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Application of Cnoidal Wave Theory in Modelling of Sediment Transport

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Abstract

A way of modelling of the net sediment transport rate under shallow water asymmetric waves is presented. In the model, wave-induced nearbed horizontal velocity is described by the cnoidal theory and the 2nd order Stokes approximation. Theoretical results of the model are compared with the IBW PAN laboratory data

1. Introduction

In the domain of coastal engineering, most problems are associated with the changes of sea bed profile. The investigations of this evolution are partly related to search of the origin of bars and the theoretical description of their migration. Ultimately, the knowledge on the sea bed dynamics leads to proper prediction of change in shoreline position. In general, the sea bed changes are theoretically modelled by means of spatial variability of net sediment transport rates. Sea bed evolution models are very sensitive to the qualitative and quantitative variability of net sediment transport. Thus the need of its precise determination.

Due to practical reasons, the phenomenon of sediment transport in coastal zones is conventionally distinguished as longshore and cross-shore transport. It can be supposed that the longshore sediment transport rates do not depend much on details of the wave profile. Contrary to it, the features of the cross-shore sediment transport are very likely to result from the nearbed interaction processes between asymmetric wave motion and wave-induced return flows. In the recent years, a number of thorough studies on the cross-shore sediment transport and sea bed profile evolution were carried out, e.g. by Brøker Hedegaard et al. (1991), O'Connor et al. (1992), Larson & Kraus (1995) and Rakha et al. (1997). In the latter, a phase-resolving wave model was engaged to include the effect of wave conditions changing across the entire coastal profile. The net sediment transport resulted from the interaction between the undertow and the wave Lagrangian drift. It was concluded – inter alia – that the onshore sediment transport rates were underestimated seawards of the bar. This could imply that the wave asymmetry effects in the model were dominated by the undertow.

In the study by Kaczmarek & Ostrowski (1996), using the water-soil mixture approach, an attempt was made to determine the bedload transport under asymmetric waves, represented by 2^{nd} and 3^{rd} Stokes theories. In that paper, however, wave asymmetry effects were considered in the context of their influence on the moveable bed roughness and on the bedload transport averaged over half wave period, not on the net sediment transport rate. The basis of the phase-resolving sediment transport model were formulated, allowing for the implementation of an arbitrary wave free stream velocity as the input.

It is well known that the Stokes theories can be applied in a limited range of wave parameters. Close to the shore, at small water depth, the Stokes approximations are not valid. In this area, the application of the cnoidal wave theory is recommended.

The present study is aimed at modelling of the net sediment transport rate under shallow water asymmetric waves, described by the cnoidal theory. The respective free stream velocity is used in the momentum integral model of the bed shear stress by Fredsøe (1984). From the shear stress distribution in the wave period, instantaneous sediment transport rates are calculated by the water-soil mixture model of Kaczmarek & Ostrowski (1998), yielding net transport in the direction of wave propagation.

2. Wave-induced Nearbed Velocity

The ultimate effect of the nearbed interaction between asymmetric wave motion and wave-induced steady flow (e.g. undertow) is implied by the wave shape, in particular by the shape of the wave free stream velocity.

The shape of waves depends on their parameters, namely height (H) and period (T), as well as on water depth (h). The wave length (L) results from the above parameters, for dispersive waves – on the basis of the dispersion relationship. There are a few quantities which help to choose the appropriate wave theory to be applied in the description of the wave motion. The wave regime is conventionally determined by use of the ratio L/h and Ursell parameter $U = H/h(L/h)^2$. An Ursell-like parameter $F_c = (H/h)^{1/2}(T\sqrt{g/h})^{5/2}$ is the other quantity assisting the evaluation of the application range of the various wave theories. According to Massel (1996), the values of $F_c = 10$ and $F_c = 500$ stand for the deep/transitional and transitional/shallow water waves limits, respectively.

