Archives of Hydro-Engineering and Environmental Mechanics
Vol. 48 (2001), No. 1, pp. 19–33

Initial Boundary Value Problems for Vortex Motion of an Ideal Fluid in Bounded Domains

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(Received December 08, 1999; revised June 26, 2000)

Abstract

The paper deals with the problem of vortex motion of an incompressible perfect fluid in bounded domains. The research is confined to chosen cases of steady velocity fields within rectangular, circular and elliptic regions with rigid boundaries. The solution to the initial-value problem of the fluid flow for the assumed velocity fields is the primary object of this paper. It is demonstrated that individual particles of the fluid have their own periods of motion and thus, one should be careful in describing such problems by means of discrete methods, especially in the Lagrangian variables. The problem discussed has its origin in numerical analysis of water waves by means of the finite difference or the finite element methods.

1. Introduction

In analysis of water gravitational waves we usually assume that the fluid is inviscid and incompressible and the flow irrotational. The main difficulty of the analysis is associated with a solution to the non-linear boundary value problem for the free surface which is a moving boundary of the fluid domain. Similar difficulties emerge in the description of a fluid-structure dynamic interaction through the common boundary of the fluid and structure. Because of the difficulties, we have, as yet, no general solution to these problems. Therefore, for many important cases we are forced to resort to approximate, discrete descriptions of the original task, for example, by means of the finite difference or the finite element method. With these methods a continuum is replaced by a discrete space of nodal points of an assumed net and the problem is reduced to a system of algebraic equations.

The literature of the subject is considerable. The problem of finite element description of incompressible material was discussed by Fried (1974). In order to improve the condition of the stiffness matrix of the considered system, the incompressibility of the material was introduced gradually into the numerical procedure developed in the paper. An important contribution to the problem of incompressible finite elements was supplied by Argyris, Dunne, Angelopoulos and Bichat (1974). In particular, the authors pointed out, that finite elements in which the incompressibility condition is satisfied are especially sensitive to boundary conditions. Gartling and Becker (1976) developed numerical procedures for the analysis of two-dimensional flows of viscous incompressible fluid by the finite element method. The finite element vibration analysis of a fluid-structure system may be found in Kiefling and Feng (1976) where both the kinetic and potential energy of the fluid were expressed as functions of nodal displacements. The formulation has lead to an eigenvalue problem of the system mentioned. The authors found out that the formulation results in spurious modes of vibrations and the number of so-called 'circulation modes' increases as the mesh becomes finer. A similar, displacement method for the analysis of vibrations of a fluid-structure system was developed by Hamdi, Ousset and Verchery (1978). An introduction to numerical treatment of the fluid-structure interaction problems may be found in Zienkiewicz and Bettes (1978).

With respect to the above, the incompressibility of the fluid and lack of shear stress within it are responsible for the condition of the stiffness matrix of the system equations. In particular, in discrete description of the irrotational motion of the incompressible fluid it is difficult to protect the solution against vortical velocity components appearing. Such a situation frequently appears in discrete analysis of water waves, especially with finite amplitudes. Certain information about the phenomenon may be obtained by analysing auxiliary problems of vortex motion of the fluid. It has therefore been found worthwhile to investigate the problem of vortex motion of the inviscid incompressible fluid. In order to simplify the discussion we confine our attention to inverse problems in which the velocity fields in the fluid domains are known in advance. With respect to the Eulerian approach only steady velocity fields are considered. The paper aims to calculate fluid paths and answer the question concerning periods of motion of individual particles. The assumed particular forms of the velocity fields enable us to derive analytical formulae describing the fluid paths. Nevertheless, since we are also interested in calculating the periods of motion for chosen particles, the derivation of the paths is obtained by means of numerical integration of the velocity fields. Thus, the most important object of this paper is to solve the initial-value problems for the steady velocity fields. The results obtained afford better understanding of difficulties in discrete formulation of water waves, especially problems associated with proper description of the free surface. At the same time, the pictures of fluid paths calculated in the present paper inform us about possible deformations of finite elements in the Lagrangian formulation.

2. Formulation of the Problem for a Rectangular Region

Let us consider the plane motion of an ideal fluid in the rectangular region shown in Fig. 1. It is assumed that we are given the following steady velocity field

$$\begin{aligned} u(x,y) &= \frac{dx}{dt} = \sum_n A_n \sin k_n x \cos l_n y, \; n = 1, 2, 3, ... \\ v(x,y) &= \frac{dy}{dt} = -\frac{H}{L}\sum_n A_n \cos k_n x \sin l_n y, \; k_n = \frac{n\pi}{L}, l_n = \frac{n\pi}{H} \end{aligned} \tag{1}$$

where (*x,y*) are Cartesian coordinates and A_n are constants.

