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Numerical Simulation of the Biochemical Oxygen Demand and Dissolved Oxygen in the Channels Network

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Abstract

The paper deals with modelling of two indicators characterizing oxygen conditions of a river, which are: biochemical oxygen demand *(BOD)* and dissolved oxygen *(DO)*. To describe the evolution of these two indicators, advection-diffusion transport equations are used. To solve these equations the decomposition technique is proposed. This approach enables the splitting up of the advection-diffusion equation into parts and independent solving of both parts in each time step. The advection equation is solved by the characteristics method with the spline function used in interpolation. To solve the diffusion equation the finite-elements method with the Crank-Nicolson scheme to integrate in time is applied. The proposed method ensures very small numerical errors. The method was tested using the known analytical solution and the results of measurements for the Reda river.

1. Introduction

Assuming that dissolved matter is fully mixed in the river cross-section the evaluation of biochemical oxygen demand *(BOD)* and dissolved oxygen *DO)* in the stream flow can be described by two one-dimensional equations (Elliot and James 1984):

$$\frac{\partial L}{\partial t} + U\frac{\partial L}{\partial x} - \frac{1}{A}\frac{\partial}{\partial x}\left(EA\frac{\partial L}{\partial x}\right) + (K_1 + K_3)L = 0, \tag{1}$$

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} - \frac{1}{A}\frac{\partial}{\partial x}\left(EA\frac{\partial C}{\partial x}\right) - K_2(C_s - C) + K_1 L + B = 0, \tag{2}$$

where:

- t – time [s],
- x – position [m],
- L – biochemical oxygen demand *(BOD)* concentration of [mgl^{-1}],
- C – concentration of dissolved oxygen *(DO)* of [mgl^{-1}],

U – average velocity of water flow $[\mathrm{ms}^{-1}]$,
E – coefficient of longitudinal dispersion $[\mathrm{m}^2\ \mathrm{s}^{-1}]$.
A – wetted cross-sectional area $[\mathrm{m}^2]$,
K_1 – the *BOD* reaction rate $[\mathrm{s}^{-1}]$,
K_2 – the reaeration rate coefficient $[\mathrm{s}^{-1}]$,
K_3 – the rate coefficient for removal of *BOD* by sedimentation and adsorption $[\mathrm{s}^{-1}]$,
C_s – the saturated dissolved oxygen concentration $[\mathrm{mgl}^{-1}]$,
B – the net removal rate of dissolved oxygen for all processes other than biochemical oxidation.

In this paper a numerical solution of presented equation is considered first of all. The forms of both equations describing *BOD* and *DO* transport are not discussed.

Usually in the rivers the advection transport dominates over the diffusion. In such a case the influence of the diffusion term is relatively small, thus it is found to assume the invariability of the longitudinal dispersion coefficient. For $E = \mathrm{const}$ Eqs. (1) and (2) can be transformed into:

$$\frac{\partial L}{\partial t} + \left(U - \frac{E}{A}\frac{\partial A}{\partial x}\right)\frac{\partial L}{\partial x} - E\frac{\partial^2 L}{\partial x^2} + (K_1 + K_3)L = 0, \tag{3}$$

$$\frac{\partial C}{\partial t} + \left(U - \frac{E}{A}\frac{\partial A}{\partial x}\right)\frac{\partial C}{\partial x} - E\frac{\partial^2 L}{\partial x^2} - K_2(C_s - C) + K_1 L + B = 0. \tag{4}$$

To solve Eqs. (3) and (4) the knowledge of average velocities in the river's cross-sections $U(x,t)$ and cross-section areas $A(x,t)$ are needed. This information can be obtained by previously solving the flow equation in an open channel. Assuming a nonuniform steady flow these variables can be defined solving the steady state flow equation in the channel network for the imposed discharges. For the unsteady flow to find the functions $U(x,t)$ and $A(x,t)$ one needs to solve the system of de Saint Venant equations with the proper initial and boundary conditions.

Eqs. (3) and (4) are parabolic type partial differential equations. In real conditions when any boundary conditions can occur and while the velocity and the cross-section vary in time and space, these equations can be solved only by numerical methods. It is known that the advection-diffusion equation is very difficult to solve when the advection transport dominates. The standard schemes of the finite-difference method, as well as the finite-element method, lead very often to non-physical solutions. In these solutions one can notice the presence of numerical diffusion or oscillations. This is caused by numerical errors of dissipation and dispersion generated by these schemes. The detailed explanation of this problem is presented by Fletcher (1991) for example. Recently, some special new algorithms, solving advection-diffusion equations having much better properties than the standard schemes, have been proposed. Unfortunately the presented algorithms are sometimes very complicated. The different approaches to solve the transport equation with the dominated advection are presented by: Pantakar (1980), Fletcher (1991), Holly and Preissmann (1977), Neumann (1984), Yeh et al. (1992), Szymkiewicz (1995).

