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Flow Friction Forces Effect on Barotropic Wave Attenuation in River Mouth

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Abstract

The paper presents an analysis of barotropic wave attenuation in the river outlet distance due to friction forces at the bottom opposing the flow. The analysis concerns various types of barotropic waves and different velocities of wave propagation. As the results in comes that barotropic waves produce significant changes in free water surface elevation. The changes depend on waves properties but it seems to be that the most significant factor is the so called incidence angle which describes way of increment of barotropic pressure.

Notations

c	–	wave propagation velocity, [m/s],
C_f	–	Chezy coefficient, [m$^{1/2}$/s],
g	–	earth gravity, [m/s^2],
$h(x,t)$	–	dimensionless change of water depth, [m],
$\bar{h}(\xi,s)$	–	Laplace transformation of the h function,
$H(x,t)$	–	water depth, [m],
H_0	–	reference depth for $\Delta P_a = 0$, [m],
I_s	–	water surface slope,
I_0, I	–	energy line slopes for undisturbed and disturbed flow,
L	–	horizontal extent of the change of atmospheric pressure, [m],
m	–	coefficient in the wave formulae), [1/s^2] Eq. (2),
$P_a(x,t)$	–	atmospheric pressure distribution, [hPa] = [kg/(ms^2)],
$P_{a,0}$	–	reference atmospheric pressure, [hPa],
$P_a^*(x,t) = P_a(x,t)/P_{a,0}$	–	relative atmospheric pressure at water surface,

$R_z(x)$	–	water surface elevation related to given reference level (for Odra river outlet this is the level –500 cm NN – Amsterdam), [m],
T_0	–	time scale of barotropic wave attenuation due to flow friction, [hour],
$u(x,t)$	–	dimensionless change of water flow velocity (v/v_0),
$\bar{u}(\xi,t)$	–	Laplace transformation of the u function,
$v(x,t)$	–	water flow velocity, [m/s],
v_0	–	reference water flow velocity, [m/s],
x, t	–	dimensional variables (distance [m] and time [s]),
X	–	water particle acceleration due to horizontal gravity component, [m/s^2],
$\alpha(x,t,\Delta t)$	–	function describing the incidence angle and gradually varied atmospheric pressure change,
$\delta(x)$	–	position of elementary water particle in crossection x, [m],
$\Delta h(x,t)$	–	change of water depth, [m],
ΔH_{max}	–	maximal change of water depth during the passage of the eye of low atmospheric pressure, [m],
ΔP_a	–	maximal difference of atmospheric pressure over the cyclone pass, [hPa],
Δt	–	time range of atmospheric pressure change defined by incidence angle,[hour]
ΔT	–	time scale of the atmospheric pressure change, [hour],
$\Delta v(x,t)$	–	change of water flow velocity, [m/s],
ΔZ	–	amplitude of water levels relevant to the given time scale ΔT, [m],
$\eta(t - x/c)$	–	step function describing rapid atmospheric pressure change,
$\vartheta(\xi,\tau)$	–	dimensionless velocity changes function $(v^2 - v_0^2)/v_0^2$,
$\vartheta_{inci}(T_0, \Delta t)$	–	maximal value of function ϑ obtained using incidence angle model,
$\vartheta_{step}(T_0, \Delta t)$	–	value of ϑ calculated from step function model at the same point,
$\bar{\vartheta}(\xi, s)$	–	Laplace transformation of function ϑ,
ξ, τ	–	dimensionless variables for x and t: $\xi = x/(v_0^2/g)$, $\tau = t/(v_0/g)$,
ρ	–	water density, [kg/m^3],
$\mathcal{L}, \mathcal{L}^{-1}$	–	direct and inverse Laplace transformation operators,

$\bar{u}, \bar{h}$ – general description of Laplace transformation:

$$\bar{u} = \mathcal{L}\{u(\xi, \tau)\} = \int_0^\infty u(\xi, \tau) e^{-s\tau} d\tau,$$

$$\bar{h} = \mathcal{L}\{h(x, t)\} = \int_0^\infty h(x, t) e^{-st} dt, \quad s = \tau + i\varpi.$$

1. Introduction

Measuring data in the lower Odra river network and Świna Strait afford several examples of significant increase in water level during the passage of low atmospheric pressure area over the region. These water level changes known as barotropic waves propagate upstream the river network. The recorded propagation velocity is about 10 m/s with height reaching 40–50 cm but as far as Widuchowa in the upper part of the river system, the waves are gradually attenuated. Field measurements of this kind of waves give approximated estimation for their time scale ΔT and water level changes ΔZ. On Fig. 1 several examples of such waves are shown, distinguished through the two parameters ΔT and ΔP.

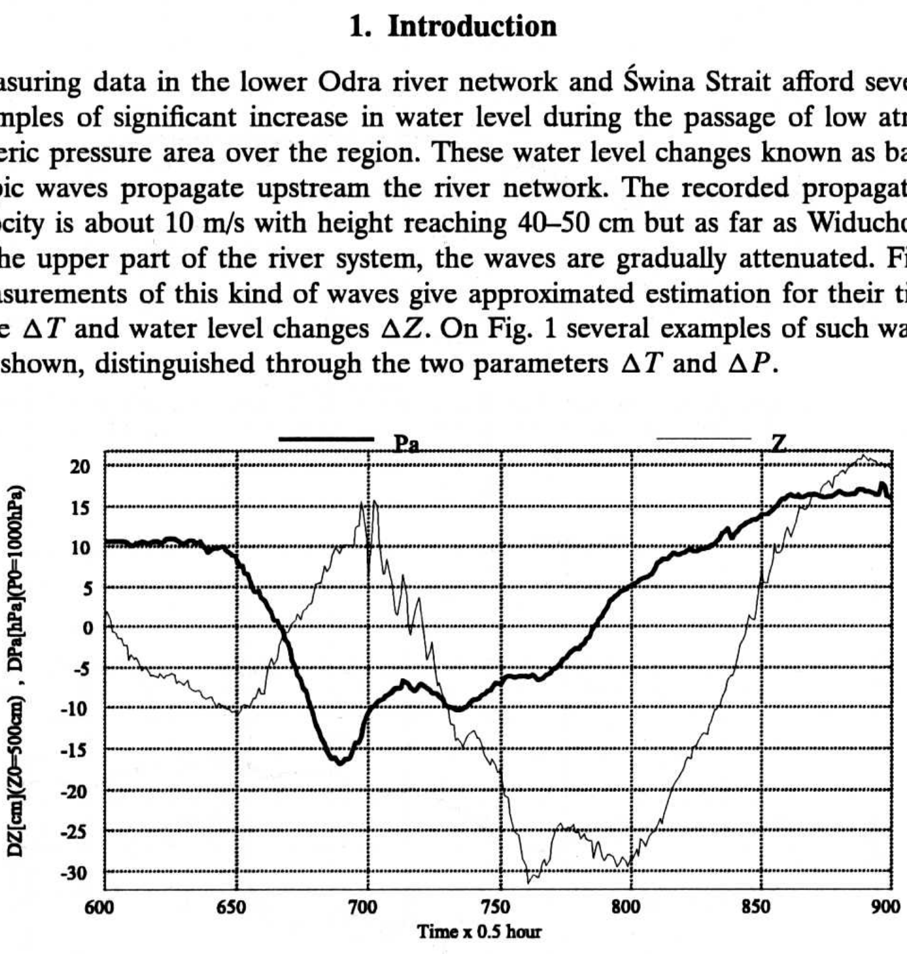

Fig. 1a. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 13.09–19.09 1994 at Police. Time scale $\Delta T = 100$ hours, $\Delta Z = 45$ cm, $\Delta P_a = 30$ hPa

Time scale ΔT in above examples varies from 30 to 150 hours, changes ΔZ of water level fluctuate in the range from 20 to 75 cm, and atmospheric pressure increment or decrement ΔP_a varies from 10 to 30 hPa. Approximated formulas for quantitative description of water level changes dependent on atmospheric pressure changes have been considered by authors in former papers (Meyer, Ewertowski 1996a, b). Taking into account and comparing these results with field measurements, one can notice that increment of water level ΔZ in some cases is considerably bigger than that one calculated from mentioned formulas. It results from the fact that theoretical consideration assumes pressure changes as a step function.