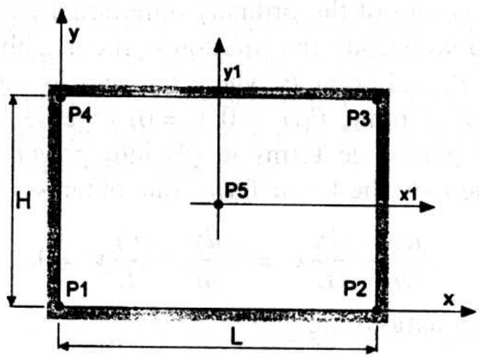

Fig. 1. Rectangular fluid region and coordinate systems

The assumed velocity satisfies the boundary condition that the normal component of the velocity is equal to zero at the domain boundary. Simultaneously, from the equations it follows that the divergence of the velocity field equals zero and thus, the condition of the fluid incompressibility is also satisfied. The trigonometric series (1) represent a set of functions depending on a particular set of the constants A_n. For our purposes it is sufficient to confine our attention to a single component of the series, for instance, the first component

$$\begin{aligned} u(x,y) &= \frac{dx}{dt} = d \sin\frac{\pi x}{L} \cos\frac{\pi y}{H}, \\ v(x,y) &= \frac{dy}{dt} = -d\frac{H}{L}\cos\frac{\pi x}{L} \sin\frac{\pi y}{H}, \end{aligned} \tag{2}$$

where *d* is the constant amplitude of the velocity.

Knowing the velocity it is a simple task to calculate the single component of the fluid vorticity

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = d\frac{\pi}{H}\left[1 + \left({}^{H}\!/_{L}\right)^2\right] \sin\frac{\pi x}{L} \sin\frac{\pi y}{H}. \tag{3}$$

Having the velocity (2) we can calculate the fluid acceleration and, from the momentum equations, the fluid pressure. Assuming, that the y axis is directed vertically upwards, the pressure is expressed as

$$p = \frac{1}{4}\rho d^2 \left[\cos\frac{2\pi x}{L} + \left(\frac{H}{L}\right)^2 \cos\frac{2\pi y}{H}\right] - \rho g y + C, \tag{4}$$

where ρ is the fluid density, g is the gravitational acceleration and C is a constant.

In order to find the fluid paths, it is assumed that at a chosen instant in time, say $t = 0$, the names of the particles are their coordinates x and y. Therefore, in order to calculate the particle positions at an arbitrary instant $t > 0$ we have to solve the non-linear system of the ordinary differential equations (2). In the first step it is necessary to investigate the solution in the neighbourhood of the corner points P_1, P_2, P_3 and P_4 and point P_5 where the velocity equals zero. Thus, let us consider now the corner point $P_1(x = 0, y = 0)$ together with its small vicinity. Expanding the right hand side terms in (2) into power series with respect to x and y and retaining only the linear terms one obtains

$$\frac{dx}{dt} - \frac{d\pi}{L}x = 0, \frac{dy}{dt} + \frac{d\pi}{L}y = 0. \tag{5}$$

The integrals of the equations are

$$x(t) = C_1 \exp\left(\frac{d\pi}{L}t\right), y(t) = C_2 \exp\left(-\frac{d\pi}{L}t\right), \tag{6}$$

where C_1 and C_2 are constants of the solutions.
Elimination of the time factor from the equations provides the trajectory equation

$$x \cdot y = \text{const.} \tag{7}$$

It can be seen that the corner point P_1 is an isolated unstable stagnation saddle point. Similar results and conclusions hold for the remaining points P_2, P_3 and P_4. In order to examine the solution in the vicinity of the point P_5 it is convenient to shift the coordinate axes to this point. Denoting the new axes as x_1 and y_1 and applying the afore-mentioned procedure to the neighbourhood of the point, the following equations are derived

$$\frac{dx_1}{dt} = -\frac{d\pi}{H}y_1, \frac{dy_1}{dt} = \frac{d\pi}{H}\left(\frac{H}{L}\right)^2 x_1, \tag{8}$$

where $x = \frac{L}{2} + x_1$ and $y = \frac{H}{2} + y_1$.
The solutions of the equations may be written in the following form

$$\begin{aligned} x_1 &= A\cos rt + B\sin rt, \\ y_1 &= \frac{H}{L}(A\sin rt - B\cos rt), \end{aligned} \tag{9}$$

where: $r = d\pi/L$ and A and B are constants.
Elimination of the time factor from the relations gives the equation of ellipse

$$\left(\frac{x_1}{L}\right)^2 + \left(\frac{y_1}{H}\right)^2 = \text{const.} \tag{10}$$

