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Dimensionless Equation for Side-Channel Weirs

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Abstract

Using the principle of conservation of momentum, an analysis of the one-dimensional description of flows of Newtonian fluids in prismatic channels with side weirs was carried out. A new form of the equation of motion – with a corrected mass decrement term and added momentum-variation term – has been derived from the principle of conservation of momentum. Following examination of relevant coefficients, the dimensionless form of the modified equation applies to the hydraulic design of a side weir with a high overfall crest and a throttling pipe, used in sewer systems.

Notation

A – cross-sectional area flow,

b – water surface width in the channel (width of rectangular channel),

D – channel diameter,

Fr_0 – Froude number in the channel at the beginning of the overflow chamber ($x = 0$),

g – acceleration of gravity,

H – depth of flow in the channel,

H_{cr} – critical depth of flow in the channel,

H_0 – depth of flow at the beginning of the overflow chamber ($x = 0$),

S – bottom slope,

S_f – hydraulic gradient,

k – ratio of longitudinal component U and mean velocity v ($k = U/v$),

K_0 – similarity number of channel shape at the beginning of the overfall ($K_0 = bH_0/A_0$),

$\vec{l}$ – unit vector having direction of local velocity vector $\vec{\nu}$ parallel to mean velocity vector $\vec{v}$,

L – length of overflow crest,

L_0 – relative length of overflow crest ($L_0 = L/H_0$),
m – mass,
n – channel roughness coefficient in Manning's formula,
p – height of weir crest,
P_h – wetted perimeter of flow section,
P_0 – relative height of overflow crest ($P_0 = p/H_0$),
q – ratio of discharge in overflow chamber ($q = Q(x)/Q_0$),
q_b – unit volume flow (per length Δx) over side weir,
q_r – ratio of flows ($q_r = Q/Q_0$),
Q – discharge of side weir,
Q_0 – discharge in inlet channel at the beginning of the overflow chamber ($x = 0$),
$Q(x)$– discharge in overflow chamber in cross-section with abscissa x,
t – time,
R_h – hydraulic radius ($R_h = A/P_h$),
U – longitudinal component of velocity of spill flow,
V – volume,
W_0 – relative head above overfall crest at the beginning of the side weir ($W_0 = (H_0 - p)/H_0$),
x – distance of any point on side weir from its beginning (abscissa),
ν – local velocity (in x directions) of stream filament in channel,
υ – mean velocity of main stream in channel,
υ_b – mean velocity of side-discharge stream,
β – momentum coefficient,
β_b – momentum coefficient of side-discharge stream,
ζ – dimensionless ordinate of depth of flow elevation in the channel ($\zeta = H/H_0$),
η – coefficient of momentum variation in the mass decrement term ($\eta = 2\beta - k\beta_b$),
χ – ratio of local value of hydraulic gradient S_f to hydraulic gradient S_{f0} in the initial section of the overflow chamber ($\chi = S_f/S_{f0}$),
Θ – angle of inclination of channel invert,
μ – weir discharge coefficient,
ξ – dimensionless abscissa of length ($\xi = x/L$),
ξ_μ – dimensionless abscissa of length for discharge coefficient ($\xi_\mu = x/H_0$),
Π – momentum of liquid mass,

ρ – liquid density,
τ – shear stress on channel wall.

Subscripts

0 – beginning cross-section of overfall chamber ($x = 0$),
1 – value normalized to the interval $< 0, 1 >$.

1. Introduction

The problem of how to compute water flow over side weirs has received considerable attention for many decades. In spite of a large number of relevant studies, none of the formulas derived so far can be applied with confidence to describe this kind of flow adequately. For convenience, use has been made of a variety of relations. Initially, front weirs were considered (after suitable adaptation; the Poleni formula). Later, preference was given to some simplified empirical formulas derived from experiments, which were mostly run within a narrow range of variation in the investigated geometrical and hydraulic parameters of side weirs (e.g. Kotowski's formulas, 1990), as well as to some theoretical expressions (e.g. those derived by Hager 1987). Further approaches to side weir computationave combined the description of the free-surface profile along the weir (using differential equations of motion) with the formulas describing flow over the side weir (e.g. de Marchi 1934, Frazer 1957, El-Khashab and Smith 1976, Ishikawa 1984, Hager 1987, 1993, Uyumaz and Smith 1991, Uyumaz 1997, Kotowski 1997, 1998).

Most of the investigators concentrating on free flow over side weirs in open channels have based their theoretical analyses on equation of motion derived from the energy solution for the energy coefficient α = constant and $U = \upsilon$, this means the longitudinal component of velocity of the spill flow (U) is equal to the mean velocity of the main stream in channel (υ). Others have used an equation of motion derived from the momentum solution for the momentum coefficient β = constant and $U \neq \upsilon$.

However, the investigations reported by El-Khashab and Smith (1976) and Kotowski (1998) revealed that the coefficients of energy and momentum were not constant and $U > \upsilon$, along the length of side weirs.