Following conventional classifications, e.g. by Massel (1989), one can assume the rough limit of $L/h \approx 10$ as the interaction between short and long waves. The wave theories stemming from Stokes approximations can be used for $L/h < 10$, while the theoretical approaches of long waves, namely cnoidal theories, should be used for $L/h > 10$. According to Fenton (1979), the above intersection lies about $L/h = 8$. However, as deducted by Fenton (1979), for lower waves there is significant overlap between the areas of validity of Stokes and cnoidal theories. For instance, a wave with $H/h = 0.2$ can be solved using either Stokes or cnoidal approximation for L/h lying between 5 and 12. The above wave conditions yield the value of Ursell parameter U lying in the range from 5 to 28.8. This allows for an arbitrary choice between the two theoretical approaches. However, although the solutions for some wave parameters can be almost identical using the cnoidal and Stokes theories, the solutions for wave-induced flow velocity can differ much. This will be visible in the following computational results.

The cnoidal wave theories are rather unpopular among coastal and ocean engineers due to elliptic functions involved, inconvenient in use for practical purposes. In order to simplify the calculations, the approximations have been introduced which, however, still require some iterative procedures, e.g. to find the unknown modulus (k) of the elliptic integrals, and the elliptic integrals themselves.

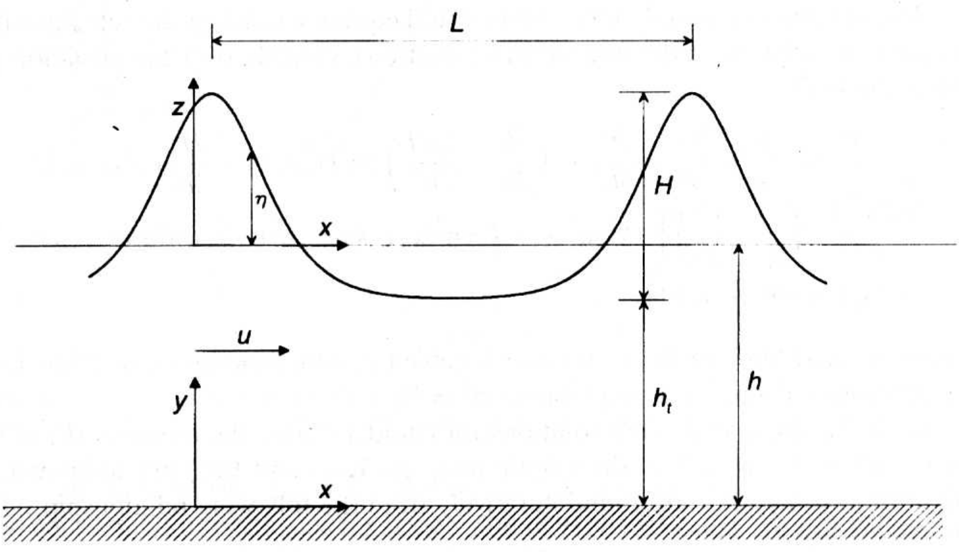

Fig. 1. Definition sketch for wave theories

Sobey et al. (1987) cited the following formulas for the free surface elevation η and horizontal velocity u (see Fig. 1 for notation):

$$\eta(x,t) = h_t + H\mathrm{cn}^2(x,t,k), \tag{1}$$

$$u(x,t) = \bar{u} + (gh_t)^{1/2}\left[-1 - \frac{H}{h_t\mathrm{cn}^2(x,t,k)}\left(-0.5 + k^2 - k^2\mathrm{cn}^2(x,t,k)\right)\right], \tag{2}$$

in which

$$h_t = h\left\{1 + \frac{H}{k^2 h}\left[1 - k^2 - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]\right\}, \tag{3}$$

$$\bar{u} = (gh_t)^{1/2}\left\{1 + \frac{H}{k^2 h_t}\left[0.5 - \frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right]\right\}, \tag{4}$$

$$\mathrm{cn}^2(x,t,k) = \mathrm{cn}^2\left[2\mathbf{K}(k)\left(\frac{x}{L} - \frac{t}{T}\right); k\right]. \tag{5}$$

In the above expressions, $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the complete elliptic integrals of the first and second kind, respectively, with modulus k. The function 'cn' is the Jacobian elliptic cosine. The function 'cn' is singly periodic provided k is real number and $0 \le k < 1$. The period becomes infinite when $k = 1$ (in which case we have the solitary wave). For $k = 0$ the wave is sinusoidal.

It is worthwhile pointing out that the flow velocity described by Eq. (2) is independent on the distance above the bed. A question arises whether the above simplified approach can be applied in modelling of sediment transport, very sensitive to distributions of nearbed velocities.