It is seen that P_5 is the neutrally stable stagnation point. Fluid particles in the small vicinity of the point (small values of $|x_1| > 0$and $|y_1| >0$ move along elliptic curves around this point and thus, this point is called a centre (Kaplan, 1958). With respect to the above results, we shall confine our attention to a solution of the non-linear differential equations (2) in the fluid domain without these singular points. To do this we apply a discrete numerical integration procedure in the time domain. In the integration, it is assumed, that the continuous parameter (time) t is substituted with a sequence of the discrete time steps: $0, \Delta t, 2\Delta t, 3\Delta t$. Denoting by n the level of time ($t_n = n \cdot \Delta t, n = 0, 1, 2, \ldots$), we can write the discrete representations of (2) in the following form

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\Delta t d \sin\frac{\pi x_{n+1}}{L}\cos\frac{\pi y_{n+1}}{H} + u_n, \\ y_{n+1} &= -\frac{1}{2}\Delta t d \frac{H}{L}\cos\frac{\pi x_{n+1}}{L}\sin\frac{\pi y_{n+1}}{H} + v_n, \end{aligned} \tag{11}$$

where

$$\begin{aligned} u_n &= x_n + \frac{1}{2}\Delta t d \sin\frac{\pi x_n}{L}\cos\frac{\pi y_n}{H}, \\ v_n &= y_n - \frac{1}{2}\Delta t d \frac{H}{L}\cos\frac{\pi x_n}{L}\sin\frac{\pi y_n}{H}. \end{aligned} \tag{12}$$

In order to solve equations (11) we make use of an iteration procedure at each level of time. If r is the number of iterations, then the numerical procedure reads

$$\begin{aligned} x_{n+1}^{(r+1)} &= \frac{1}{2}\Delta t d \sin\frac{\pi x_{n+1}^{(r)}}{L}\cos\frac{\pi y_{n+1}^{(r)}}{H} + u_n, \\ y_{n+1}^{(r+1)} &= -\frac{1}{2}\Delta t d \frac{H}{L}\cos\frac{\pi x_{n+1}^{(r)}}{L}\sin\frac{\pi y_{n+1}^{(r)}}{H} + v_n. \end{aligned} \tag{13}$$

With respect to the equations, the time step Δt should be small enough to protect the stability of the procedure mentioned. The stability is preserved if the following inequality holds (Björck, Dahlquist 1983)

$$\rho(\mathbf{A}) = \max|\lambda_i(\mathbf{A})| < 1, \ \ 1 \le i \le 2, \tag{14}$$

where $\rho(\mathbf{A})$ is the spectral radius and λ_i are eigenvalues of the matrix $\mathbf{A}$ defined as

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial x_{n+1}^{(r+1)}}{\partial x_{n+1}^{(r)}} & \dfrac{\partial x_{n+1}^{(r+1)}}{\partial y_{n+1}^{(r)}} \\ \dfrac{\partial y_{n+1}^{(r+1)}}{\partial x_{n+1}^{(r)}} & \dfrac{\partial y_{n+1}^{(r+1)}}{\partial y_{n+1}^{(r)}} \end{bmatrix}. \tag{15}$$

From substitution of (13) into (15) it follows that

$$\mathbf{A} = \frac{1}{2}\Delta t d \begin{bmatrix} \dfrac{\pi}{L}\cos\dfrac{\pi x_{n+1}^{(r)}}{L}\cos\dfrac{\pi y_{n+1}^{(r)}}{H} - \dfrac{\pi}{H}\sin\dfrac{\pi x_{n+1}^{(r)}}{L}\sin\dfrac{\pi y_{n+1}^{(r)}}{H} \\ \dfrac{\pi H}{L^2}\sin\dfrac{\pi x_{n+1}^{(r)}}{L}\sin\dfrac{\pi y_{n+1}^{(r)}}{H} - \dfrac{\pi}{L}\cos\dfrac{\pi x_{n+1}^{(r)}}{L}\cos\dfrac{\pi y_{n+1}^{(r)}}{H} \end{bmatrix}. \tag{16}$$

It is a simple task to calculate the eigenvalues of the matrix and finally, the relation (14) is transformed into the following form

$$\rho(\mathbf{A}) = \frac{1}{2}\Delta t d\frac{\pi}{L}\max\left|\sqrt{\left(\cos\frac{\pi x_{n+1}^{(r)}}{L}\right)^2 - \left(\sin\frac{\pi y_{n+1}^{(r)}}{H}\right)^2}\right| < 1. \tag{17}$$

The stability condition of the numerical procedure leads to the inequality:

$$\Delta t < \frac{2L}{\pi d}, \tag{18}$$

which is not a strong limitation for the time step.