2. Derivation of the Equation of Motion for Channels with Side Weirs

2.1. Balance of Momentum and Outside Forces

From the principle of conservation of momentum in Newtonian continuous-medium mechanics it follows that the change of momentum ($d\vec{\Pi}$) with time ($dt \rightarrow 0$) is equal to the sum of body and surface forces. Thus, the change of momentum is equal to the sum of forces acting on the control liquid volume (ΔV) between cross-sections I – I and II – II of the channel (Fig. 1). Momentum was balanced for this volume of the liquid and then the sum of forces acting on the liquid was calculated. On this basis, using the continuity and momentum equations, the equation of motion for side weir flow was derived (Kotowski 2000b).

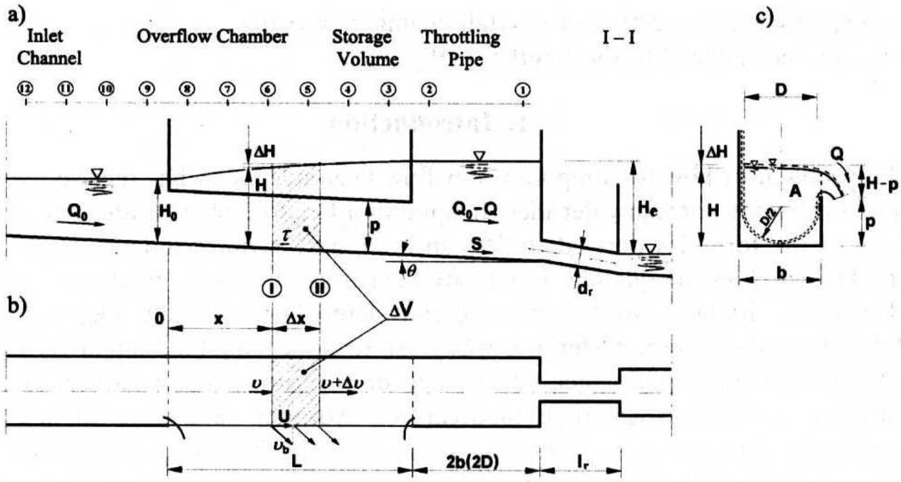

Fig. 1. Definition sketch for channel with discharge over side weir and throttling pipe: (a) elevation; (b) plan; (c) section I – I

The momentum brought in by a liquid mass m flowing through an element having area dA in time $dt \to 0$ is:

$$d(m\vec{v}) = (\rho v \, dA \, dt)\vec{v} \quad (1)$$

where $\vec{v}$ – a local velocity vector perpendicular to area dA cut out from flow area A. Thus the total momentum brought in by the liquid mass flowing through a flow section I – I of area A in time dt is (Fig. 1):

$$d\vec{\Pi}_I = \rho \, dt \int_A v\vec{v} \, dA = \vec{\imath}\rho \, dt \int_A v^2 \, dA \quad (2)$$

where $\vec{\imath}$ – a unit vector, whose direction is the same as that of the local velocity vector $\vec{v}$ parallel to the mean velocity vector $\vec{\upsilon}$. Momentum (Eq. 2), expressed by the mean liquid flow velocity, can be written as:

$$d\vec{\Pi}_I = \beta \, (\rho\upsilon A \, dt) \, \vec{\upsilon} = \beta \, (\rho\upsilon A \, dt) \, \upsilon\vec{\imath} = \beta\rho A\upsilon^2 dt \, \vec{\imath} \quad (3)$$

where β – dimensionless (corrective) coefficient of momentum.

It follows from equations 2 and 3 that:

$$\vec{\imath}\rho \, dt \int_A v^2 dA = \beta \, \vec{\imath}\rho \, dt \, \upsilon^2 A \quad (4)$$

hence momentum coefficient β, can be expressed by equation:

$$\beta = \frac{\int_A v^2 dA}{v^2 A}. \tag{5}$$

It should be noted that the solid of velocities varies along path x, as the mean velocity does. This means that coefficient β is constant only for uniform flow. Here it varies from section to section. This observation is based on the results of experiments carried out by the author (1998) on non-conventional side weirs in rectangular and U-shaped channels.

Total momentum $d\vec{\Pi}_I$ introduced into the control space ΔV with a liquid flowing through a flow section of area A in time $dt \to 0$ assumes the following value:

$$d\vec{\Pi}_I = \beta\,\rho\,A\,v^2 dt\,\vec{\imath} = \beta\,\rho\,Q\,v\,dt\,\vec{\imath} \tag{6}$$

where $Q = Q(x)$ – a discharge in cross-section I – I of the area $A : Q = Av$. The momentum of the liquid leaving the interior of the control space through flow section II – II of area $A + \Delta A$ in time $dt \to 0$ is:

$$\begin{aligned} d\vec{\Pi}_{II} &= (\beta + \Delta\beta)\rho\,(A + \Delta A)\,(v + \Delta v)^2 dt\,\vec{\imath} \\ &= (\beta + \Delta\beta)\rho\,(Q + \Delta Q)\,(v + \Delta v) dt\,\vec{\imath} \end{aligned} \tag{7}$$

At the same time ($dt \to 0$) a mass of liquid flows out from the control space, flowing over the weir length Δx at mean velocity v_b. The velocity changes its value and direction along the overflow edge. The momentum of the liquid mass flowing over length Δx of the overflow edge is:

$$d\vec{\Pi}_b = \beta_b\,\rho\,\Delta x q_b\,dt\,\vec{v}_b \tag{8}$$

where β_b – a lateral stream momentum (correction) coefficient, q_b – a unit volume flow over the side weir (per length Δx of the overflow crest).