Wiegel (1960) provided more sophisticated equation yielding the velocity u (for the same description of the free surface elevation), variable over the elevation (y) above the bed:

$$\begin{aligned}\frac{u(x,y,t)}{(gh)^{1/2}} &= -\frac{5}{4} + \frac{3h_t}{2h} - \frac{h_t^2}{4h^2} + \left(\frac{3h}{2h} - \frac{h_t H}{2h^2}\right)\mathrm{cn}^2(x,t,k) - \frac{H^2}{4h^2}\mathrm{cn}^4(x,t,k)\\ &-\frac{8H\mathbf{K}^2(k)}{L^2}\left(\frac{h}{3} - \frac{y^2}{2h}\right)\left[-k^2\mathrm{sn}^2(x,t,k)\mathrm{cn}^2(x,t,k) + \mathrm{cn}^2(x,t,k)\mathrm{dn}^2(x,t,k)\right.\\ &\left.-\mathrm{sn}^2(x,t,k)\mathrm{dn}^2(x,t,k)\right].\end{aligned} \tag{6}$$

where 'sn' and 'dn' are the other two Jacobian elliptic functions (available from the relations $\mathrm{sn}^2+\mathrm{cn}^2 = 1$ and $k^2\mathrm{sn}^2+\mathrm{dn}^2 = 1$).

In all the above and other solutions for cnoidal waves, the modulus (k) of the elliptic integrals, as well as the elliptic integrals $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are unknown. In the present study, they are found iteratively from the following relationship, after Massel (1989):

$$\left(\frac{H}{h}\right)\left(\frac{gT^2}{h}\right) = \frac{16}{3}k^2\mathbf{K}^2(k), \tag{7}$$

while the cnoidal wave length is calculated from the following equation (assuming the wave celerity $c = L/T = (gh)^{1/2}$):

$$\left(\frac{H}{h}\right)\left(\frac{L}{h}\right)^2 = \frac{16}{3}k^2\mathbf{K}^2(k). \tag{8}$$

The exemplary results of computations of the free surface elevation by Eq. (1), the depth-independent velocity by Eq. (2), as well as the velocities at the bottom and at the wave trough by Eq. (6) are plotted in Figures 2 and 3. For comparison, the distributions of free surface and velocity by 2^{nd} Stokes approximation (the formulas of which are not cited herein) are also shown.

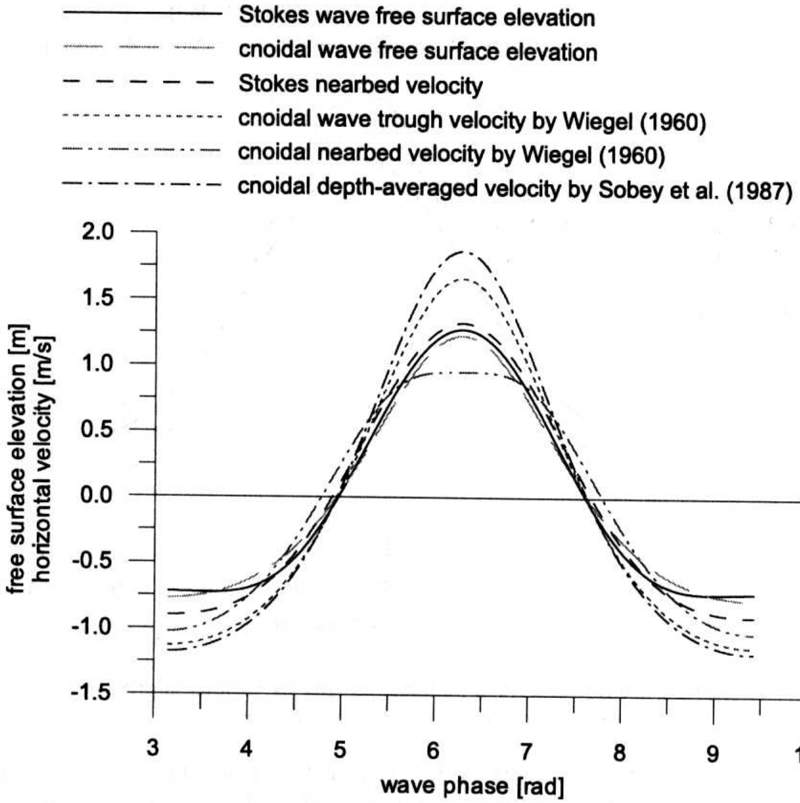

Fig. 2. Free surface elevation and wave-induced velocity by various approaches for $h = 5$ m, $H = 2$ m, $T = 6$ s; $L/h \approx 8$, $U \approx 23$