Following the equations derived, numerical calculations have been performed. Some of the results obtained in this way are presented in the subsequent figures 2, 3 and 4. In Fig. 2 the trajectories of chosen particles are depicted. Each of the particles has its own period of motion. The last feature is illustrated in Figs. 3 and 4 where the plots show the horizontal and vertical components of displacements of the particles as functions of time. It can be seen from the plots that the distance between any two neighbouring particles may grow in time to a diameter of the corresponding path.

3. Fluid Paths in Circular and Elliptic Regions

To learn more about the subject, solutions to the flows in circular and elliptic regions are presented. Now we have smooth boundaries of the fluid domains. In accordance with the shape of the regions we apply the polar and elliptic systems of coordinates, respectively. Let us consider first the circular area shown in Fig. 5a. With respect to the polar coordinates r and φ, the velocity vector is expressed as

$$\vec{\mathbf{v}} = \dot{r}\cdot\vec{\mathbf{a}}_1 + \dot{\varphi}\cdot\vec{\mathbf{a}}_2, \tag{19}$$

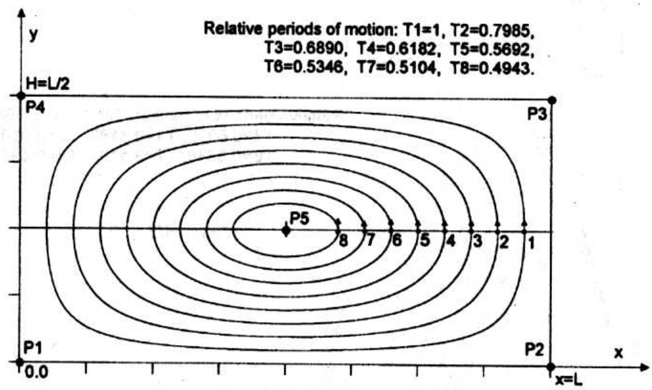

Fig. 2. Particle paths in rectangular region

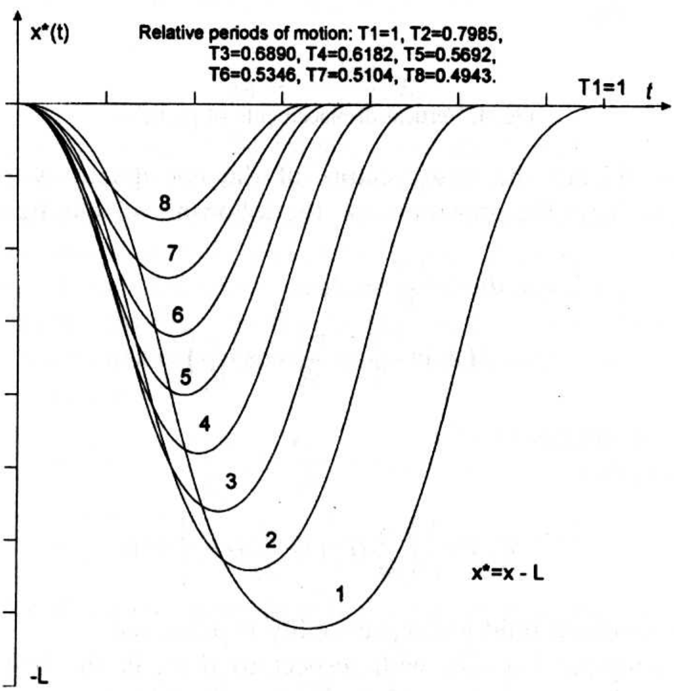

Fig. 3. Horizontal displacements of particles

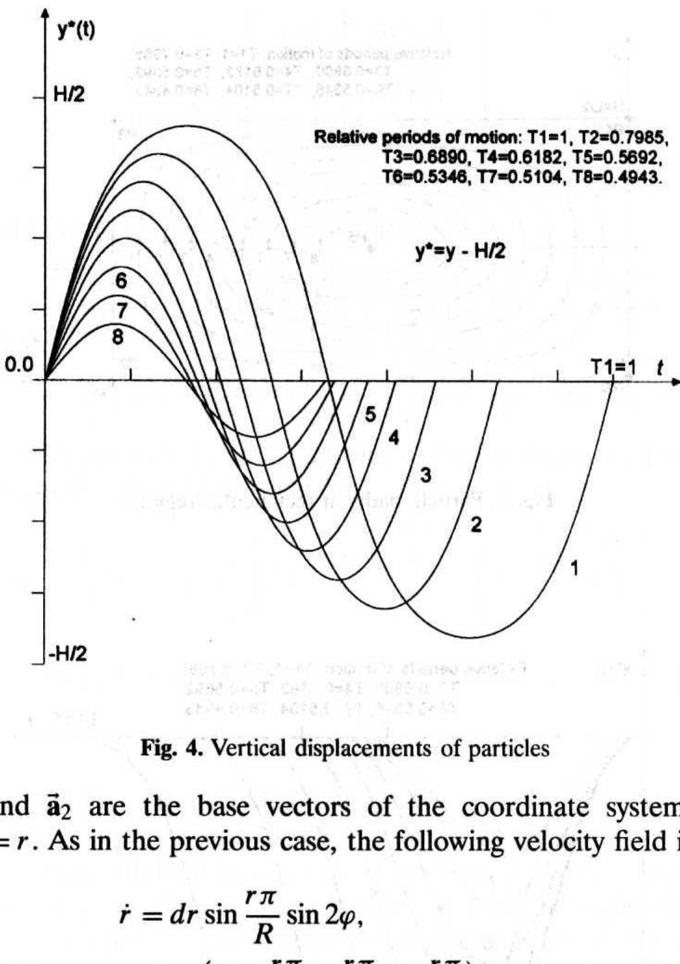

Fig. 4. Vertical displacements of particles

where $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ are the base vectors of the coordinate system and $|\vec{\mathbf{a}}_1| = 1$ and $|\vec{\mathbf{a}}_2| = r$. As in the previous case, the following velocity field is assumed

$$\begin{aligned} \dot{r} &= dr \sin\frac{r\pi}{R} \sin 2\varphi, \\ \dot{\varphi} &= d\left(2\sin\frac{r\pi}{R} + \frac{r\pi}{R}\cos\frac{r\pi}{R}\right)\cos^2\varphi, \end{aligned} \tag{20}$$

where d has the dimension s^{-1}.