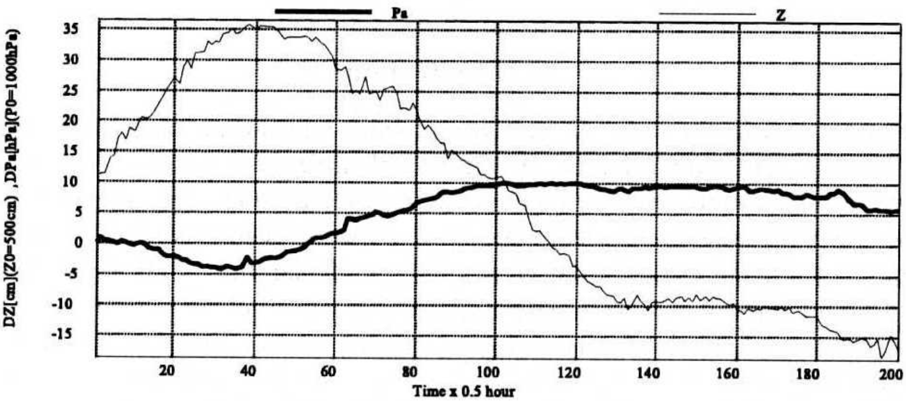

Fig. 1b. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 19.05–23.05 1994 at Police. Time scale $\Delta T = 50$ hours, $\Delta Z = 45$ cm, $\Delta P_a = 15$ hPa

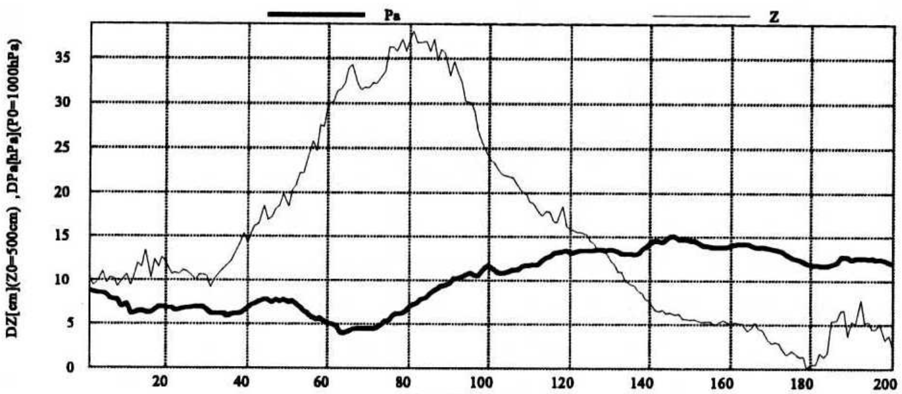

Fig. 1c. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 01.06–05.06 1995 at Police. Time scale $\Delta T = 40$ hours, $\Delta Z = 25$ cm, $\Delta P_a = 11$ hPa

Another important factor, which plays a significant role in shaping of barotropic wave, can be flow friction. In previous papers (Meyer, Ewertowski 1996a, b) the effect of friction forces has been neglected. Shapes of crests observed in the falling phase of barotropic waves indicate that friction forces have significant

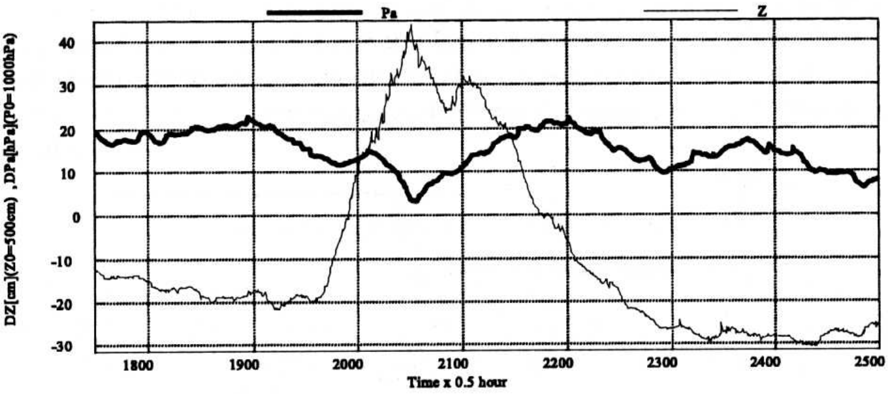

Fig. 1d. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 06.04–22.04 1996 at Police. Time scale $\Delta T = 150$ hours, $\Delta Z = 75$ cm, $\Delta P_a = 18$ hPa

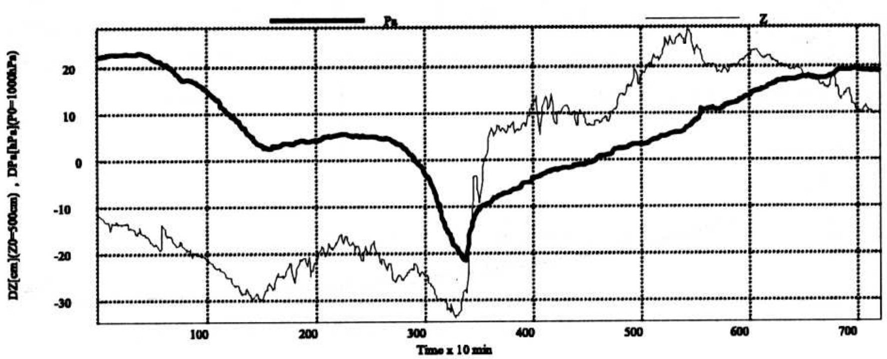

Fig. 1e. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 26.03–30.03 1997 at Police. Time scale $\Delta T = 35$ hours, $\Delta Z = 50$ cm, $\Delta P_a = 25$ hPa

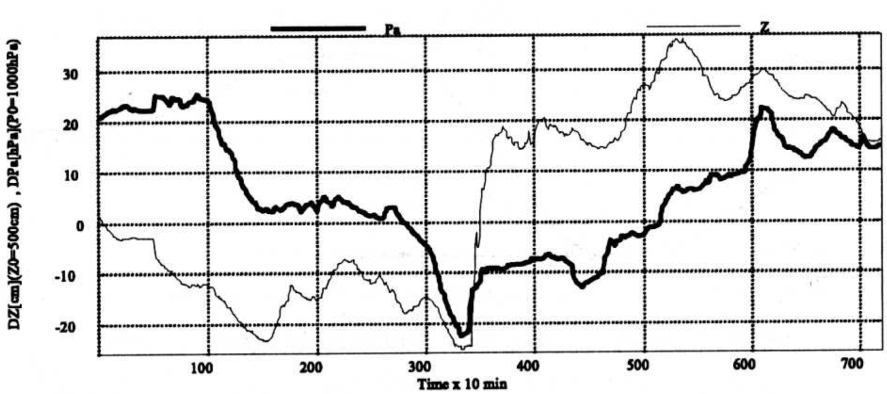

Fig. 1f. Atmospheric pressure (around 1000 hPa) and water level around the reference 500 cm in period 26.03–30.03 1997 at Szczecin. Time scale $\Delta T = 30$ hours, $\Delta Z = 45$ cm, $\Delta P_a = 25$ hPa

meaning and should be considered in theoretical models. This should be combined with the function of pressure distribution in time and space. In the previous papers (Meyer, Ewertowski 1996a) a simple model of the atmospheric pressure changes with the step function (Fig. 2a) was considered.

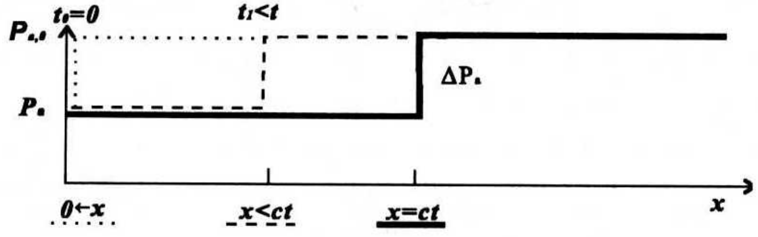

Fig. 2a. Step-wise model of the atmospheric pressure distribution function

For this function the solution for water level changes and flow velocity has been obtained in straight channel. Another model of atmospheric pressure distribution over a flowing river, which has been considered, was that one shown in Fig. 2b (Meyer, Ewertowski 1996b)

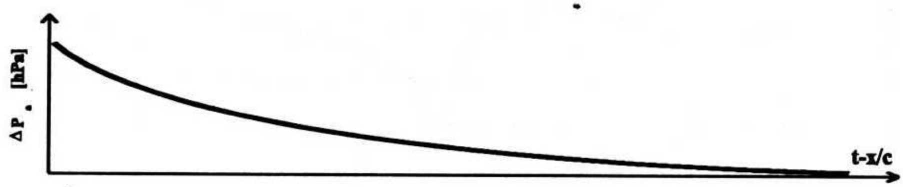

Fig. 2b. Exponent model of atmospheric pressure distribution

and in Fig. 2c (Meyer, Ewertowski 1996c):

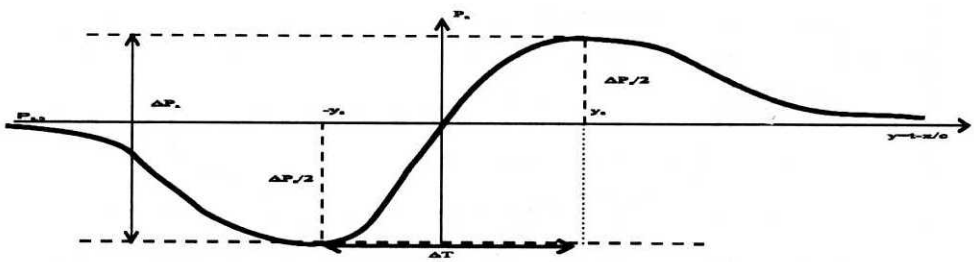

Fig. 2c. An example of the general function of atmospheric pressure

It is a matter of discussion how exactly the theoretical solution of water level and flow velocity changes obtained for these solutions fit the real observations. In the previous paper by Meyer, Ewertowski (1997) it was shown that the best fit was given when using the exponential model. However, differences between calculations and observations are not always acceptable. For this reason the authors decided to check the influence of flow friction on generation, shaping and propagating of barotropic waves caused by low atmospheric pressure passing over the channel.