One of the most efficient techniques to solve Eqs. (3) and (4) is the splitting method. This approach is very often used to solve many different problems. It consists of splitting the transport equation into the advection part and the diffusion part in each time step and solving the two equations obtained separately. To solve the advection-diffusion equation this approach has already been introduced by Holly and Priessmann (1977) using the characteristic method for both parts of the split equation. For the interpolation they used the third degree polynomial based on two nodes. Another example of the splitting method has been presented by Szymkiewicz (1993). In this case the advection equation was solved by the method of the characteristic also but with third degree spline function to interpolation whereas the diffusion equation by the finite elements method. This algorithm was elaborated for the advection-diffusion equation with constant parameters, i.e. for the steady and uniform flow in the channel ($U =$ const) and for the constant coefficient of a longitudinal dispersion (E = const). The algorithm presented here is more general because it deals with the equations with variable parameters solved for the channels network.

2. The Initial-Boundary Conditions

For Eqs. (3) and (4) the initial-boundary problem is formulated. Functions $L(x,t)$ and $C(x,t)$ must satisfy the equations for $0 \leq x \leq X$ (where X is the length of the river reach) and $t \geq 0$, and the imposed initial-boundary conditions. The initial conditions are described as follows:

$$\text{for } t = 0 \; L(x,t) = L_i(x) \text{ and } C(x,t) = C_i(x) \text{ for } 0 \leq x \leq X$$

where $L_i(x)$ describes the initial distribution of BOD and $C_i(x)$ describes the initial distribution of DO along the river reach considered.
The boundary conditions are defined as follows:
– for $x = 0$ the following functions are imposed:

$$\left.\begin{aligned} L(0,t) &= L_0(t) \\ C(0,t) &= C_0(t) \end{aligned}\right\} \text{for } t \geq 0 \tag{5}$$

– for $x = X$ are imposed the functions:

$$\left.\begin{aligned} L(X,t) &= L_x(t) \\ C(X,t) &= C_x(t) \end{aligned}\right\} \text{for } t \geq 0 \tag{6a}$$

or their gradients:

$$\left.\frac{\partial L}{\partial x}\right|_x = \phi_L(t) \text{ for } t \geq 0 \tag{6b}$$

and

$$\left.\frac{\partial C}{\partial x}\right|_x = \phi_C(t) \text{ for } t \geq 0. \tag{6c}$$

As in practice there are no possibilities of satisfing the required conditions at the boundary $x = X$, it is usually assumed that the diffusion flux is equal to zero at the downstream end of the river reach, i.e. $\phi_L(t), \phi_C(t) = 0$. This means that at the downstream end the diffusive flux is neglected and only the advection transport occurs. Where the advection transport dominates, the influence of this simplification is insignificant. For transport in the channels network, with the exception of the boundary conditions imposed at the external points, as presented above, the additional conditions should be introduced at the points of the channels connection (Fig. 1a) as wellas at pollutants entrance (Fig. 1b). In both types of the inner boundaries the additional condition results from the mass conservation law, which includes the advection transport only. The accepted initial and boundary conditions enable of the solving Eqs. (3) and (4) for any open channels network.

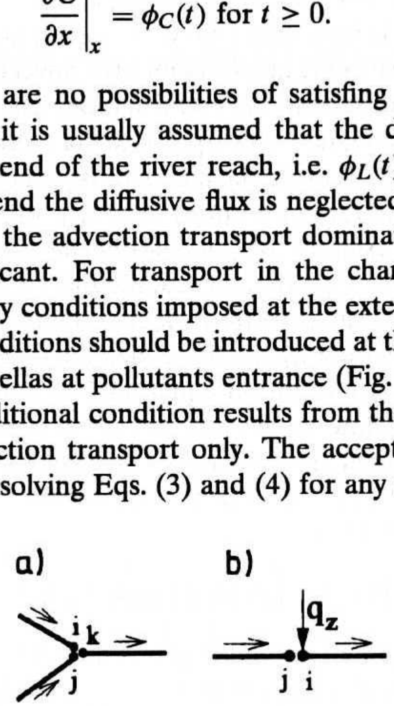

Fig. 1. The kinds of internal boundaries: a) the point of connection of channels; b) the point of entrance of pollutants

3. Solution of Advection-Diffusion Transport Equation

The splitting method

Eqs. (3) and (4) can be rewritten as follows:

$$\frac{\partial \phi}{\partial t} + F = 0 \tag{7}$$

where:

ϕ – is the function L or C,

F – includes all terms of the equation (1) or (2) except time derivative.