One can check, that

$$div\,\vec{\mathbf{v}} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\dot{r}) + \frac{\partial}{\partial \varphi}(r\dot{\varphi})\right] = 0 \tag{21}$$

and thus, the assumed fluid incompressibility is preserved.
In order to integrate Eq (20) with respect to time, in the first step we have to investigate solutions of the equations in small vicinities of stagnation points at which the velocity is equal to zero. It can be seen that at points $\varphi = 0$ and $\varphi = \pi/2$ the radial component of the velocity equals zero. At the same time, from

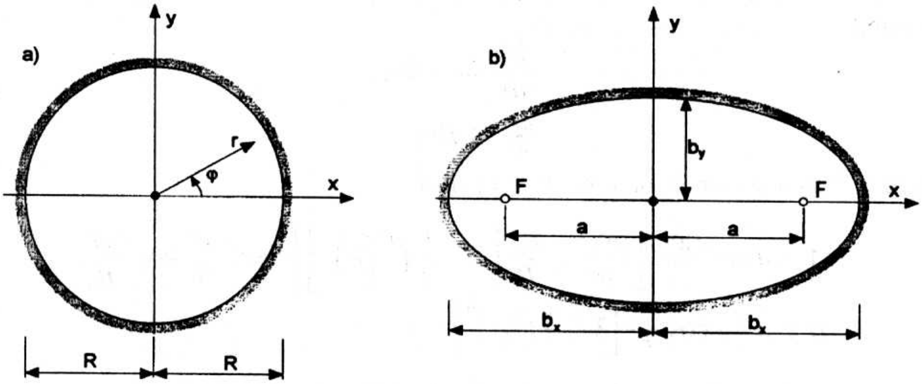

Fig. 5. Circular and elliptic regions of fluid

the second equation of (20) it follows that at point $r = r_0$ for which

$$2\sin\frac{r_0\pi}{R} + \frac{r_0\pi}{R}\cos\frac{r_0\pi}{R} = 0, \tag{22}$$

the second component of the velocity is also equal to zero. A numerical solution to this equation gives

$$r_0 \cong \frac{2.28893}{\pi}R. \tag{23}$$

The point P $(\varphi = 0, r = r_0)$ is the stagnation point. Therefore, because of the symmetry, we have also the second singular point $Q(r = r_0, \varphi = \pi)$ of the same features. Apart from these isolated points we have zero velocity at points $\varphi = \pi/2$ and thus the vertical diameter of the circle is the stagnation line. As in the previous cases we examine the solution within small neighbourhoods of the singular points. Thus let us consider now the fluid flow in the small vicinity of point P. For our purposes it is convenient to calculate the velocity components with respect to the Cartesian coordinate axes

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = -\frac{dr\pi}{R}\cos\frac{r\pi}{R}r\sin\varphi\cos^2\varphi, \\ \dot{y} &= \frac{dy}{dt} = dr\cos\varphi(2\sin\frac{r\pi}{R} + \frac{r\pi}{R}\cos\frac{r\pi}{R}\cos^2\varphi). \end{aligned} \tag{24}$$