2. Mathematical Description of the Phenomenon

It has been assumed here that the only important forces acting on water flow in a straight rectangular channel are those of: inertia, friction, acceleration due to gravity and atmospheric pressure. The basic schema of forces acting on water volume are shown in Fig. 3.

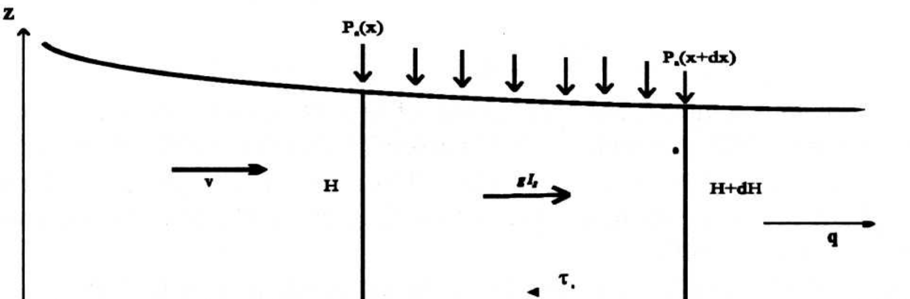

Fig. 3. Schema of the forces acting on water flow in test channel

Undisturbed uniform flow of water element bounded by abscissas x and $x + dx$ is described using a one-dimensional equation of motion $\frac{\partial v_0}{\partial t} + v_0 \frac{\partial v_0}{\partial x} = g\,I_s - g\frac{v_0^2}{C_f^2 H}$. Next, the disturbance of water flow caused by change of atmospheric pressure is: $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = g\,I_s - g\frac{v^2}{C_f^2 H} - \frac{1}{\rho}\frac{\partial P_a}{\partial x}$ and subtracting them, one can obtain the following equation:

$$\frac{\partial (v - v_0)}{\partial t} + \frac{\partial (v^2 - v_0^2)}{\partial x} = -g\frac{v^2 - v_0^2}{C_f^2 H} - \frac{1}{\rho}\frac{\partial P_a}{\partial x} \tag{1}$$

where bottom stresses in both cases are approximated by Chezy formula:

$\frac{\tau_{b,0}}{\rho H} = g\frac{v_0^2}{C_f^2 H} = g\,I_0$ – justified for undisturbed uniform flow in a flat-bed rectangular channel,

$\frac{\tau_{b,0}}{\rho H} = g \frac{v^2}{C_f^2 H}$ – results from flow velocity change caused by the atmospheric pressure.

Here is made an the assumption of small disturbances propagation, and thus, water level slopes ($g I_s$) change slighty. That assumption allows also to linearize Eq. (1) by introducing the following simplification:

$$\frac{\partial (v - v_0)}{\partial t} \approx \frac{v_0}{2} \frac{\partial}{\partial t} \left(\frac{\partial v^2 - v_0^2}{\partial v_0^2} \right) \tag{2}$$

For further analysis, dimensionless independent variables τ, ξ and a unknown ϑ have been used

$$\tau = \frac{t}{v_0/g}, \quad \xi \frac{x}{v_0^2/g}, \quad \vartheta = \frac{v^2 - v_0^2}{v_0^2} \tag{3}$$

where ϑ denotes dimensionless difference of instant square velocity and steady square velocity. This approach is justified assuming that the velocity changes $\Delta v = v - v_0$ are small and backwater currents will not appear. For greater changes of atmospheric pressure, which can reverse flow direction, numerical methods should be used to obtain solution.

Using (3), the friction term in Eq. (1) can be expressed now as follow:

$$g \frac{v^2 - v_0^2}{C_f^2 H} = g \frac{v_0^2}{C_f^2 H} \frac{v^2 - v_0^2}{v_0^2} = g I_0 \vartheta$$

As initial condition there was assumed the following equations:

$$t = 0, \quad \vartheta(0_+) = \frac{v^2(0) - v_0^2}{v_0^2} = 0 \tag{4}$$

We assume that change of atmospheric pressure is similar to the wave of disturbance which propagates with c velocity along the channel. It has been expressed by step function:

$$P_a(x, t) = P_{a,0} + \Delta P_a \eta \left(t - \frac{x}{c} \right)$$

where: $\eta(t - \frac{x}{c}) = \begin{cases} 1, & t > x/c \\ 0, & t \le x/c \end{cases}$ and $P_{a,0}$ is the constant atmospheric pressure over the channel before the disturbance.

In dimensionless co-ordinates it can be written as follows:

$$P_a(\xi, \tau) = P_{a,0} + \Delta P_a \eta \left(\tau - \frac{v_0}{c} \xi \right), \quad \eta \left(\tau - \frac{v_0}{c} \xi \right) = \begin{cases} 1, & \tau > \frac{v_0}{c} \xi \\ 0, & \tau \le \frac{v_0}{c} \xi \end{cases} \tag{5}$$

After introducing these assumptions equation (1) takes the form:

$$\frac{\partial \vartheta}{\partial \tau} + \frac{\partial \vartheta}{\partial \xi} = -2I_0\vartheta - \frac{2\Delta P_a}{\rho v_0^2}\frac{\partial}{\partial \xi}\eta\left(\tau - \frac{v_0}{c}\xi\right) \tag{6}$$

Convenient way of solutions for equation (6) is to apply the Laplace transformation (Bobrowski 1981, Carslow 1948, Romanowski 1968). Laplace transformation $\mathcal{L}$ of the atmospheric pressure derivative in equation (6) is equal to:

$$\frac{2\Delta P_a}{\rho v_0^2}\frac{\partial}{\partial \xi}\mathcal{L}\left\{\eta\left(\tau - \frac{v_0}{x}\xi\right)\right\} = -2\frac{\Delta P_a}{\rho v_0 c}\exp\left(-s\frac{v_0}{c}\xi\right)$$

and further transformation of Eq. (6) gives the following ordinary differential equation:

$$s\overline{\vartheta} - \vartheta(\xi, 0_+) + \frac{\partial \overline{\vartheta}}{\partial \xi} = -2I_0\overline{\vartheta} + 2\frac{\Delta P_a}{\rho v_0 c}\exp\left(-s\frac{v_0}{c}\xi\right) \tag{7}$$

Taking into account the initial condition (4), the equation (7) can be rewritten in the form:

$$\frac{\partial \overline{\vartheta}}{\partial \xi} + (s + 2I_0)\overline{\vartheta} = 2\frac{\Delta P_a}{\rho v_0 c}\exp\left(-s\frac{v_0}{c}\xi\right) \tag{8}$$

The solution of equation (8) is equal to:

$$\begin{aligned}\overline{\vartheta}(s, \xi) &= \exp\left(-(s + 2I_0)\xi\right) \times \\ &\times \left(\int 2\frac{\Delta P_a}{\rho v_0 c}\exp\left(-s\frac{v_0}{c}\xi\right)\exp\left(-(s + 2I_0)\xi\right)d\xi + \text{const}\right)\end{aligned} \tag{9}$$

after integrating one can obtain that:

$$\begin{aligned}\overline{\vartheta}(x, \xi) &= 2\frac{\Delta P_a}{\rho(c - v_0)v_0}\exp\left(-m\frac{v_0}{c}\xi\right)\frac{\exp\left(-\frac{v_0}{c}\xi(s + m)\right)}{s + m} + \\ &+ \text{const } \times \exp(-s + 2I_0\xi)\end{aligned} \tag{10}$$

where: $m = \dfrac{2I_0}{1 - v_0/c}$.