The general formula of integration of this equation at time interval Δt is as follows:

$$\phi_{t+\Delta t} = \phi_t + \Delta t \int_{t}^{t+\Delta t} F dt \tag{8}$$

where: ϕ_t and $\phi_{t+\Delta t}$ are the values at time level t and $t + \Delta t$.

Most of the known methods of integration for differential equation are based on the above formula. They differ in the manner of spatial approximation of the differential operators included in F, and in the method of integration in Eq. (8). From the derivation of the transport equation it results that it constitutes the superposition of the two different elementary processes: advection and diffusion. The variable F that occurs in Eq. (7) can be presented in the form of two components:

$$F = F^{(1)} + F^{(2)} \tag{9}$$

where:

$F^{(1)}$ – represents the advection part,
$F^{(2)}$ – represents the diffusion part including the source terms.

Consequently this equation can be written in the following form:

$$\frac{\partial \phi}{\partial t} + F^{(1)} + F^{(2)} = 0. \tag{10}$$

Its solution is as follows:

$$\phi_{t+\Delta t} = \phi_t + \Delta t \int_{t}^{t+\Delta t} \left(F^{(1)} + F^{(2)}\right) dt = \phi_t + \Delta t \int_{t}^{t+\Delta t} F^{(1)} dt + \Delta t \int_{t}^{t+\Delta t} F^{(2)} dt. \tag{11}$$

Introducing the additional variable:

$$\phi^{(1)}_{t+\Delta t} = \phi_t + \Delta t \int_{t}^{t+\Delta t} F^{(1)} dt \tag{12}$$

the Eq. (11) will assume a form which is the solution of Eq. (7):

$$\phi_{t+\Delta t} = \phi^{(1)}_{t+\Delta t} + \Delta t \int_{t}^{t+\Delta t} F^{(2)} dt. \tag{13}$$

The above algorithm can be written in a more general form. The solution of advection-diffusion equation in the time interval $[t, t + \Delta t]$ can be obtained as the result of solutions of the following series of equations:

$$\frac{\partial \phi^{(1)}}{\partial t} + F^{(1)} = 0 \tag{14a}$$

with the initial conditions: $\phi_t^{(1)} = \phi_t$,

$$\frac{\partial \phi^{(2)}}{\partial t} + F^{(2)} = 0 \tag{14a}$$

with the initial condition: $\phi_t^{(2)} = \phi_{t+\Delta t}^{(1)}$.

Finally the unknown value ϕ at time level $t + \Delta t$ equals: $\phi_{t+\Delta t} = \phi_{t+\Delta t}^{(2)}$. With reference to Eqs. (3) and (4) the described process of solution is as follows:

– for the biochemical oxygen demand (BOD):

$$\frac{\partial L^{(1)}}{\partial t} + \left(U - \frac{E}{A}\frac{\partial A}{\partial x}\right)\frac{\partial L^{(1)}}{\partial x} = 0 \tag{15a}$$

with the initial condition: $L_t^{(1)} = L_t$

$$\frac{\partial L^{(2)}}{\partial t} - E\frac{\partial^2 L^{(2)}}{\partial x^2} + (K_1 + K_3)L^{(2)} = 0 \tag{15b}$$

with the initial condition: $L_t^{(2)} = L_{t+\Delta t}^{(1)}$

– for dissolved oxygen (DO):

$$\frac{\partial C^{(1)}}{\partial t} + \left(U - \frac{E}{A}\frac{\partial A}{\partial x}\right)\frac{\partial C^{(1)}}{\partial x} = 0 \tag{16a}$$

with the initial condition: $C_t^{(1)} = C_t$

$$\frac{\partial C^{(2)}}{\partial t} - E\frac{\partial^2 C^{(2)}}{\partial x^2} - K_2(C_s - C) + K_1 L^{(2)} + B = 0 \tag{16b}$$

with the initial condition: $C_t^{(2)} = C_{t+\Delta t}^{(1)}$.