The coefficient β_b relates here to velocity $\vec{v}_b$ and can be calculated using Eq. 5 and integrating along depth $(H - p)$ of the layer of liquid above the weir ($A = \Delta x \cos\theta(H - p)$ for $\Delta x \to 0$). It thus refers to the profile of velocity $\vec{v}_b$ at the place indicated by the abscissa x, and not to a solid of velocities as in sections I – I and II – II.

It follows from Eqs. 6, 7 and 8 that the following change in momentum will occur in time dt:

$$d\vec{\Pi}_{II} + d\vec{\Pi}_b - d\vec{\Pi}_I = d\vec{\Pi} \tag{9}$$

which, after ordering and neglecting the terms containing products of two quantities with infinitesimal values, can be written as:

$$d\vec{\Pi} = \rho\,(\beta\,Q\Delta v\vec{\imath} + \beta\,\Delta Q v\vec{\imath} + \Delta\beta\,Q v\vec{\imath} + \beta_b q_b \Delta x\,\vec{v}_b)\,dt. \tag{10}$$

The unit surface forces here are: the hydrostatic pressure, increasing linearly towards the centre of the liquid, acting perpendiculary to the areas of flow sections I – I and II – II and the shear stress $\vec{\tau}$ acting on the wetted surface of the channel's walls and the invert between sections I – I and II – II. The vector of a unit body force is the gravitational acceleration vector $\vec{g}$. The resultant $(\Delta\vec{F})$ of the surface and body forces can be expressed as follows (Kotowski 1998, 2000b):

$$\Delta\vec{F} = -A\,\Delta H\,\rho\,g\,\vec{\iota} + A\,\Delta x\,\cos\Theta\,\rho\,\vec{g} - P_h\Delta x\,\cos\Theta\,\vec{\tau} \tag{11}$$

where P_h – the wetted perimeter in section I – I.

2.2. Equation of Motion

The vector of momentum variation in time dt, calculated using of Eq. 10, is equal to the resultant vector of forces $\Delta\vec{F}$ (Eq. 11) acting on the infinitesimal liquid volume ΔV considered. This means that the projection of these vectors onto axis x satisfies the following equation (after dividing both sides by $\rho g A\Delta x\cos\Theta$):

$$\begin{aligned}\frac{1}{gA}\left(\frac{\beta Q\Delta v + \beta\Delta Q v + \Delta\beta Q v}{\Delta x} + \beta_b\,q_b\,U\right) = \\ = -\frac{\Delta H}{\Delta x} + \sin\Theta - \frac{P_h\tau}{\rho g A}\end{aligned} \tag{12}$$

where U – a coordinate of the longitudinal component $\vec{U}$ of the velocity vector $\vec{v}_b$ of a lateral stream along the direction of the mean velocity vector $\vec{v}$ of the main liquid stream in the channel, $\sin\Theta \equiv S$ – bottom slope of the channel, $P_h\tau/\rho g A = S_f$ – a hydraulic gradient at $\tau \equiv |\vec{\tau}|$. By introducing the above notations and performing lim operations at $\Delta x \to 0$ we obtain:

$$\frac{1}{gA}\left(\beta Q\,\frac{dv}{dx} + \beta v\,\frac{dQ}{dx} + vQ\,\frac{d\beta}{dx} + \beta_b\,q_b\,U\right) = -\frac{dH}{dx} + S - S_f. \tag{13}$$

It follows from the equation of the continuity of motion that $v = Q/A$, hence:

$$\frac{dv}{dx} = \frac{1}{A}\,\frac{dQ}{dx} - b\,\frac{Q}{A^2}\,\frac{dH}{dx} \tag{14}$$

where $b\,dH/dx = dA/dx$, i.e. the area increment ΔA along path Δx occurs as the result of an increase in height by ΔH, while the width of the overflow chamber is b (Fig. 1).

From the volume flow balance it follows that (Kotowski 1998, 2000b):

$$Q(x) = Q_0 - \int_0^x q_b dx \tag{15}$$

where Q_0 – discharge in the channel at the beginning of the overflow chamber ($x = 0$), and:

$$\frac{dQ}{dx} = -q_b = \frac{2}{3}\mu\sqrt{2g}\,(H-p)^{3/2} \tag{16}$$

after inserting it into Eq. 13 and ordering we get:

$$\frac{dH}{dx} = \frac{S - S_f - \left[(2\beta - k\beta_b)\,Q\,\frac{dQ}{dx} + Q^2\,\frac{d\beta}{dx}\right]\frac{1}{gA^2}}{1 - \beta\,\frac{Q^2 b}{gA^3}} \tag{17}$$

where k – ratio of the longitudinal component U and the mean velocity $v(k = U/v)$.