To obtain the inverse Laplace transformation of equation (10) we shall apply the shift property (Carslaw 1948, Bobrowski 1981) and so:

$$\mathcal{L}^{-1}\left\{\frac{1}{s + M}\right\} = \exp(-m\tau)\eta(\tau),$$

$$\mathcal{L}^{-1}\left\{\exp\left(-\frac{v_0}{c}\xi(s + m)\right)\frac{1}{s + m}\right\} = \exp(-m\tau)\eta\left(\tau - \frac{v_0}{c}\xi\right)$$

The constant value in equation (10) can be obtained by assuming the presence of internal condition which occurs at moment of passage of low atmospheric pressure "eye" at the given point x. That can be formulated as follows:

$$\begin{aligned} v(x,t) = v_0, \quad H(x,t) = H_0 \quad \text{for} \quad t = \frac{x}{c} \\ v(\xi,\tau) = v_0, \quad H(\xi,\tau) = H_0 \quad \text{for} \quad \tau = \xi\frac{v_0}{c} \end{aligned} \tag{11}$$

Imposing the boundary condition gives const = 0, and thus solution for $\vartheta(\tau,\xi)$ is equal to:

$$\vartheta(\tau,\xi) = 2\frac{\Delta P_a}{\rho(c-v_0)v_0}\exp\left(-m\left(\tau - \frac{v_0}{c}\xi\right)\right)\eta\left(\tau - \frac{v_0}{c}\xi\right) \tag{12}$$

Turning back to dimensional velocity function $v(\tau,\xi)$ we have:

$$v(\tau,\xi) = v_0\sqrt{1 + 2\frac{\Delta P_a}{\rho(c-v_0)v_0}\exp\left(-m\left(\tau - \frac{v_0}{c}\xi\right)\right)\eta\left(\tau - \frac{v_0}{c}\xi\right)} \tag{13}$$

and, coming back to x, t variables:

$$v(t,x) = v_0\sqrt{1 + \frac{2\Delta P_a}{\rho v_0(c-v_0)}\exp\left(-\frac{2gI_0c}{v_0(c-v_0)}\left(t - \frac{x}{c}\right)\right)\eta\left(t - \frac{x}{c}\right)} \tag{14}$$

To simplify the formulas, it is convenient to put: $A = 2\frac{\Delta P_a}{\rho(c-v_0)v_0}$. Then the solution of v takes final form:

$$v(\tau,\xi) = v_0\sqrt{1 + A\exp\left(-m\left(\tau - \frac{v_0}{c}\xi\right)\right)\eta\left(\tau - \frac{v_0}{c}\xi\right)} \tag{15}$$

Changes of water level which take place during wave propagation can be estimated using an simplified version of water flow continuity equation (Meyer, Ewertowski 1996b):

$$\Delta H = -\frac{H_0}{c}\Delta v = -\frac{H_0}{c}(v - v_0) \tag{16}$$

Thus, the derivative of ΔH with respect to τ is equal to:

$$\begin{aligned} \frac{\partial(\Delta H)}{\partial\tau} = -\frac{H_0}{c}\frac{\partial v}{\partial\tau} = 2\frac{H_0 I_0 \Delta P_a}{\rho(c-v_0)^2} \times \\ \times \frac{\exp\left(-m\left(\tau - \frac{v_0}{c}\xi\right)\right)}{\sqrt{1 + A\exp\left(-m\left(\tau - \frac{v_0}{c}\xi\right)\right)\eta\left(\tau - \frac{v_0}{c}\xi\right)}} \end{aligned} \tag{17}$$

The rate of water level changes at some finite stretch of time Δt can be described as follows:

$$\frac{\partial H}{\partial t} = -2\frac{g v_0 \Delta P_a}{\rho(c-v_0)^2 C^2} \times \tag{18}$$
$$\times \frac{\exp\left(-\frac{2g I_0 c}{v_0(c-v_0)}\left(t-\frac{x}{c}\right)\right)}{\sqrt{1+\frac{2\Delta P_a}{\rho v_0(c-v_0)}\exp\left(-\frac{2g I_0 c}{v_0(c-v_0)}\left(t-\frac{x}{c}\right)\right)\eta\left(t-\frac{x}{c}\right)}}$$

It is also possible to assume that time is running from culmination of water level at point x. We have then $\Delta t = t - x/c$, and from Eq. (14):

$$v(x,\Delta t) = v_0\sqrt{1+\frac{2\Delta P_a}{\rho v_0(c-v_0)}\exp\left(-\frac{2g I_0 c}{v_0(c-v_0)}\Delta t\right)} \tag{19}$$

It arises that the time scale of wave attenuation is equal to:

$$\frac{1}{T_0} = \frac{2g I_0 c}{v_0(c-v_0)},\quad T_0 = \frac{v_0(c-v_0)}{2gcI_0} = \frac{(c-v_0)C_f^2 H_0}{2g v_0 c} \approx \frac{C_f^2 H_0}{2g v_0} \tag{20}$$

where C_f is velocity constant according to Chezy formula.
The changes of water velocity can be expressed as:

$$\Delta v = v_0\left(\sqrt{1+A\exp\left(-\frac{\Delta t}{T_0}\right)}-1\right) \tag{21}$$

and thus, combining Eq. (19) and (21), relative change of water level is given by formulae:

$$\frac{\Delta H}{H_0} = -\frac{v_0}{c}\left(\sqrt{1+A\exp\left(-\frac{\Delta t}{T_0}\right)}-1\right) \tag{22}$$

In further part of the paper solutions $\Delta v(x,t)$, $\Delta H(x,t)$ have been analysed in connection with admissible solution for ΔP_a that has been described in earlier paper (Meyer, Ewertowski 1997). Here, the admissible condition is: $\Delta P_a \geq \Delta P_{a,\min} = \dfrac{\rho(c-v_0)v_0}{v_0}e^{\Delta t/T_0}$.

One can also solve an inverse problem assuming, that extreme value of water level is known as $\Delta H_{\max}$ and we are looking for parameter A. In that case we have from Eq. (22) $\Delta t = 0$, $\Delta H = \Delta H_max$ and so:

$$\Delta H_{\max} = -\frac{H_0 v_0}{c}\left(\sqrt{1+A}-1\right)$$

and further

$$A = -\frac{2c\Delta H_{\max}}{H_0 v_0} + \frac{c^2\Delta H_{\max}^2}{H_0^2 v_0^2} = \frac{c\Delta H_{\max}}{H_0 v_0}\left(\frac{c\Delta H_{\max}}{H_0 v_0}-2\right)$$

The approximated expression for the formula (22) can be obtained after putting $\sqrt{1+x} \approx 1 + x/2$. It results in:

$$\Delta H(\Delta t) = \Delta H_{\max} \left(\frac{c}{v_0} \frac{\Delta H_{\max}}{H_0} \right) \exp \left(\frac{-\Delta t}{T_0} \right) \tag{23}$$

3. Examples of Calculations

Including the friction forces of motion in this wave flow gives an additional, exponential term, which arises in theoretical solutions. This term causes gradual attenuation of the wave. The rate of water level and velocity change, which has been generated by sudden change of atmospheric pressure field, is described by time scale T_0. Time scale depends on hydraulic parameters of the flow.

In the case of river Odra outlet it has been assumed that mean flow velocity varies in the range of $v_0 \in \langle 0.05,\ 0.2 \rangle$ m/s, hydraulic slope $I_0 \in \langle 10^{-6},\ 10^{-5} \rangle$, mean depth $H_0 \in < 5,\ 10 >$ m, and Chezy coefficient $C_f \in < 30,\ 40 >$ $m^{1/2}$/s. Atmospheric pressure has been modelled using the step function for several values of ΔP_a. To investigate the function of relative velocity and level changes, computations have been made for different cases. Results of these calculations are presented on Figs. 4a, b, 5a, b. Title of each figure describes the kind of dependence being investigated.

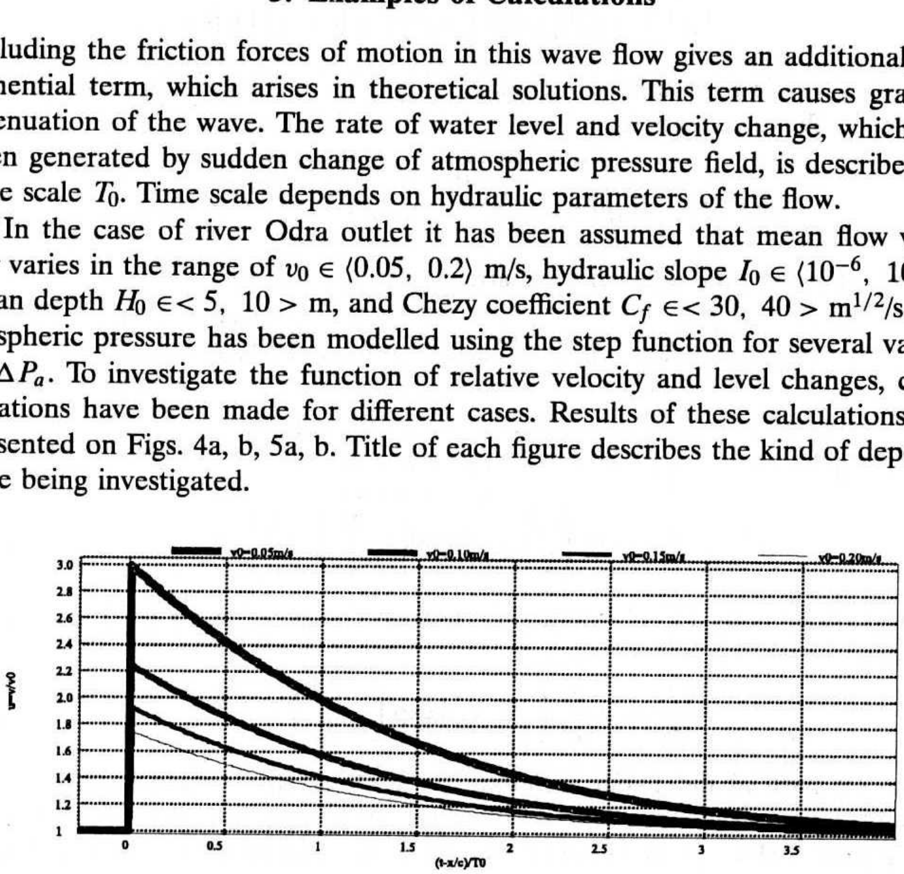

Fig. 4a. Dependence of relative velocity changes with regard to mean velocity flow calculated for $H_0 = 10$ m, $C_f = 40$ $m^{1/2}$/s, $c = 10$ m/s, $\Delta P_a = 20$ hPa, $T_0 = 4.5$ hour