In the small vicinity of the point $(\varphi = 0, r = r_0)$ we may use the approximate relations

$$\begin{aligned} \frac{dx}{dt} &\cong -d\left(\frac{r\pi}{R}\cos\frac{r\pi}{R}\right)y, \\ \frac{dy}{dt} &\cong d\left(2\sin\frac{r\pi}{R} + \frac{r\pi}{R}\cos\frac{r\pi}{R}\right)x. \end{aligned} \tag{25}$$

Shifting the Cartesian coordinate axes to this point, the following equations are obtained

$$\frac{dx_1}{dt} \cong -Ay_1, \qquad \frac{dy_1}{dt} \cong Bx_1, \tag{26}$$

where $r \cong r_0 + \delta \cong x \cong r_0 + x_1$, $y = y_1$ and

$$A = d\cos\frac{r_0\pi}{R}\left\{\frac{r_0\pi}{R} + \frac{\delta\pi}{R}\left[1 + \frac{1}{2}\left(\frac{r_0\pi}{R}\right)^2\right]\right\} \cong d\frac{r_0\pi}{R}\cos\frac{r_0\pi}{R}, \qquad B = d\frac{r_0\pi}{R}\cos\frac{r_0\pi}{R}\left[3 + \frac{1}{2}\left(\frac{r_0\pi}{R}\right)^2\right]. \tag{27}$$

The solution Eqs (26) can be written in the following form

$$x_1(t) = C_1\cos\beta t + C_2\sin\beta t, \qquad y_1(t) = \sqrt{\frac{B}{A}}\left(C_1\sin\beta t - C_2\cos\beta t\right), \tag{28}$$

where $\beta^2 = AB > 0$, and C_1 and C_2 are constants.
Elimination of the time factor from the equations leads to the equation of ellipse. It means that this isolated stagnation point is neutrally stable and the flow around it is similar to the one in the vicinity of P_5 in Fig. 2. Examination of the solution in the vicinity of the vertical diameter of the circle leads to the conclusion that the points of the line are also neutrally stable. The fluid paths do not touch this line. The solution of the non-linear equations (20) within the circle domain, except for the singular points, may be obtained with the help of numerical integration of the equations. The numerical procedure of the solution is similar to that applied to the rectangular region. In order to establish the stability condition for the procedure mentioned, one may apply the Gerschgorin theorem (Bodewig 1959) which enables us to estimate the bounds for the spectral radius corresponding to the relevant matrix **A** of the iteration procedure applied. In the case discussed, the Gerschgorin theorem leads to the following restriction for the time step:

$$\Delta t < \frac{2}{d\cdot(5+2\pi)}. \tag{29}$$

The inequality obtained may be improved by more detailed examination of the eigenvalues of the matrix **A**. Numerical integration of Eqs. (20) gives the fluid paths of chosen particles. Some of the paths obtained in this way are depicted in Fig. 6. Each of the marked particles has its own period of motion.

Let us consider now the fluid motion within the elliptic region shown in Fig. 5b. In this case it is convenient to introduce the elliptical system of coordinates $(\eta, \varphi))$ and write

$$\vec{\mathbf{r}} = x\cdot\vec{\mathbf{e}}_1 + y\cdot\vec{\mathbf{e}}_2 = a\left(\cosh\eta\cos\varphi\vec{\mathbf{e}}_1 + \sinh\eta\sin\varphi\vec{\mathbf{e}}_2\right), \tag{30}$$

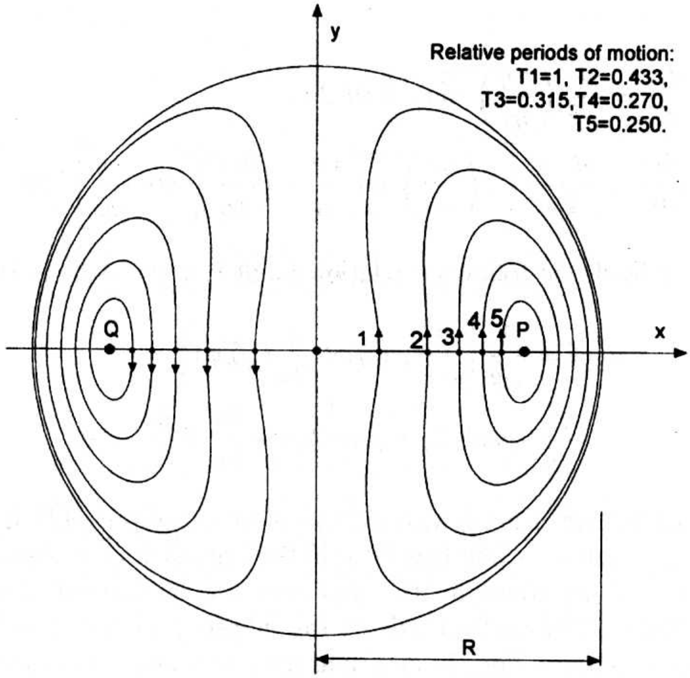

Fig. 6. Particle paths in circular region

where $\eta = const.$ defines an ellipse and φ =const. describes a hyperbola. With respect to the coordinate system introduced, the boundary of the region is defined by the ellipse $\eta = \eta_0$. The base vectors $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ of the system are

