński (2002), and certainly needs further investigations, both theoretical and experimental. As the global strains were determined from experiments, it is possible to use these data for testing other hypotheses. The strains were measured locally by special modern transducers, hence it is believed that the experimental data are realistic.

c) For example, let us consider one aspect of the behaviour of dense sand, dealing with directions of plastic strain increments, which is important in the analysis of plastic potential. The plastic strains developed during particular loading stages are given by simple formulae (29), (33) and (36) from which one can determine the directions of plastic strain increments. The ratios of $d\varepsilon_v^{pl}/d\varepsilon_q^{pl}$ are the following: $-4.153$, $1.25$ and $-1.5$ for sectors OA, AC and AB respectively, cf. Fig. 1. This result is illustrated in Fig. 10, showing coupling between the volumetric and deviatoric strains. In the case of loose sand, a similar picture is more complex because the equations describing the development of plastic strains are non-linear, cf. Eq. (16), etc.

d) The results obtained are consistent with general soil mechanics knowledge, i.e. dense sand is stiffer than loose, dissipation of energy during plastic flow is positive, etc.

## 6. Conclusions

An original empirical method for studying the development of plastic strains in sands has been proposed. Experimental results are presented by simple analytical

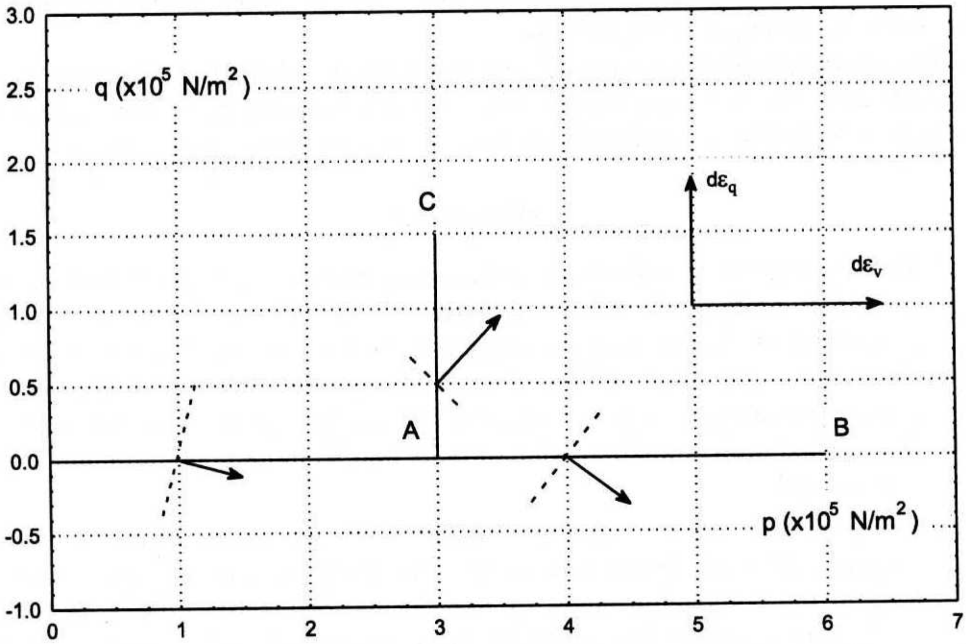

**Fig. 10.** Directions of plastic strain increments during different stages of loading. Dense sand

formulae which may be used for testing various theories and hypotheses. This paper is of a purely experimental character, and therefore theoretical analysis of these results is beyond its scope. An important aspect of the paper is that the strains were determined using local strain gauges, which leads to more realistic results than those obtained from classical external measurement. Such a realistic determination of plastic strains is important for more precise determination of soil parameters. The experimental programme will be continued.

## Acknowledgement

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