Similar results have been obtained for different values of propagation velocity c, where smaller values of c give bigger extreme values of u (from 2.5 for $c = 8$ m/s to 1.8 for $c = 16$ m/s), and h (from 0.98 for $c = 8$ m/s to 0.995 for $c = 16$ m/s).

Also results for different values of atmospheric pressure changes have similar shape but in that case, bigger values of ΔP_a give bigger extreme values of u (from 1.4 for $\Delta P_a = 5$ hPa to 2.2 for $\Delta P_a = 20$ hPa), and h (from 0.98 for $c = 8$ m/s to 0.995 for $c = 16$ m/s).

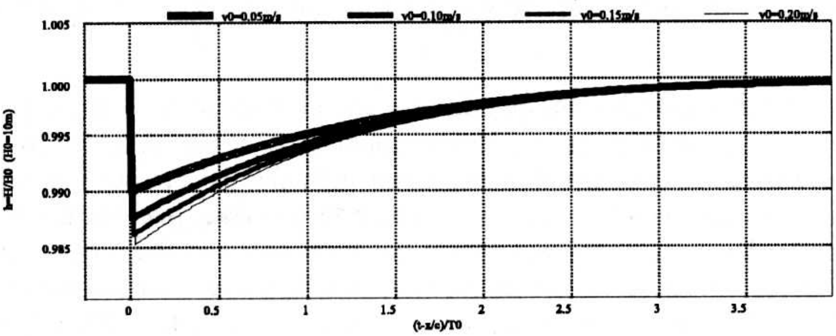

Fig. 4b. Dependence of relative depth changes with regard to mean velocity flow calculated for $H_0 = 10$ m, $C_f = 40$ m$^{1/2}$/s, $c = 10$ m/s, $\Delta P_a = 20$ hPa, $T_0 = 4.5$ hour

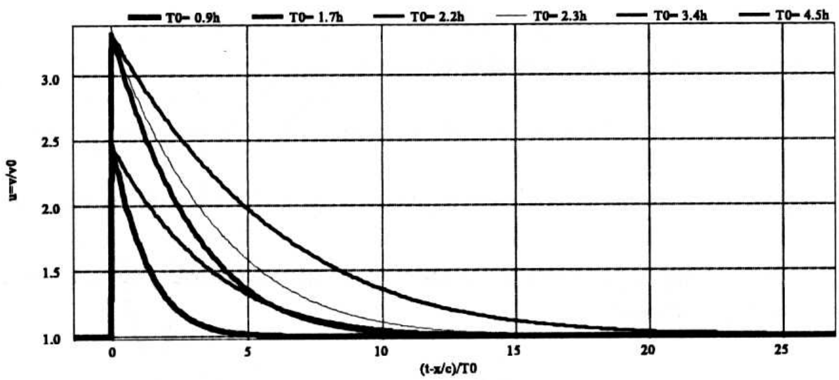

Fig. 5a. Dependence of relative velocity changes with regard to time scale of disturbance calculated for $H_0 = 5.10$ m, $C_f = 35.40$ m$^{1/2}$/s, $v_0 = 0.10$ m/s, $c = 8$ m/s, $\Delta P_a = 20$ hPa

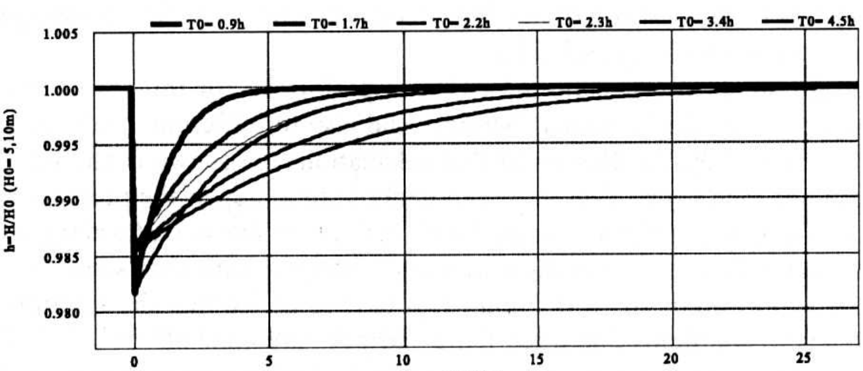

Fig. 5b. Dependence of relative depth changes with regard to time scale of disturbance calculated for $H_0 = 5.10$ m, $C_f = 35.40$ m$^{1/2}$/s, $v_0 = 0.10$ m/s, $c = 8$ m/s, $\Delta P_a = 20$ hPa

4. Influence of the Atmospheric Pressure Incidence Angle on Barotropic Wave

Modelling of the temporal atmospheric pressure changes at fixed point x using step function given in Fig. 2a is rather simplification of the nature. From the authors investigations it comes that important influence on the wave shape has the pressure gradient while low or high pressure "eye" passes over the given river point (Figs 1a–1f). In the present paper it has been called as the time incidence angle. To make the model of atmospheric pressure more realistic, method of superposition of solutions for family of step functions gradually shifted by some time increment was introduced and it is shown on Fig. 6.

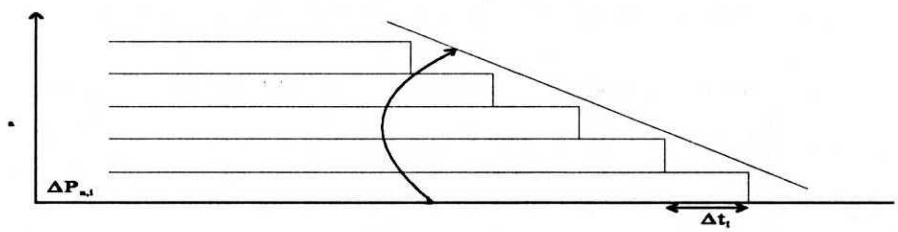

Fig. 6. Superposition of step function and incidence angle for atmospheric pressure changes

In the case of superposition total change of atmospheric pressure at point x can be written as follows:

$$\Delta P_a(x,t) = \sum_{i=1}^{N} \left(\Delta P_{a,i} \eta \left(t + \Delta t_i - \frac{x}{c} \right) \right) \tag{24}$$

where: $\Delta P_{a,i} = \Delta P_a(\max)/N$. Atmospheric pressure function expressed by function (24) has been used for evaluating of velocity and water levels changes accordingly to formulas (21) and (22).

Several numerical calculations have been performed for three different time steps Δt_1, Δt_2, Δt_3, for individual "stripes" and also for different incidence angles defined by ratio $\Delta P_a/\Delta t$. Results of that calculations are shown in the Fig. 7a, b. The legend of each figure gives information about time step of the pressure stripes $(\Delta t_i = dt_i)$ and the incidence angle (knt) defined as $knt = (\Delta P_a(\max)/(N\Delta t_i)$. The case in the Fig. 7 assumes division into 64 "stripes" with time steps $\Delta t_1 = 0.1$ hour, $\Delta t_2 = 0.15$ hour, $\Delta t_3 = 0.2$ hour. It is easy to notice that the solution for u and h tends to continuos one when more "stripes" are applied.