It can be seen that the velocities by Eq. (2) and Eq. (6) differ significantly from each other, in particular for large value of H/h ratio (Fig. 2). For $H/h = 0.1$ (Fig. 3), the solution by Eq. (2) coincides with the result obtained using Eq. (6) for $y = h_t$, while the 2^{nd} Stokes approximation yields the velocity almost identical as Eq. (6) for $y = 0$. As mentioned earlier, both the cnoidal and Stokes theory can be applied in the latter wave regime. The above findings show that Eq. (6) produces reasonable nearbed velocities under shallow water wave motion.

In Fig. 2, the nearbed velocity by Eq. (6) at the phase of wave crest is much less than the velocities by the other solutions. To clarify the above discrepancy, the computations were carried out for the conditions of the laboratory experiments of Iwagaki & Sakai, as given by Fenton (1979). The tests (a)–(f) have been chosen, representing 'long waves', with the ratio L/h changing from 15 to 9.5, respectively.

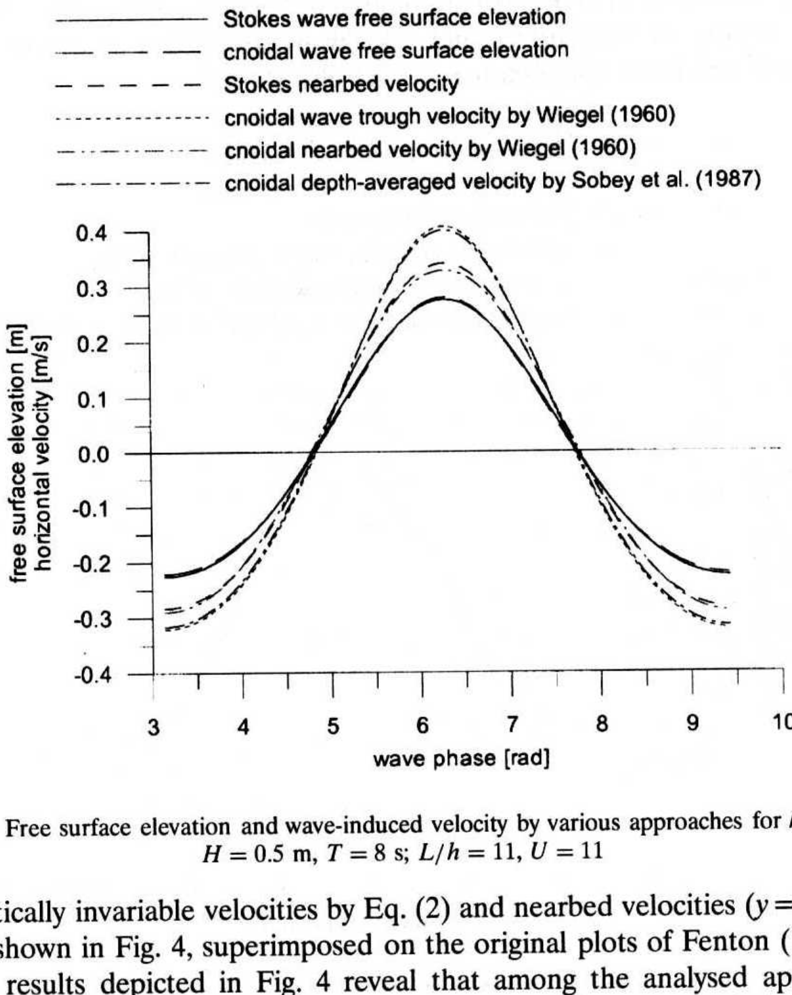

Fig. 3. Free surface elevation and wave-induced velocity by various approaches for $h = 5$ m, $H = 0.5$ m, $T = 8$ s; $L/h = 11$, $U = 11$

The vertically invariable velocities by Eq. (2) and nearbed velocities ($y = 0$) by Eq. (6) are shown in Fig. 4, superimposed on the original plots of Fenton (1979).