Solving the Eqs. (15a, b) and (16a, b) in every time step the demanded values at time level $t + \Delta t : L_{t+\Delta t} = L_{t+\Delta t}^{(2)}$ and $C_{t+\Delta t} = C_{t+\Delta t}^{(2)}$ are obtained.

As mentioned before, to solve the advection equation the characteristic method and for the diffusion equation, the finite-elements method are proposed.

The solution of the advection equation

The advection equation in general form is considered:

$$\frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x} = 0 \tag{17}$$

where $\nu = \nu(x, t)$ is a velocity.

Equations of this type are (15a) as well as (16a). It is well known that the standard schemes of the finite-difference and finite-element methods do not give satisfactory results. For example the backward difference scheme generates the great numerical diffusion and the Crank-Nicolson scheme leads to solutions with oscillations. In this paper the characteristics method will be applied. Equation (17) in the Lagrange system moving with velocity ν can be written as follows:

$$\frac{D\phi}{Dt} = 0 \tag{18a}$$

for

$$\frac{dx}{dt} = \nu \tag{18b}$$

where $D\phi/Dt$ represents the material derivative of the function ϕ.

This means that the concentration does not change along the curve (18b) which is the characteristic of Eq. (17). The shape of this characteristic passing through any node of the numerical mesh is shown in Fig. 2. The characteristic passing through the node (x_j, t_{n+1}) crosses time level t_n, at point x^*. Thus if the concentration at this point at time t_n is known, then, according to the Eq. (18a), the concentration at point x_j at level t_{n+1} will be known also. To define x^* one needs to integrate the characteristic equation in time interval $[t_n, t_{n+1}]$. The characteristic Eq. (18b) is the ordinary differential equation. Its solution should satisfy the initial condition, i.e:

$$x(t = t_{n+1}) = x_j$$

One can obtain the approximate solution, using, for example, the Runge-Kutta fourth order method (Stoer and Bulirsh 1980):

$$k_1 = \Delta t f(t_{n+1}, x_j), \tag{19a}$$

$$k_2 = \Delta t f\left(t_{n+1} + \frac{\Delta t}{2}, x_j + \frac{k_1}{2}\right), \tag{19b}$$

$$k_3 = \Delta t f\left(t_{n+1} + \frac{\Delta t}{2}, x_j + \frac{k_2}{2}\right), \tag{19c}$$

$$k_4 = \Delta t f(t_{n+1} + \Delta t, x_j + k_3), \tag{19d}$$

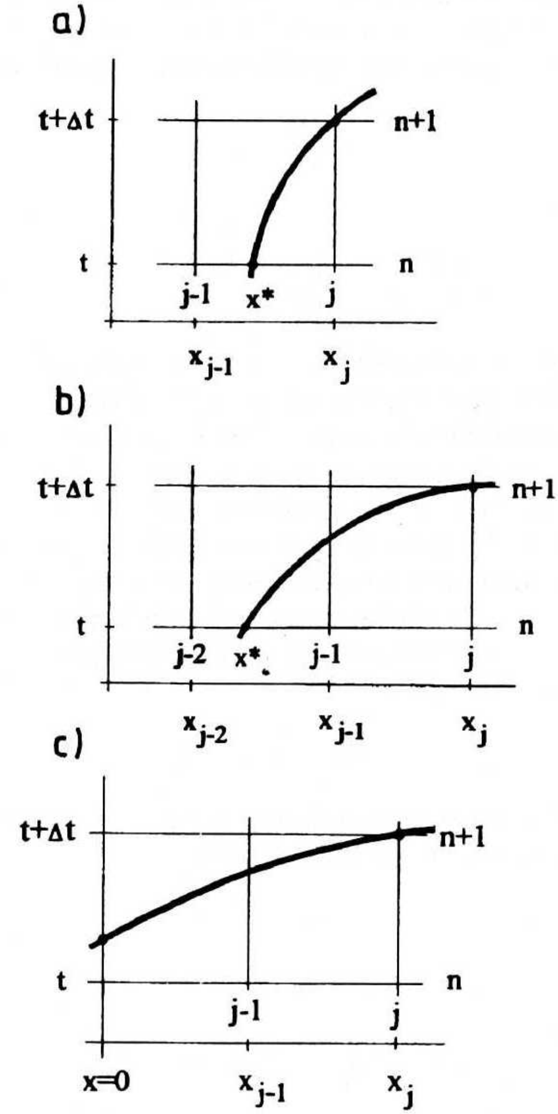

Fig. 2. The possible location of characteristic passing through the node (x_j, t_{n+1}) depending on time step Δt