An attempt was made to obtain an analytical solution for the case of gradually varying pressure. We have

$$P_a(x,t) = P_{a,0} + \Delta P_a \alpha(x,t,\Delta t) \tag{25}$$

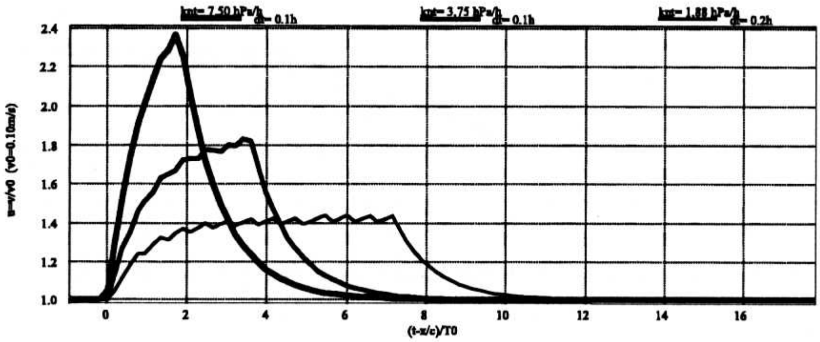

Fig. 7a. Relative change of water velocity for partition of $\Delta P_a = 20$ hPa into 64 stripes calculated for $H_0 = 10$ m, $C_f = 40$ m$^{1/2}$/s, $v_0 = 0.10$ m/s, $c = 10$ m/s, $T_0 = 2.2$ hour

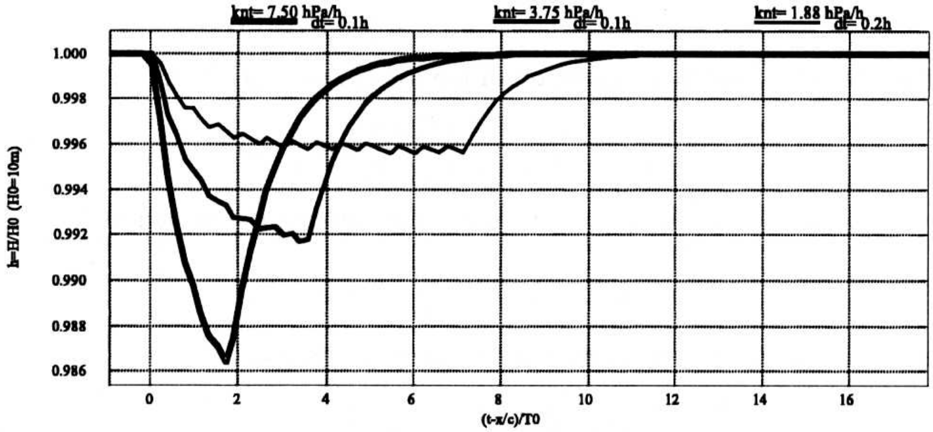

Fig. 7b. Relative change of water depth for partition of $\Delta P_a = 20$ hPa into 64 stripes calculated for $H_0 = 10$ m, $C_f = 40$ m$^{1/2}$/s, $v_0 = 0.10$ m/s, $c = 10$ m/s, $T_0 = 2.2$ hour

where:

$$\alpha(x, t, \Delta t) = \begin{cases} 0 & , \ t < x/c \\ \frac{t - x/c}{\Delta t} & , \ \frac{x}{c} \le t \le \frac{x}{c} + \Delta t \\ 1 & , \ t > x/c + \Delta t \end{cases}$$

The $P_a(x, t)$ function is illustrated by the following schema in Fig. 8.

In dimensionless form relationships (25) can be written as follows:

$$P_a^*(\xi, \tau) = \frac{P_a}{P_{a,0}} = 1 + \frac{\Delta P_a}{P_{a,0}} \alpha(\xi, \tau, \Delta \tau) \tag{26}$$

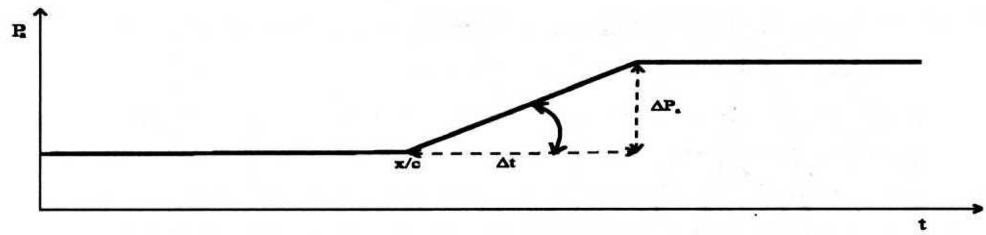

Fig. 8. Schema of atmospheric pressure change using time incidence angle

where: $\Delta\tau = \dfrac{\Delta t}{v_0/g}$,

$$\alpha(\xi, \tau, \Delta\tau) = \begin{cases} 0 & , \ \tau < \frac{v_0}{c}\xi \\ \frac{\tau-(v_0/c)\xi}{\Delta\tau} & , \ \frac{v_0}{c}\xi \le t \le \frac{v_0}{c}\xi + \Delta\tau \\ 1 & , \ \tau > \frac{v_0}{c}\xi + \Delta\tau \end{cases}$$

After putting this formula to equation of motion (6) we can obtain:

$$\frac{\partial \vartheta}{\partial \tau} + \frac{\partial \vartheta}{\partial \xi} = -2I_0\vartheta - \frac{2\Delta P_{a,0}}{\rho v_0^2}\frac{\partial}{\partial \xi}\alpha(\xi, \tau, \Delta\tau) \tag{27}$$

The solution was obtained by using Laplace transformation of function:

$$\mathcal{L}\left\{\alpha(\xi, \tau, \Delta\tau)\right\} = \frac{e^{-s(v_0/c)\xi}}{s^2\Delta\tau}\left(1 - e^{-s\Delta\tau}\right)$$

and Laplace transformation of the whole equation (27) gives:

$$\frac{\partial \overline{\vartheta}}{\partial \xi} + (s + 2I_0)\overline{\vartheta} = D\frac{e^{-s(v_0/c)\xi}}{s}\left(1 - e^{-s\Delta\tau}\right) \tag{28}$$

with $D = \dfrac{2\Delta P_a}{\rho v_0 c \Delta\tau}$.

Solution of this ordinary differential equation is equal to:

$$\overline{\vartheta} = D'\frac{e^{-s(v_0/c)\xi}}{s(s+m)} - D'\frac{e^{-s((v_0/c)\xi+\Delta\tau)}}{s(s+m)} \tag{29}$$

where: $m = \dfrac{2I_0}{1 - v_0/c}$, $D' = \dfrac{D}{1 - v_0/c}$.

Inverse Laplace transformation of function $F(s) = 1/s(s+m)$ gives:

$$\mathcal{L}^{-1}\{F(s)\} \equiv f(\tau) = \frac{1}{m}\left(1 - e^{-m\tau}\right)$$

Taking into account the shift law, one can obtain that

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} \equiv f(\tau - a) = \frac{1}{m}\left(1 - e^{-m(\tau-a)}\right) \quad \text{for} \quad \tau > a \ge 0$$

what can be rewrite as:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} \equiv f(\tau) = \frac{1}{m}\left(1 - e^{-m(\tau-a)}\right)\eta(\tau - a)$$

Following the above rule the solution for ϑ is equal to:

$$\begin{aligned}\vartheta(\xi, \tau, \Delta\tau) &= D'\mathcal{L}^{-1}\left\{\frac{e^{-s(v_0/c)\xi}}{s(s+m)}\right\} - D'\mathcal{L}^{-1}\left\{\frac{e^{-s((v_0/c)\xi+\Delta\tau)}}{s(s+m)}\right\} = \\ &= \frac{D'}{m}\left[\left(1 - e^{-m(\tau-(v_0/c)\xi)}\right)\eta\left(\tau - \frac{v_0}{c}\xi\right) + \right. \\ &\quad \left. - \left(1 - e^{-m(\tau-(v_0/c)\xi-\Delta\tau)}\right)\eta\left(\tau - \frac{v_0}{c}\xi - \Delta\tau\right)\right]\end{aligned} \tag{30}$$

In term of variables x, t the solution takes the following form:

$$\begin{aligned}\vartheta(x, t, \Delta t) = D''\left[\left(1 - e^{(t-x/c)/T_0}\right)\eta\left(x - \frac{x}{c}\right) + \right. \\ \left. - \left(1 - e^{(t-x/c-\Delta t)/T_0}\right)\eta\left(x - \frac{x}{c} - \Delta t\right)\right]\end{aligned} \tag{31}$$

where: $T_0 = \dfrac{C_f^2 H_0(c - v_0)}{2g\, v_0 c}$ [s], $D'' = \dfrac{\Delta P_a C_f^2 H_0}{\rho v_0^2 cg\, \Delta t} = \dfrac{\Delta P_a}{\rho cg\, I_0 \Delta t}$.

Finally, velocity of water flow with friction complying Chezy law and upon the influence of atmospheric pressure described by equation (25) is given by relationship:

$$v(x, t) = v_0\sqrt{1 + \vartheta} \tag{32}$$

where ϑ is given by (31). Change of water depth can be evaluated from equation (16) giving:

$$H_{inci}(x, t) - H_0 = \Delta H_{inci} = -\frac{H_0 v_0}{c}\left(\sqrt{1 + \vartheta} - 1\right) \tag{33}$$

Several computations of relative velocity $u = v(x, t)/v_0$ and depth $h = H(x, t)/H_0$ have been made using lately obtained solutions (32) and (33). One of the parameters is time incidence angle described as $\Delta P_a/\Delta t$. Some of results are shown in Fig. 9 and it can be compared to Fig. 7a, b. It can noticed similarity between analytical (Fig. 9) and numerical ones from Fig. 7a, b.