The results depicted in Fig. 4 reveal that among the analysed approaches, Eq. (6) yields the best accuracy in comparison to the experimental data (except for the cases 'c' and 'd'). The other solutions, in particular by Eq. (2), seems to overestimate the nearbed velocity at the phase of wave crest. Therefore the velocity by Eq. (6) has been taken for further computations of sediment transport under shallow water waves.

3. Sediment Transport: Computational Results Versus Laboratory Data

It is assumed in accordance with most of the conventional approaches that the bed shear stress is the driving force for the sediment movement. In the present study, this shear stress is calculated using the momentum integral model of the bed shear stress by Fredsøe (1984). The input wave free stream velocity in this model is provided by Eq. (6). From the shear stress distribution in the wave period, instantaneous sediment transport rates are calculated by the water-soil mixture

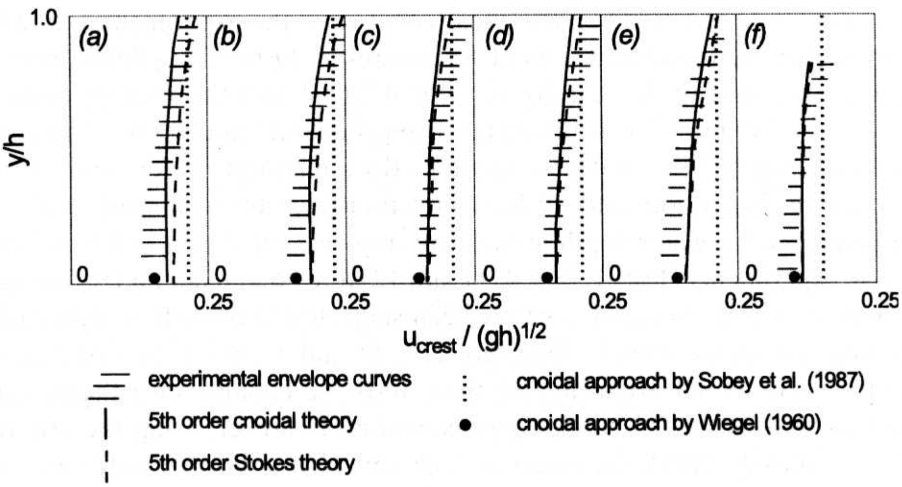

Fig. 4. Velocities at the phase of wave crest calculated by Equations (2) and (6), representing the approaches of Sobey et al. (1987) and Wiegel (1960), respectively, versus 5^{th} order cnoidal and Stokes approximations by Fenton (1979) and the experimental data of Iwagaki & Sakai, after Fenton (1979)

model of Kaczmarek & Ostrowski (1998), yielding net transport in the direction of wave propagation. The full set of the equations used in this model, as well as the theoretical background, are given in the paper by Kaczmarek & Ostrowski (1998), while the concise scheme of the model layers is plotted in Fig. 5.