$$x^* = x_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{19e}$$

Because the integration is being carried out in decreasing time, Δt has a negative value. The velocity v inside the mesh can be computed by interpolation between the neighbouring nodes of coordinates: (x_{j-1}, t_n), (x_{j-1}, t_{n+1}), (x_j, t_{n+1}) and (x_j, t_n). The concentration at point x^* can be solved by interpolation betweeen mesh nodes at time level t_n. As the standard ways of interpolation do not afford satisfactory results (Cunge et al. 1980) application of interpolation using the third degree spline function is proposed in the form:

$$\phi(x) = \phi_j + \alpha_j (x - x_j) + \beta_j (x - x_j)^2 + \gamma_j (x - x_j)^3 \tag{20}$$

$$\text{for } x_j \le x \le x_{j+1} \text{ and } j = 1, 2, \ldots, M - 1$$

where:

ϕ_j – is the value of the function at j node,

$\alpha_j, \beta_j, \gamma_j$ – are the coefficients.

This formula coincides with Taylor's expansion of the function ϕ at points x_j. The coefficients of Eq. (20) are defined as follows:

$$\alpha_j = \left.\frac{\partial \phi}{\partial x}\right|_{x_j}, \beta_j = \left.\frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}\right|_{x_j}, \gamma_j = \left.\frac{1}{6}\frac{\partial^3 \phi}{\partial x^3}\right|_{x_j},$$

and are calculated using the nodal values $\phi_j (j = 1, 2, 3, \ldots, M)$. If the values of ϕ are known at all nodes at level t_n, we can describe the polynomial (20) solving its coefficient $\alpha_j, \beta_j, \gamma_j$ in all intervals $j = 1, 2, 3, \ldots, M - 1$.

The spline function and the manner of solving its coefficients is presented in greater detail for example by Stoer (1979). Finding the coefficients requires the solving of the algebraic system of linear equations with tridiagonal matrix. It is therefore a similar effort of calculation as needed by the implicite scheme, which also involves solving the system of equations with tradiagonal matrix in every time step.

Knowing the coefficients of the polynomial (20) one can define the concentration at all nodes at time level t_{n+1}:

$$\phi_{j+1,n+1} = \phi_j + \alpha_j (x^* - h_j) + \beta_j (x^* - x_j)^2 + \gamma_j (x^* - x_j)^3 \tag{21}$$

The solving of a diffusion equation

The diffusion equation in the following form is considered:

$$\frac{\partial \phi}{\partial t} - E\frac{\partial^2 \phi}{\partial x^2} - \varphi = 0 \tag{22}$$

where:

E – is the longitudinal dispersion coefficient,

φ – is the source term, which for Eq. (1) is $\varphi = -(K_1 + K_3)L$ and for Eq. (2) is $\varphi = K_2(C_s - C) - K_1L - B$.

Both (16b) and (15b) are equations of this kind. The diffusion equation can be solved by applying the well known methods. Most of them guarantee satisfactory accuracy and do not cause any numerical problems. Here, the finite element method with the linear shape functions is applied. The standard operation, according to Galerkin's procedure (Zienkiewicz 1972) leads to the following system of ordinary differential equations:

$$\mathbf{S}\frac{d\boldsymbol{\phi}}{dt} + \mathbf{A}\boldsymbol{\phi} + \mathbf{F} = 0 \tag{23}$$

with the initial condition $\boldsymbol{\phi}_t = \boldsymbol{\phi}_{t+\Delta t}$
where: $\mathbf{S}$ and $\mathbf{A}$ are the tridiagonal matrices,

$$\boldsymbol{\phi} = col\{\phi_1\phi_2, \dots \phi_M\},$$

$$\frac{d\boldsymbol{\phi}}{dt} = col\left\{\frac{d\phi_1}{dt}, \frac{d\phi_2}{dt}, \dots \frac{d\phi_M}{dt}\right\},$$

$\mathbf{F}$ is the vector representing the source terms.

To solve this system the trapezoidal rule is applied:

$$\boldsymbol{\phi}_{t+\Delta t} = \boldsymbol{\phi}'_t + \frac{\Delta t}{2}\left(\boldsymbol{\phi}'_t + \boldsymbol{\phi}_{t+\Delta t}\right) \tag{24}$$

with: $\boldsymbol{\phi}' = d\boldsymbol{\phi}/dt$.