3. Dimensionless Form of Equation of Motion

Introducing the following dimensionless variables:

$$\zeta = H/H_0;\ \xi = x/L;\ q = Q(x)/Q_0 \tag{18}$$

where ζ – dimensionless ordinate of depth of flow, ξ – dimensionless abscissa of length and q – ratio of discharge in overflow chamber, area A of the flow section in the overflow chamber can be written in a generalized form (for the adopted shape of the chamber cross-section). It has been assumed that above the crest height of the side weir, the overflow chamber has a constant width equal to b when the channel is prismatic in shape, e.g. rectangular channels, and a constant width which is equal to the diameter $D \equiv b$ when the channel is e.g. U-shaped. This conforms with the conditions encountered in sewage-engineering (Kotowski 1997: $p > D/2$; Fig. 1). Hence,

$$A = A_0 + b(H - H_0) = A_0\left[1 + \frac{bH_0}{A_0}(\zeta - 1)\right] = A_0\left[K_0\zeta - (K_0 - 1)\right] \tag{19}$$

where A_0 – upstream surface area of the flow ($x = 0$), K_0 – coefficient which can be defined as a similarity number of the channel shape ($K_0 = 1$ for a rectangular channel and $K_0 > 1$ for other typical shapes of the channel, e.g. U-shaped channels): $K_0 = b\,H_0/A_0$.

After substitution of Eqs. 18 and 19 into Eq. 17 and after suitable arrangement, we obtain the following dimensionless form of the modified equation of motion:

$$\frac{d\zeta}{d\xi} = \frac{L_0(S - \chi S_{f0}) - \left[\eta\,q\frac{dq}{d\xi} + q^2\frac{d\beta}{d\xi}\right]\frac{Fr_0^2}{[K_0\zeta-(K_0-1)]^2}}{1 - \frac{\beta Fr_0^2\,K_0 q^2}{[K_0\zeta-(K_0-1)]^3}} \tag{20}$$

where L_0 – relative length of overflow crest ($L_0 = L/H_0$), χ – ratio of local value of hydraulic gradient S_f to hydraulic gradient S_{f0} at the initial section of the overflow chamber ($\chi = S_f/S_{f0}$), η – coefficient of momentum variation in the mass decrement term which can be determined experimentally for the set weir, channel shape and motion parameters ($\eta = \eta\ \xi$); $\eta = 2\beta - k\beta_b$), and $q = q(\xi)$ in the overflow chamber $0 \leq \xi \leq 1$ in conventional formulation:

$$q = 1 - \frac{2}{3}\mu\,\frac{L\,H_0\sqrt{2g\,H_0}}{Q_0}\int_0^{\xi}(\zeta - P_0)^{3/2}d\xi \tag{21}$$

where P_0 – relative height of overflow crest at the beginning of the side weir: $P_0 = p/H_0$, Fr_0 – Froude number at the first cross-section ($x = 0$) of the overflow chamber:

$$Fr_0 = \frac{Q_0}{A_0\sqrt{g\,H_0}}. \tag{22}$$

Assuming that the hydraulic gradient (S_f) in nonuniform flow can be calculated in terms of the Manning equation derived for uniform flow, and considering the real value of H in the set cross-section of the overflow chamber (n = constant), we can write:

$$S_f = \frac{(n\,Q(x))^2}{A^2\,R_h^{4/3}}. \tag{23}$$

Assuming furthermore that $S_f = \chi S_{f0}$, and using Eqs. 18 and 19, we obtain:

$$\frac{n^2\,q^2\,Q_d^2}{A_0^2\,[1 + K_0\,(\zeta - 1)]^2\left(\frac{A_0[1+K_0(\zeta-1)]}{P_h}\right)^{4/3}} = \chi\,\frac{n^2\,Q_d^2}{A_0^2\left(\frac{A_0}{P_{h0}}\right)^{4/3}} \tag{24}$$

where S_{f0} – hydraulic gradient at the beginning of the overflow chamber. The wetted perimeter (P_h) in the cross-section of the overflow chamber with the unilateral weir can be written as: $P_h = P_{h0} + (H - H_0) = P_{h0} + H_0(\zeta - 1)$. For the bilateral weir $P_h = P_{h0}$. And finally for $0 \leq \xi \leq 1$ we have $1 \geq \chi > 0$:

$$\chi = \frac{[P_{h0} + H_0(\zeta - 1)]^{4/3}}{[1 + K_0(\zeta - 1)]^{10/3}\,P_{h0}^{4/3}}\,q^2 \tag{25}$$

for $1 \geq q \geq 1 - q_r$, where q_r – ratio of flows: $q_r = Q/Q_0$.

In general, the weir discharge coefficient μ in Eq. 21, is affected by abscissa x, as the head of the free surface varies along the weir edge, and so does the contraction of the stream along the weir length. In practice, it is impossible to determine the behaviour of the value of μ along the weir edge. The rate of flow over the side weir can however, be calculated when use the following equation (Kotowski 2000a):

$$Q = \frac{2}{3}\mu\sqrt{2g}\int_0^{L}(H - p)^{3/2}\,dx \tag{26}$$

where μ – discharge coefficient (mean) calculated for a weir of length L.