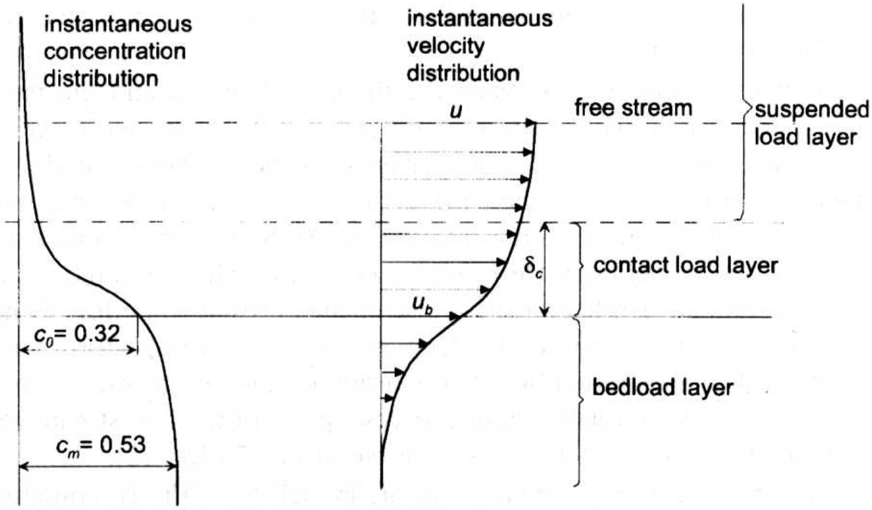

Fig. 5. Definition sketch for the sediment transport system

The comparisons between theoretical results and the experimental data have been made for the laboratory tests of Kaczmarek & Ostrowski (1996), who used natural quartz sand of the density $\rho_s = 2650$ kg/m^3 and the median grain diameter $d_{50} = 0.21$ mm. The experiments comprised measurements of waves (by string wave gauges), wave-induced velocities (by a micro-propeller) and sediment transport rates in onshore and offshore direction (by a two-cell sand trap).

For all tests, the water depth at the sand trap amounted to $h = 0.5$ m. Among all twelve experiments, Tests 1, 2, 3, 4, 11 and 12 were conducted under the regular wave motion. These tests have recently been subjected to more thorough analysis.

In four test series, namely Tests No. 1, 2, 11 and 12, the values of L/h ratio amount to 8 or 10. Theoretically, for these tests the cnoidal wave theory can be applied, as well as the 2^{nd} Stokes approximation. However, using the criteria of Fig. 2.2 in Massel (1989), the values of L/h and H/h show that only Test 11 lies in the range of applicability of the cnoidal theory and Test 12 lies very close to the intersection between the cnoidal and Stokes approaches. The cited diagram suggests that Tests 1 and 2 lie rather beyond the applicability of the cnoidal theory.

Therefore, in the first stage of the comparisons, the intermediate check of the model versus the experimental data has been done for Tests 11 and 12 only. Within this check, the nearbed free stream velocities calculated by Eq. (6) and the 2^{nd} Stokes approximation have been compared with the values registered by the micro-propeller (raw data). The free surface elevations according to the cnoidal theory given by Eq. (1) and according to the 2^{nd} Stokes theory have also been compared with the laboratory data. The full sets of the above comparisons are given in Figures 6–7 for Tests 11 and 12, respectively. For both cases, it can be seen that the nearbed velocity is better reproduced by the cnoidal theory. Similarly, the wave shape by the cnoidal theory better fits to the measured free surface elevation, in particular for Test 11.

The ultimate comparisons between the theoretical results and the measurements have been made with respect to the net sediment transport rates. The results obtained for all analysed cases, given in Table 1, show that the model net sediment transport rates by the cnoidal theory (q_{cn}) are in better agreement with the experimental data, than the rates by the 2^{nd} Stokes approximation (q_{St}). Unfortunately, the model rates are generally larger than the measured quantities (q_{meas}). A question arises whether these discrepancies result from disturbances to sediment transport process caused by the edges of the sand trap, disturbances of the micro-propeller registration by the suspension of sand, inaccuracy of the sediment transport model or, finally, imprecise description of the free stream velocity in the computations. The latter is clearly visible in Fig. 7 (Test 12).

To concentrate on the sediment transport model itself, the computations of net sediment transport rate for Test 12 have been repeated with the modified free stream input. Namely, the wave-induced nearbed velocity has been determined by