Comparison between v_{inci} solution (Eq.32) and v_{step} solution (Eq.14) is given in the Fig. 10 for $c = 10$ m/s, $v_0 = 0.1$ m/s, $\Delta P_a = 20$ hPa, $\Delta t = 10$ h and for five different values of friction slope I_0.

One can easily find that maximum of function $\vartheta(\xi, \tau, \Delta\tau)$ given by equation (30), which appeared at time $\tau = (v_0/c)\xi + \Delta\tau$, and is equal to

$$\vartheta_{inci}^{\max} = \frac{\Delta P_a}{\rho I_0 v_0 c \Delta\tau}\left(1 - e^{-m\Delta\tau}\right) \tag{34}$$

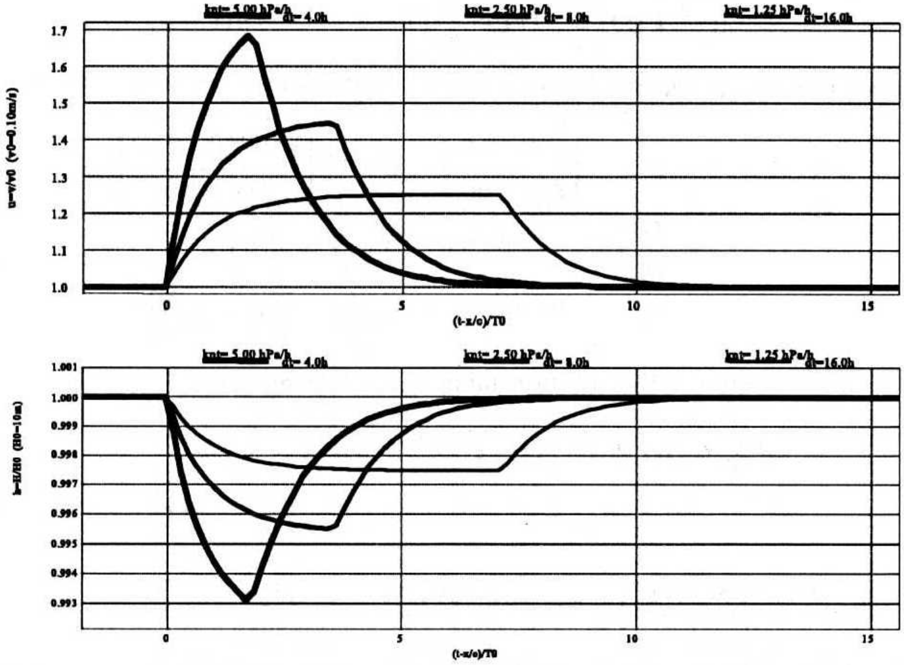

Fig. 9. Relative change of water velocity and depth from analytical solution of incidence angle for $\Delta P_a = 20$ hPa calculated for $H_0 = 10$ m, $C_f = 40$ m$^{1/2}$/s, $v_0 = 0.10$ m/s, $c = 10$ m/s, $T_0 = 2.2$ hour

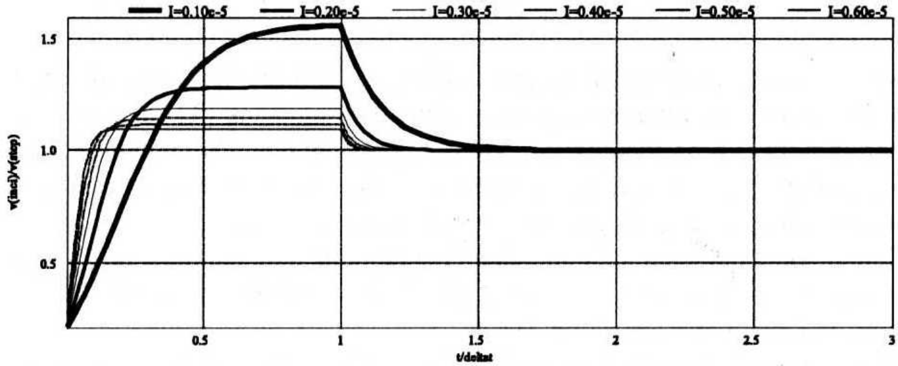

Fig. 10. Velocity relation v_{inci}/v_{step} during 30 hours for several friction slopes I_0

On the other hand, solution (12) of $\vartheta(\xi, \tau, \Delta\tau)$ at the same time for the step function gives:

$$\vartheta_{step} = Ae^{-m\Delta\tau} = \frac{2\Delta P_a}{\rho(c - v_0)v_0} e^{-m\Delta\tau}$$

and relative difference of these values can be expressed as:

$$\frac{\vartheta_{step} - \vartheta_{inci}^{\max}}{\vartheta_{step}} = 1 - \frac{1}{I_0 \Delta\tau}\left(e^{m\Delta\tau} - 1\right) \tag{35}$$

When $\Delta\tau \to 0$ then the expression on right hand side of (35) is equal to $1 - m/2I_0 \approx 0$ what confirms that both solutions give the same value for $\Delta\tau \to 0$.

Instead of $\vartheta_{inci}^{\max}$, ϑ_{step} we can calculate the relative difference of velocity according to the following formula (in dimensional form):

$$\frac{\vartheta_{step} - \vartheta_{inci}^{\max}}{\vartheta_{step}}(T_0, \Delta t) = 1 - \frac{\sqrt{D''\left(1 - e^{-\Delta t/T_0}\right) + 1}}{\sqrt{e^{-\Delta t/T_0} + 1}} \tag{36}$$

where attenuation time scale T_0 is given by equation (20) and coefficients D'', A have been defined by Eq. (14) and (31).
Using continuity equation (16) and performing similar transformations as above, one can obtain relative depth difference as:

$$\frac{H_{step} - H_{inci}^{\max}}{H_{step}} = \frac{\frac{v_0}{c}\left(\sqrt{D''\left(1 - e^{-\Delta t/T_0}\right) + 1} - \sqrt{Ae^{-\Delta t/T_0} + 1}\right)}{1 - \frac{v_0}{c}\left(\sqrt{A - e^{-\Delta t/T_0} + 1} - 1\right)} \tag{37}$$

Functions given by formulas (36) , (37) has been examined for several values of T_0 assuming $\Delta P_a = 20$ hPa. Results of the calculations of the velocity changes (Eq. 36) are shown in Fig. 11a.

From the figure it comes that extreme values of $v(x, t)$ function calculated from step function model (Eq. 32) at $t = x/c + \Delta t$ are generally much smaller than those from the incidence angle model. That results from the fact that step function solution quickly decreases with time from its extreme value at $t = x/c$.

Results of the calculations of water depth using equation (37) are shown in Fig. 11b. The values of water depth, calculated from the step function model, are generally greater than those from the incidence angle model, because velocity quickly increases with time.

Numerical calculations were proceeded to compare extreme values for step function model (at $t = x/c$) and incidence angle model (at $t = x/c + \Delta t$) as a function of Δt for different values of friction slope I_0:

$$\frac{v_{inci}^{\max}}{v_{step}^{\max}} = \frac{1 + D''\left(1 - \exp(-\Delta t/T_0)\right)}{1 + D'} \tag{38}$$

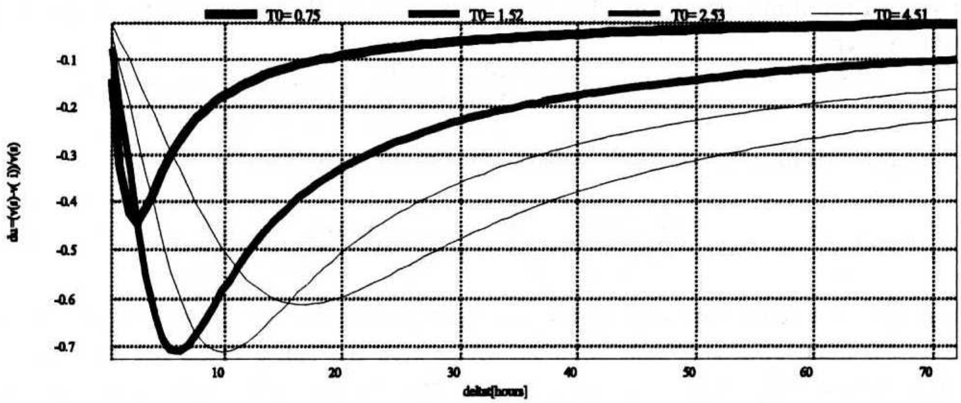

Fig. 11a. Relative difference of water velocity between the step function solution and maximum of the time incidence angle solution as a function of Δt , T_0