$$\begin{aligned} \vec{\mathbf{a}}_1 &= a\,(\sinh\eta\cos\varphi\vec{\mathbf{e}}_1 + \cosh\eta\sin\varphi\vec{\mathbf{e}}_2)\,, \\ \vec{\mathbf{a}}_2 &= a\,(-\cosh\eta\sin\varphi\vec{\mathbf{e}}_1 + \sinh\eta\cos\varphi\vec{\mathbf{e}}_2)\,. \end{aligned} \tag{31}$$

On the basis of the equations the covariant components of the metric tensor are obtained

$$g_{11} = g_{22} = a^2\left[(\cosh\eta)^2 - (\cos\varphi)^2\right],\; g_{12} = g_{21} = 0 \tag{32}$$

The determinant of the tensor is

$$g = a^4\left[(\cosh\eta)^2 - (\cos\varphi)^2\right]^2. \tag{33}$$

The velocity vector for this case is expressed in the form

$$\vec{\mathbf{v}} = \dot\eta\vec{\mathbf{a}}_1 + \dot\varphi\vec{\mathbf{a}}_2, \tag{34}$$

where $|\vec{\mathbf{a}}_1| = |\vec{\mathbf{a}}_2| = \sqrt[4]{g}$.
In the case discussed the velocity components in the equation are assumed as

follows

$$\dot{\eta} = \frac{d\eta}{dt} = \frac{d}{\sqrt{g}}\left(\frac{\pi\eta}{\eta_0}\right)^2 \sin\frac{\pi\eta}{\eta_0}\sin 2\varphi,$$
$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{d}{\sqrt{g}}\frac{\pi}{\eta_0}\left[2\left(\frac{\pi\eta}{\eta_0}\right)\sin\frac{\pi\eta}{\eta_0} + \left(\frac{\pi\eta}{\eta_0}\right)^2\cos\frac{\pi\eta}{\eta_0}\right](\cos\varphi)^2. \tag{35}$$

For the velocity field, the following relation holds (Green, Adkins 1964)

$$div\,\vec{v} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial\eta}(\sqrt{g}\dot{\eta}) + \frac{\partial}{\partial\varphi}(\sqrt{g}\dot{\varphi})\right] =$$
$$= \frac{a^2}{\sqrt{g}}(\dot{\eta}\sinh 2\eta + \dot{\varphi}\sin 2\varphi) + \frac{\partial\dot{\eta}}{\partial\eta} + \frac{\partial\dot{\varphi}}{\partial\varphi} = 0. \tag{36}$$

This means that we are dealing with motion of an incompressible fluid. One can check, that in the limit $\eta \to 0$ in Eqs (35) (in the Cartesian co-ordinates $0 \le x \le a$ and $y = 0$) the velocity components equal zero and thus, points $\eta = 0$ form the stagnation segment. The vertical axis of the ellipse $\varphi = \pi/2$ ($x = 0$, $-b \le y \le -b$) is also the stagnation line. In addition, for $\varphi = 0$ and $\eta = \eta_s$ for which

$$2\sin\frac{\pi\eta_s}{\eta_0} + \frac{\pi\eta_s}{\eta_0}\cos\frac{\pi\eta_s}{\eta_0} = 0 \tag{37}$$

we have the stagnation point $\eta_s \cong 2.28893 \cdot \eta_0/\pi$. In accordance with the Cartesian coordinate system, for the stagnation point the following inequality holds

$$a < x_s < b_x = a\cosh\eta_0. \tag{38}$$

As in the previous case, we have two symmetrical stagnation points: $P(\eta = \eta_s, \varphi = 0)$ and $Q(\eta = \eta_s, \varphi = \pi)$. To examine the solution within the small vicinity of P it is convenient to substitute $\eta = \eta_s + \varepsilon$ into Eqs (35). Then, expanding the right hand side terms of the equations into power series with respect to small quantities ε and φ and retaining only the linear terms, the following is obtained

$$\frac{d\varepsilon}{dt} - A\varphi = 0,\ \frac{d\varphi}{dt} + B\varepsilon = 0, \tag{39}$$

where

$$A = A(\eta_s) \cong \frac{2d}{a^2\sinh^2\eta_s}\left(\frac{\pi\eta_s}{\eta_0}\right)^2\sin\frac{\pi\eta_s}{\eta_0} > 0,$$
$$B = B(\eta_s) \cong -\frac{d}{a^2\sinh^2\eta_s}\left(\frac{\pi}{\eta_0}\right)^2\frac{\pi\eta_s}{\eta_0}\left[3+\frac{1}{2}\left(\frac{\pi\eta_s}{\eta_0}\right)^2\right]\cos\frac{\pi\eta_s}{\eta_0} > 0. \tag{40}$$