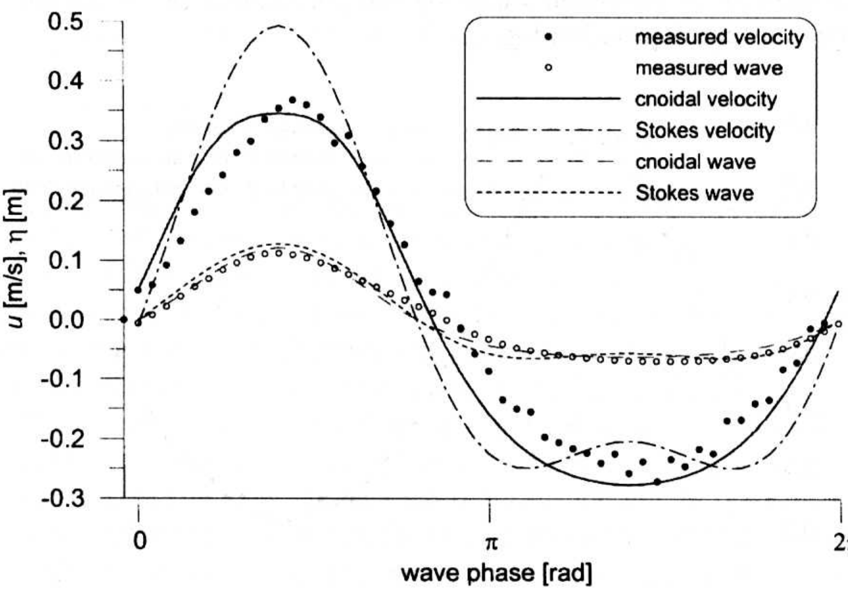

Fig. 6. Nearbed velocities by cnoidal theory (Eq. 6) and 2nd Stokes approximation, free surface elevations by cnoidal theory (Eq. 1) and 2nd Stokes approximation versus laboratory data of Test 11 by Kaczmarek & Ostrowski (1996)

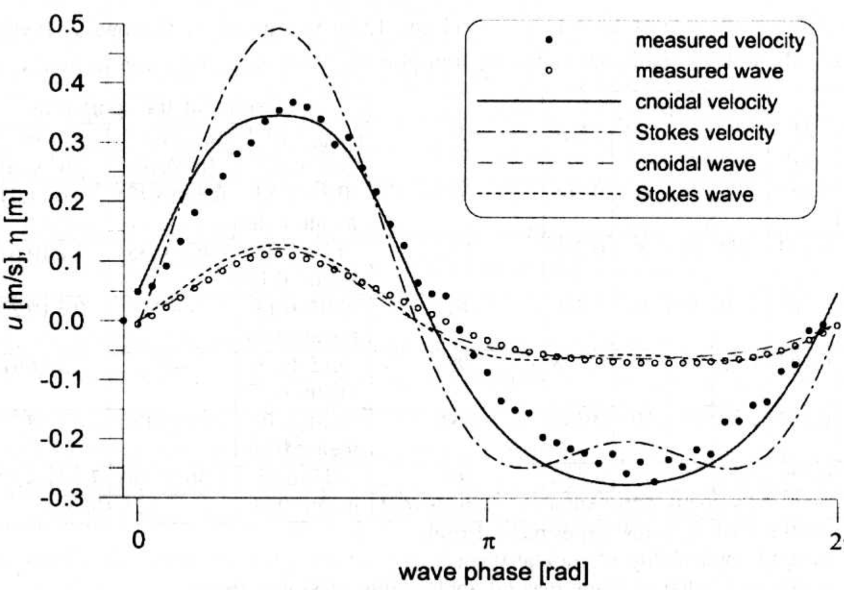

Fig. 7. Nearbed velocities by cnoidal theory (Eq. 6) and 2nd Stokes approximation, free surface elevations by cnoidal theory (Eq. 1) and 2nd Stokes approximation versus laboratory data of Test 12 by Kaczmarek & Ostrowski (1996)

approximation of the measured values by two polynomials of 2nd degree. These approximations are depicted in Fig. 8.

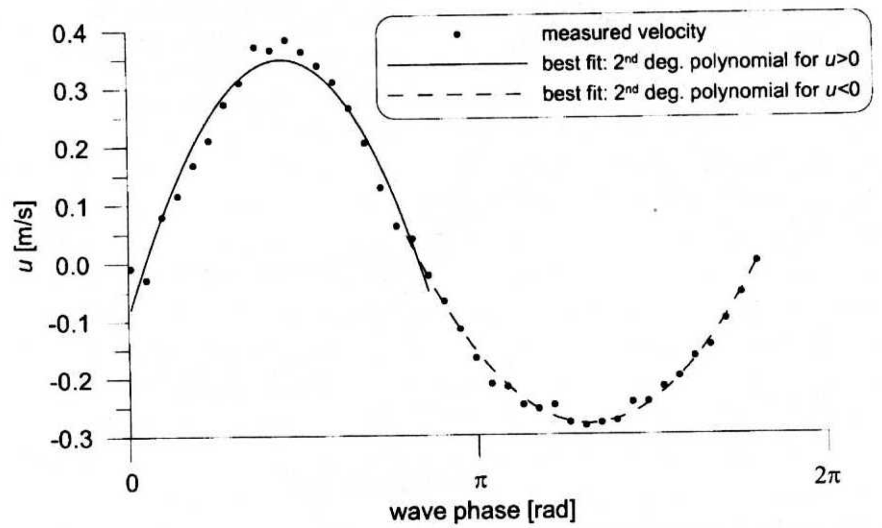

Fig. 8. Measured nearbed velocity in Test 12 approximated by least-square method for precise computations of net sediment transport rate