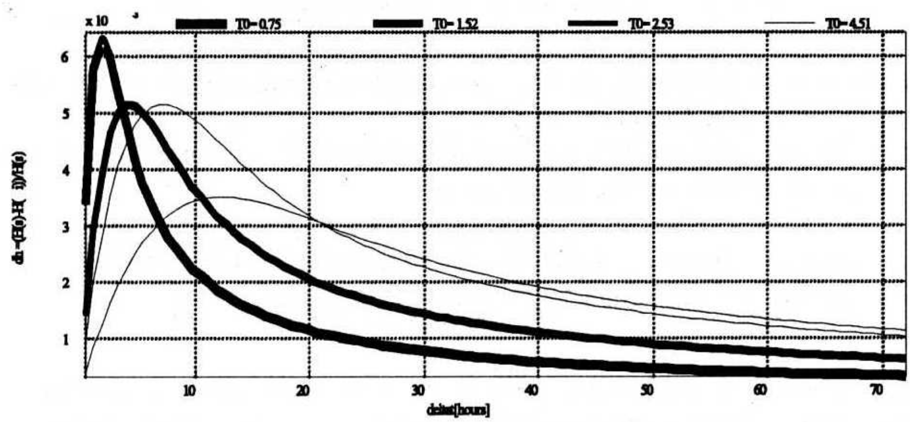

Fig. 11b. Relative difference of water depth between the step function solution and maximum of the time incidence angle solution as a function of Δt , T_0

Results of calculations are shown on Fig. 12 where we can observe that extreme values of velocity calculated from step function model are greater than those from the incidence angle model. The difference is high for small values of Δt. For Δt greater than 10 hours and for the time of attenuation T_0 smaller than 1 hour, the difference reaches range 0.5 to 0.6. That means that incidence angle model gives velocity half of that from step function model.

Similar calculations were performed for depth extreme values of step function model (at $t = x/c$) and incidence angle model (at $t = x/c + \Delta t$) as a function of Δt:

$$\frac{H_{inci}^{\max}}{H_{step}^{\max}} = \frac{1 - v_0/c\left(\sqrt{1 + D''\left(1 - \exp(-\Delta t/T_0)\right)} - 1\right)}{1 - v_0/c\left(\sqrt{1 + A} - 1\right)} \tag{39}$$

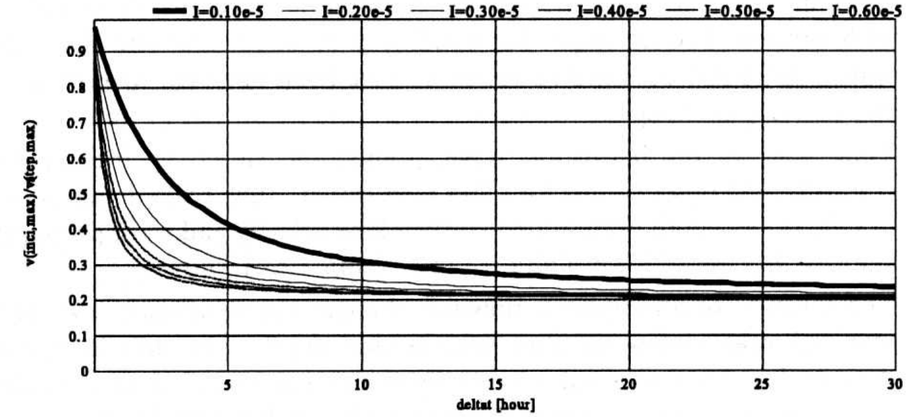

Fig. 12a. Relation of maximal flow velocity from incidence angle model to the maximum from step function model as a function of Δt, I_0

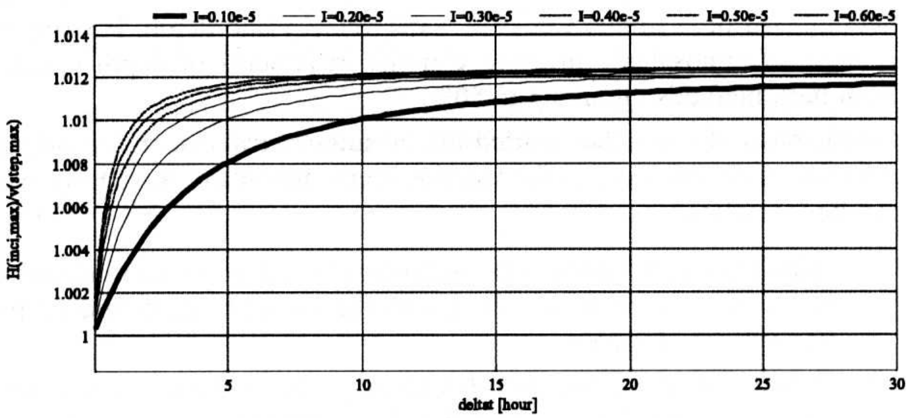

Fig. 12b. Relation of maximal flow depth from incidence angle model to the maximum from step function model as a function of Δt, I_0

In Fig. 12b we can see, that extreme water depth differences calculated from step function model are smaller than those from the incidence angle model. The greatest difference between the both solutions amounts to 1.2% of the water depth calculated from step function model.

5. Concluding Remarks

1. The paper presents the mathematical model of barotropic river wave induced by atmospheric pressure changes. The model takes into account wave attenuation due to friction forces of water flow, which exist in any real flow. To simulate the rapid atmospheric pressure variation the step function has

been applied. The phenomenon was analysed in simplified straight channel with constant bottom slope. For conditions of steady and one-dimensional water movement a simplified solutions have been obtained describing velocity and depth changes.

2. Time scale of the barotropic wave attenuation caused by friction forces confirms field observations for Odra river outlet. Time scale depends on hydraulic slope, depth of flow, undisturbed flow velocity and wave propagation velocity. In case of lower Odra it ranges from 1 to 10 hours.
3. The function of atmospheric pressure changes has significant influence on maximal values of water flow velocity and depth. The influence can be described in approximated manner by evaluation so called time incidence angle. In this paper the time incidence angle has been analysed in to ways. First, using superposition of stripes pressure function with time steps. The second, the analytical solution for applying gradually varying pressure.
4. Detailed analysis of the influence of time incidence angle for the wave elements indicates, that gradual increase of atmospheric pressure over a channel diminish the extreme values of water velocity and depth. For very slow changes of atmospheric pressure extreme increments of depth calculated from both methods differs up to 50%.
5. On the basis of researches carried out by authors, two essential conclusions referring to the influence of flow friction on the barotropic wave propagation can be formulated:

   (a) Taking into consideration in calculations the friction forces causes barotropic wave attenuation. Along with the wave its height and propagation velocity decrease.

   (b) Depending on the function describing pressure changes if the friction forces are included the resulting barotropic wave in the river is of different shape. Two functions were taken for analysis: step function (Eq. 5) and gradually varying function (Eq. 25). If we assume that both functions gives the same $\Delta P_{a,\max}$ the resulting wave will be with different flow elements ie: depth and velocity. The case of gradually varying pressure gives wave with depth changes smaller by comparison to step function case. It confirms field observations that there exist no direct relation between changes of depth and pressure. Additional factor is needed to explain the differences. This factor in the present research was introduced as incidence angle $\Delta P_a/\Delta t$.

6. Problems for further researching are listed bellow:

   – analysis of real atmospheric pressure field evaluation over the river area based upon recorded data,

- channel cross section shape influence on the barotropic wave propagation,
- formulation of the numerical solution of the considered unsteady water flow in the river network with practical applications.

References

Bobrowski D., Ratajczak Z. (1981) *Przekształcenia Laplace'a i jego zastosowania*, Wyd. Pol. Poznańskiej nr 1016, Poznań.

Carslaw H., Jaeger J. (1948) *Operational methods in applied mathematics*, Oxford Univ. Press, London.

Druet C. (1978) *Hydromechanika morskich budowli i akwenów portowych*, Wydawnictwo Morskie, Gdańsk.

Meyer Z. (1995) Problemy hydrauliczne ujściowego odcinka Odry, *XV Ogólnopolska Szkoła Hydrauliki*, Wrocław-Trzebieszowice, Wyd. P.S., Szczecin.

Meyer Z., Ewertowski R. (1996a) Analiza wpływu zmian cionienia atmosferycznego na stany wody w ujściu rzeki Odry, *Ogólnopolska Konf. Naukowa "Współczesne ..."* – 50-lecie WBiA PS, Szczecin.

Meyer Z., Ewertowski R. (1996b) Approximated Method for Description of Barotropic Waves Penetrating Upstream Lower Odra River, *IV Int. Conf. on Hydrodynamics and Hydrology for River's Mouths ...*, Instytut Morski, Szczecin.

Meyer Z., Ewertowski R. (1997) An Approximation of Barotropic Wave Penetrating River Outlet Using General Function for Description of Atmospheric Pressure Effect, *IV Int. Seminar on Renovation and Improvements to existing Quay Structures*, Tech. Univ. of Gdańsk.

Romanowski P. (1968) *Szeregi Fouriera, funkcje analityczne i specjalne, przekształcenia Laplace'a*, BNI PWN, Warszawa.