Incorporating the dimensionless variables of Eqs. 18, and defining the dimensionless variable of the length in a different way: $\xi_\mu = x/H_0$, yields the following expression which describes the derivative dQ/dx of Eq. 16 ($0 \le \xi_\mu \le L_0$) as

$$\frac{dq}{d\xi_\mu} = \frac{2}{3}\,\mu\,\frac{H_0^{5/2}}{Q_0}\,\sqrt{2g}\,(\zeta - P_0)^{3/2}\,. \tag{27}$$

Defining

$$\frac{2}{3}\,\frac{H_0^{5/2}}{Q_0}\,\sqrt{2g} = V_0 \tag{28}$$

where V_0 – dimensionless similarity number determined from the conditions of motion at the beginning of the overflow chamber ($x = 0$), we obtain for $\xi_\mu = L_0$:

$$\mu\,V_0 \int_0^{L_0} (\zeta - P_0)^{3/2}\,d\xi_\mu = \frac{Q}{Q_0} = q_r\,. \tag{29}$$

Thus,

$$\mu = \frac{q_r}{V_0 \int_0^{L_0} (\zeta - P_0)^{3/2}\,d\xi_\mu}\,. \tag{30}$$

The dimensionless form of the equation of motion (Eq. 20) is an ordinary first-order differential equation with the dimensionless abscissa ξ (counted from the initial section of the weir; $0 \le \xi \le 1$) as an independent variable, and the dimensionless depth ζ in the overflow chamber axis (generally, $\zeta \ge 1$ for the water rise curve (Fig.1) and $\zeta \le 1$ for the drawdown curve along the weir) as a dependent variable. This nonlinear equation cannot be solved analytically, and it is necessary to use numerical methods. This requires knowledge of the functions that relate the coefficients χ (Eq. 25), β, η and the term q (Eq. 21) to the dimensionless parameters of motion (similarity numbers) q_r, L_0, P_0, S_{f0}, Fr_0^2 and K_0, and to the independent variable ξ. The initial condition takes the form of $\zeta(0) = 1$. The ratio of discharge inside the overflow chamber is defined by Eq. 21, from which it follows that a formula is needed to describe the weir discharge coefficient (Eq. 30).

The usefulness of Eq. 20 in describing the motion of a liquid in channels with side weirs and throttling pipes for the adjustment of discharge from a storage volume located after the overflow chamber has been verified by experiments.

4. Example of Solution of the Equation of Motion

4.1. Program and Description of Experimental Studies

Experiments were conducted on a hydraulic model (Kotowski 1998). Two basic series of experimental investigations into unilateral and bilateral side weirs (in six design versions) were carried out. The first series was conducted on side weirs in a channel with a rectangular cross-section (b = 315 mm; Fig. 1). The second series involved U-shaped channels: circular in the lower part (up to a height equal to half the channel diameter D = 287 mm) and rectangular in the upper part (above this height). The bottom slope was constant ($S = 3.3^0/_{00}$, and so was the height of the weir edges $p > H_{cr}(Q_o)$), i.e. p = 210 mm (= $2b/3$) for the channel with a rectangular cross-section and p = 204 mm ($\approx 5D/7$) for the channel with a complex cross-section. Such assumptions are based on the results obtained by the author in his previous studies (Kotowski 1990, 1997). Thus, the present study was focused on the conditions of subcritical flow (water rise curve; Fig. 1), as well as the conditions of free flow over the weir crest. A 2.6 m long throttling pipe 152 mm in diameter was mounted on a slope $6.6^0/_{00}$. A gate valve was used for discharge adjustment. The length (l_s) of the storage volume downstream of the weir was assumed to be constant, l_s = 600 mm ($\approx 2b \approx 2D$) – after Saul and Delo (1981). The model was made of PVC with a roughness coefficient in Manning's formula $n \cong 0,01$ s/m$^{1/3}$. The weir crest was 5 mm wide.

The model studies included measurements of motion parameters in 12 cross-sections located in the storage volume, overflow chamber and inlet channel:

- Variant 1 – a unilateral weir L = 600 mm ($\approx 2b$) in a rectangular channel,
- Variant 2 – a unilateral weir L = 900 mm ($\approx 3b$) in a rectangular channel,
- Variant 3 – a unilateral weir L = 1200 mm ($\approx 4b$) in a rectangular channel.

Three subvariants of discharge to the weir: Q_0 = 16.9, 33.8 and 50.8 dm^3/s, were planned for each variant. For each subvariant a different number of measurements was planned for the coefficient of the separation of flow on the weir: $q_r = Q/Q_0$ = 1.0, 0.8, 0.6 and 0.5 at Q_0 = 33.8 dm^3/s and q_r = 0.8 at Q_0 = 16.9 and 50.8 dm^3/s.

- Variant 4 – L = 2 x 600 mm, in the range as above (bilateral weir in a rectangular channel b = 315 mm).

In the second series of experiments for side weirs in a U-shaped channel ($b \equiv D$ = 287 mm) the following variants were investigated:

- Variant 5 – a unilateral weir L = 1200 mm, in the range as above,
- Variant 6 – a bilateral weir L = 2 x 600 mm, in the range as above.