Table 1. Laboratory data of Tests 1, 2, 3, 4, 11 and 12 by Kaczmarek & Ostrowski (1996) and present theoretical results for sediment transport (bedload rates are given in brackets)

Test No.	H [m]	T [s]	U	L/h	H/h	F_c	net sediment transport rate $q_{meas} \cdot 10^{-6}$ [m^3/s/m]*	$q_{cn} \cdot 10^{-6}$ [m^3/s/m]	$q_{st} \cdot 10^{-6}$ [m^3/s/m]
1	0.183	2.00	24	8	0.366	145	0.23–0.33 mean: 0.28	0.73 (0.39)	2.6 (1.3)
2	0.144	2.00	19	8	0.288	128	0.10–0.26 mean: 0.16	0.62 (0.35)	1.1 (0.6)
3	0.135	1.50	9	6	0.270	61	0.04–0.13 mean: 0.08	—**	0.3 (0.2)
4	0.164	1.75	16	7	0.328	98	0.15–0.38 mean: 0.25	—**	1.1 (06)
11	0.184	2.50	40	10	0.368	254	0.20–1.75 mean: 0.64	2.14 (1.09)	6.0 (2.7)***
12	0.236	2.00	31	8	0.472	164	0.15–0.75	1.36 (0.66)	7.5 (3.4)***
12	nearbed velocity approximated as measured						mean: 0.36	1.25 (0.69)	

* on the basis of a few experimental runs
** beyond applicability of cnoidal theory
*** series concluded as being beyond applicability of Stokes theory

The last row in Table 1 comprises the results of net transport computations for the newly assumed nearbed velocity. The agreement between measured and computed values, although better than for velocity calculated from the cnoidal theory, still reveals a slight theoretical overestimation of the laboratory data. This is believed to result from inaccuracies of the measuring techniques.

4. Conclusions

Sediment transport process is highly non-linear with respect to water flow, which is its driving force. For flow patterns produced by wave motion, sediment moves onshore and offshore at the phases of wave crest and trough, respectively. The net transport results from the differences of wave shape during these phases, which occur for asymmetric waves. Although the mean Eulerian velocity for such waves at any ordinate above sea bed amounts to zero, the nearbed net transport is always directed onshore, due to non-linear effects. Roughly, due to these effects, more sand is transported onshore at wave crest than offshore at wave trough. Certainly, this concerns only the nearbed sediment motion, in the bedload layer and the contact load layer (see Fig. 5). Higher above the sea bed, the influence of wave-induced currents (e.g. undertow) can be significant and can cause the sediment to move offshore.

As proved by the computations, the net sediment transport rate is very sensitive to changes in the nearbed wave-induced velocities. The correct description of the nearbed free stream velocity, namely its distribution over the wave period, is necessary for precise prediction of sediment transport. To do this, the cnoidal and Stokes approaches can be used within their ranges of applicability. However, further theoretical and experimental studies would be desirable to check the correctness of modelling of the nearbed wave-driven flows. These studies should comprise investigation of free stream velocities caused not only by horizontally asymmetric waves but by the waves vertically asymmetric as well.

The Author treats the present study as the intermediate stage in formulation of a quasi-phase-resolving cross-shore sediment transport model. In such a model, any classical solution of wave-current field in the coastal zone can be followed by the computation of net sediment transport rates at all locations of the cross-shore transect. Consequently, sea bed profile evolution can be modelled. Within this approach, the description of wave-induced nearbed velocity will be carried out using various wave theories, depending on the regime of wave motion, indicated by the quantities such as Ursell parameter and the ratio L/h, as well as by other features (for instance F_c). This way of modelling does not exclude more sophisticated methods, namely fully phase-resolving models for the entire cross-shore profile, using the presently tested sediment transport module. If any of the above concepts are successful the sediment transport module can be applied in a 2DH model, which will enable a reliable prediction of coastal changes in the cross-shore and longshore direction.

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