The solutions of the differential equations (39) may be expressed as follows

$$\varepsilon(t) = C_1 \cos\beta t + C_2 \sin\beta t,$$
$$\varphi(t) = -\sqrt{\tfrac{B}{A}}\,(C_1 \sin\beta t - C_2 \cos\beta t)\,, \tag{41}$$

where C_1 and C_2 are constants and $\beta^2 = AB > 0$.
Knowing the Cartesian coordinates: $x_s = a\cosh\eta_s$ and $y_s = 0$ of the point P and shifting the coordinate axes to this point ($x_1 = x - x_s, y_1 = y$) we obtain

$$x_1(t) = x - x_s \cong a\sinh\eta_s\varepsilon(t),$$
$$y_1(t) \cong a\sinh\eta_s\varphi(t). \tag{42}$$

Elimination of the time terms in Eqs (41) and (42) leads to an ellipse equation. As in the previous cases this point is the neutral stability stagnation point. The same character of the solution is valid for point Q.
In order to perform numerical integration of Eqs (35) within the elliptic region, except for the stagnation points, it is convenient to introduce the new variable

$$\xi = \pi\eta/\eta_0 \tag{43}$$

and rewrite the equations in the following form

$$\dot{\xi} = \frac{d\xi}{dt} = \frac{\pi d}{\eta_0\sqrt{g}}\xi^2 \sin\xi \sin 2\varphi,$$
$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{\pi d}{\eta_0\sqrt{g}}(2\xi\sin\xi + \xi^2\cos\xi)\cos^2\varphi. \tag{44}$$

In numerical integration of the equations in the time domain we apply a procedure similar to that used in the case of the rectangular region. In order to ensure the stability of the procedure mentioned one can apply the Gerschgorin theorem which enable us to calculate the relevant bound for the time step. Numerical integration performed for this case gives paths of fluid particles. Some of the results obtained in computations are illustrated in Fig. 7, where the chosen paths are plotted. As in previous cases, the individual particles of fluid have their own periods of motion.

4. Concluding Remarks

In the preceding sections numerical solutions to the plane, initial bundary value problems of the steady vortex motion of the fluid have been presented. Numerical calculations have shown that for the considered cases of closed fluid paths, the individual particles of fluid have their own periods of motion. It means that any two particles which at a certain moment of time are only at a small, but finite

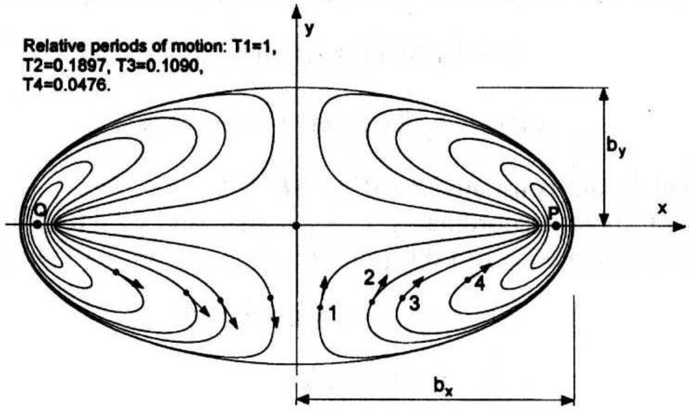

Fig. 7. Particle paths in eliptic region

distance from each other, may depart to a relatively large distance from one another at a later moment in time. The last feature may be important in discrete description of the phenomenon. For instance, in calculating the fluid motion a natural way is to choose some material particles as nodal points of finite elements and follow the points in the time domain. It may happen, however, that the displacements of the nodal points will lead to such gross distortion of the elements that it will be impossible to get a unique numerical solution. Another difficulty may appear in the neighbourhood of the singular points where the concentration of the fluid paths take place. In discrete integration of such cases in the time domain it may occur that a nodal material point of a moving net may jump from one path to another. Such cases may appear in numerical calculations of water gravitational waves, especially with finite amplitude, described in the Lagrangian variables. Therefore, in discrete description of water waves a better way is to use the Eulerian formulation corresponding to nodal points fixed in space.

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