A total of 36 combinations of weir design and hydraulic parameters was investigated using the model. In the adopted range of changes of Q_0 and q_r, the

Reynolds number (Re) in the throttling pipe varied as follows: $23800 < Re < 129500$, whereas the Froude number (Eq. 22) in the inlet channel directly before the weir fell within $0.14 < Fr_0 < 0.35$ in the rectangular channel, and within $0.17 < Fr_0 < 0.46$ in the U-shaped channel.

4.2. Important Experimental Results

The values of the momentum coefficient β were established in terms of Eq. 5. Integration was carried out after the areas of partial sections between consecutive local velocity isolines in the investigated cross-section of the channel had been calculated. Local velocities (ν) were measured with a hydrometric current meter. The interpretation of the variability of coefficient β was limited to three variants: 3, 5 and 6 at $Q_0 = 33.8\ \mathrm{dm^3/s}$ and $q_r = 1.0, 0.8, 0.6, 0.5$, which involved about 5600 local velocity measurements (in 88 cross-sections). Measured profile of velocity in cross-sections along the longitudinal axis of the channel was shown in Fig. 2 – for example in variant 5. In the adopted range of model parameter variations, it was the ratio of flows (q_r) that had the strongest influence on the behaviour of β along the length of the channels with side weirs (in inlet channel, overflow chamber and storage volume). A statistical measure for the coefficient was found to be the Froude number (Fig. 3), which – after suitable transformation – gives:

$$\beta = 1.06(Fr_0/Fr_{(x)})^{0.22} \tag{31}$$

where $1.01 < \beta < 1.6$ (correlation coefficient $R = 0.92$ and standard error $\delta = 0.04$). For coefficient β (for the solution of the equation of motion), use was made of another formula, which related β to $\xi (0 \le \xi \le 1)$:

$$\beta = 0.287 + 0.180q_r + 0.116q_r^2 + 0.807W_0 - 3.43W_0^2 - 0.622\xi + 0.573exp\xi. \tag{32}$$

To calculate the discharge coefficient (μ) Eq. 30 was used for 36 free-surface profiles (for 6 variants) measured along the longitudinal axis of the overflow chamber. The profiles were approximated with a third-degree polynomial as

$$(\zeta - P_0)^{3/2} = \left(W_0 + W_1\xi_\mu + W_2\xi_\mu^2 + W_3\xi_\mu^3\right)^{3/2} \tag{33}$$

where W_0 – free term of the polynomial ($W_0 = 1 - P_0$): $W_0 = (H - p)/H_0$. From Eqs. 20, 30 and 33 the coefficient μ is a function of the dimensionless parameters $\mu = \mu(q_r, V_0, L_0, W_0, Fr_0, K_0)$. The partial dependence of μ on particular motion parameters was tested and it was found that (Fig. 4):

$$\mu = 0.644 - 0.052q_r + 0.0088L_0 + 0.035W_0 - 0.075Fr_0 - 0.065K_0 \tag{34}$$

as a result of multiple regression at the significance level of 0.05 (Kotowski 1998) at $0.52 \le \mu \le 0.59$ and $\bar{\mu} = 0.55$ for subcritical flow ($Fr_0 < 1$). Similar values were reported e.g. by Ishikawa (1984) and Uyumaz (1997).

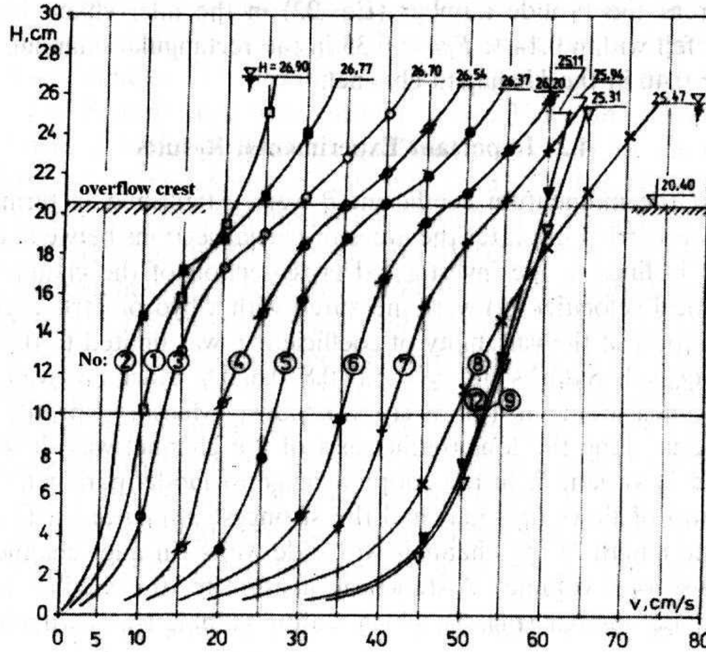

Fig. 2. Example of profile of velocity in 10 cross-sections measured along the longitudinal axis of the channel in variant 5 (U-shaped channel, L = 1.2 m, Q_0 = 33.8 dm^3/s and q_r = 0.8)

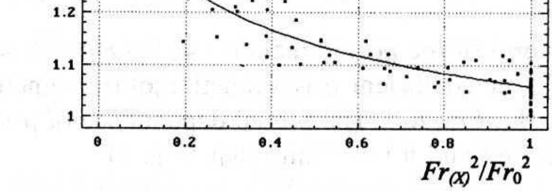

Fig. 3. Regression of coefficient β versus $Fr_{(x)}^2/\ Fr_0^2$ (along channels with side weirs in variants 3, 5 and 6)

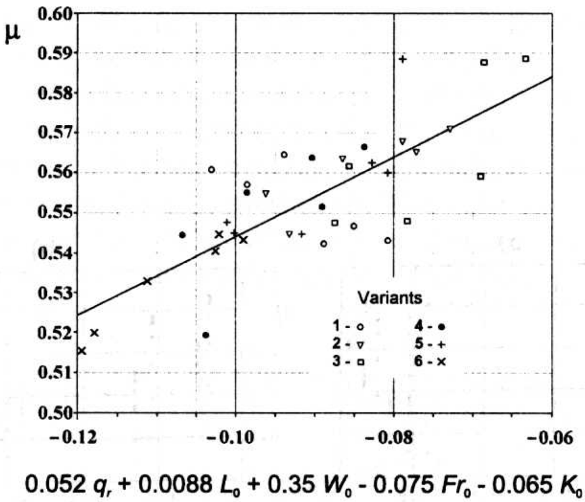

Fig. 4. Regression of weir discharge coefficient μ from dimensionless parameters of motion q_r, L_0, W_0, Fr_0 and K_0 (in variants $1 \div 6$)

The equation of motion (Eq. 20) for the investigated side weirs with a throttling pipe were solved in terms of the formulas (Eqs. 21, 25, 32 and 34 for similary numbers q_r, L_0, P_0, W_0, S_{f0}, $Fr_0{}^2$ and K_0) derived in this paper. The last coefficient of Eq. 20, denoted as $\eta = 2\beta - k\beta_b$, was calculated directly from Eq. 20, since the water-surface profiles along the longitudinal axis of the overflow chamber were measured, and the other coefficients were known. Coefficients β_b and k could not be calculated on the basis of model measurements as in the adopted model scale the weir layer was several centimeters thick. Thus, it was impossible to directly measure the distribution of the velocities of the side-discharge streams (β_b). The values of η calculated in terms of Eq. 20 were related to the dimensionless parameters of motion L_{01}, W_{01}, q_{r_1}, Fr_{01}^2 and K_{01} was normalized to the interval $< 0.1 >$ (Fig. 5a $\div$ e), as well as to the abscissa ξ (Fig. 5f). It was now possible to adopt appropriate classes of functions. After approximation, using Chebyshev polynomials normalized to $< 0, 1 >$, the equation describing the behavior of η over dimensionless overflow chamber length $0 \leq \xi \leq 1$ assumes the form

$$\begin{aligned} \eta = 6.46 + 5.61 q_r - 1.30 q_r^2 - 0.0531 L_0 - 59.2 W_0 + \\ +80.4 W_0^2 - 4.94 Fr_0^2 - 0.460 K_0 + 2.11\xi - 1.27\xi^2 \end{aligned} \tag{35}$$

for the following ranges of variation: $0.3 \leq \eta \leq 2.2$, $0.5 \leq q_r \leq 1.0$, $1.8 \leq L_0 \leq 5.1$, $0.13 \leq W_0 \leq 0.35$, $0.65 \leq P_0 \leq 0.87$, $0.14 \leq Fr_0 \leq 0.46$, $1.0 \leq K_0 \leq 1.15$, $0.0001 \leq S_{f0} \leq 0.001$.

The accuracy (related to measurements) with which the equation is solved (and thus the quality of the proposed mathematical model of the flow of a liquid in the overflow chamber of the investigated side weirs with a throttling pipe)

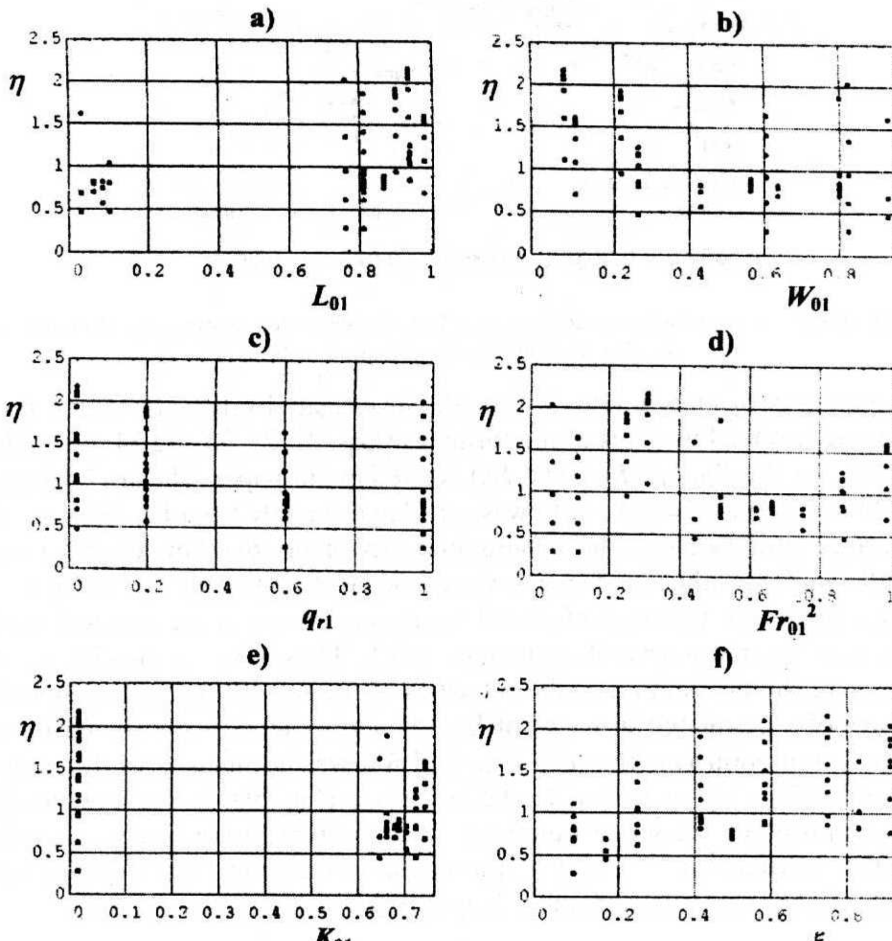

Fig. 5. Distribution of η-values versus parameters of motion L_{01} (a), W_{01} (b), q_{r1} (c), $Fr_{01}^2 (d)$, K_{01} (e) normalized to $< 0.1 >$ and versus parameter ξ (f) (variants 3, 5 and 6)

is illustrated in Fig. 6. The figure shows a diagram of dimensionless water surface height $\zeta(\xi)$ along the longitudinal axis of the overflow chamber ($0 \le \xi \le 1$), calculated numerically (the initial condition takes the form of $\zeta(0) = 1$) for variant 5: U-shaped channel, $L = 1200$ mm, $Q_0 = 33.8$ dm^3/s and $q_r = 0.8$. Figure 6 shows that conformity between calculated (numerically) and measured results (with a depth measuring error) is satisfactory. The water rise curve for the weir, calculated in terms of the mathematical model, describes the hydraulic model measurements within the water-surface height measuring error.

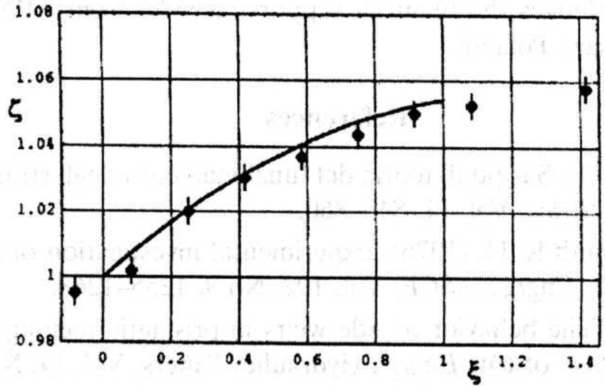

Fig. 6. Dimensionless elevation of water-surface ζ along overflow chamber $0 \le \xi \le 1$, numerically calculated (—) using equation of motion (20) and measured in model ♦ with marked measuring error (variant 5, for $q_r = 0.8$, $L_0 = 4.73$, $P_0 = 0.803$, $W_0 = 0.197$, $Fr_0^2 = 0.112$, $K_0 = 1.14$, $S_{f0} = 0.000578$ and $\mu = 0.552$)

5. Conclusions

Using on the principle of conservation of momentum, a new form of the equation of motion (Eq. 17) which describes the free-surface profiles in the overflow chamber, has been derived. It differs from the available equations of motion in that it incorporates a corrective mass decrement term $(2\beta - k\beta_b)[Q/(gA^2)]dQ/dx$ and a momentum variation term $[Q^2/(gA^2)]d\beta/dx$.

Studies of local velocity distributions in overflow channels and chambers differing in cross-sectional profiles have shown that momentum coefficient β varies markedly along the weir, as regards its value ($1.01 < \beta < 1.6$) and the value of its derivative $d\beta/dx$. And this indicates that the use of the new form of the equation of motion is correct from the point of view of the physics of the phenomenon.

The dimensionless form of the modified differential equation of motion (Eq. 20) describes liquid flow in the overflow chamber of a defined geometry. Equation

20 applies to the hydraulic design of a side weir with a high crest $p > H_{cr}(Q_0)$ and a throttling pipe. Model studies have substantiated the accuracy of Eq. 20 in determining the value of $d\zeta/d\xi$ (and consequently the value of dH/dx which is within the measuring error for height H in physical models). A mathematical model which describes the behavior of such weirs, as well as a numerical procedure enabling their dimensioning, has been developed in a previous study (Kotowski 1998).

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