It leads to the following system of algebraic equations with the tridiagonal matrices:

$$\left(\mathbf{S} + \frac{1}{2}\Delta t\mathbf{A}\right)\boldsymbol{\phi}_{t+\Delta t} = \left(\mathbf{S} - \frac{1}{2}\Delta t\mathbf{A}\right)\boldsymbol{\phi}_t - \frac{1}{2}\Delta t\,(\mathbf{F}_t + \mathbf{F}_{t+\Delta t}). \tag{25}$$

The solution of this system of equations is the vector $\boldsymbol{\phi}_{t+\Delta t}$, which is the solution of the diffusion equation at time level $t + \Delta t$ required.

4. The Stability and Accuracy of the Solution

The applied approach towards solving the transport advection-diffusion equation is based mainly on using the different techniques to solve the equations obtained by the decomposition of the governing equation. Namely the advection equation has been solved using the characteristic method and the diffusion one by the Crank-Nicolson scheme with the finite element method.

As far as the characteristics method is concerned, the version used in this paper does not restrict the applied time step Δt. The interpolation formula (20) holds for the situation shown in Fig. 2a. In this case the characteristic passing through the node (x_j, t_{n+1}) crosses the axis x between two nodes x_{j-1} and x_j. Should the applied integration step be bigger, the characteristic point of passage (x_j, t_{n+1}) would cross axis x between two other nodes, for example as shown in Fig. 2b. The concentration at the node (x_j, t_{n+1}) would therefore be the result of interpolation between a pair of nodes other than shown in Fig. 2b. Thus, a large time step of integration would only cause a change of interpolation formula (20). If the characteristic passing through the point (x_j, t_{n+1}) does not cross line $t = t_n$ but crosses the boundary $x = 0$ (Fig. 2c), the value of concentration at point (x_j, t_{n+1}) can be obtained on the basis of an imposed boundary condition, at the boundary $x = 0$. As far as the numerical stability of Crank-Nicolson scheme is concerned, it is known (Fletcher 1991), that for the diffusion equation it is absolutely stable. Therefore the proposed algorithm of the advection – diffusion equation solution does not require any limitations of Δt because of the numerical stability.

Another important feature of the numerical scheme is the accuracy of approximation. In the case of the advection equation interpolation by the spline function is used. Its equation (21) is otherwise an expansion of the ϕ function into Taylor's series including terms with the third order derivative. It is therefore the approximation of the third order with regard to x. The Runge-Kutta method applied for integrating of the characteristic is an approximation of the fourth order. Whereas the Crank-Nicolson scheme applied with the finite-elements to time integration of the diffusion equation is an approximation of the second order with regard to time t and third order to x (Fletcher 1991). Thus one can assume that the proposed algorithm ensures the second order approximation with regard to t and third order with regard to x.

5. Examples of Calculations

The efficiency of the proposed algorithm was checked by a series of test calculations which are compared with the analytical solutions. These tests verify the numerical diffusion produced by the method. The calculations were carried out for a simple channel network with the structure created by four arms as shown in Fig. 3. Each arm is divided into constant intervals of length Δx. It is assumed that a steady and uniform flow exists in the channels and that the flow discharge in the reach A – C and B – C is $Q = 5$ m^3/s and the cross-sectional area $A = 10$ m^2, whereas for C – D and D – E $Q = 10$ m^3/ s and $A = 20$ m^2 are accepted.

First the pure advection equation was solved. The initial condition was assumed in the form of triangular distribution of concentration along A – C and B – C. At nodes A and B the boundary conditions in the form $\phi(t) = 0 = \text{const}$ were imposed. As the velocities in the channels are constant, the analytical solution is the initial distribution shifted without any deformation along the channel

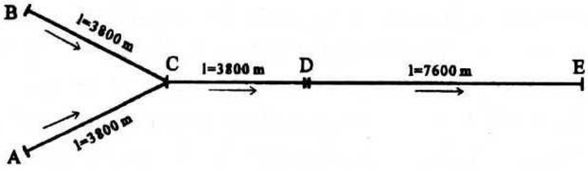

Fig. 3. Channel network accepted for testing

axis. The calculations were carried out for the above data, $\Delta x = 100$ m and various values of time steps. The results obtained for the advective Courant numbers $C_a = U\Delta t/\Delta x = 1.5$, 1.0 and 0.25 corresponding to time $t = 7200$ s and $t = 14400$ s are presented in Fig. 4. One can state that only a slight reduction of the peak concentration and slight oscillations are observed. With the decreasing of Δx the accuracy of solutions increases significantly. It results from the properties of the spline function, which for $\Delta x \to 0$ tends to the interpolated function.

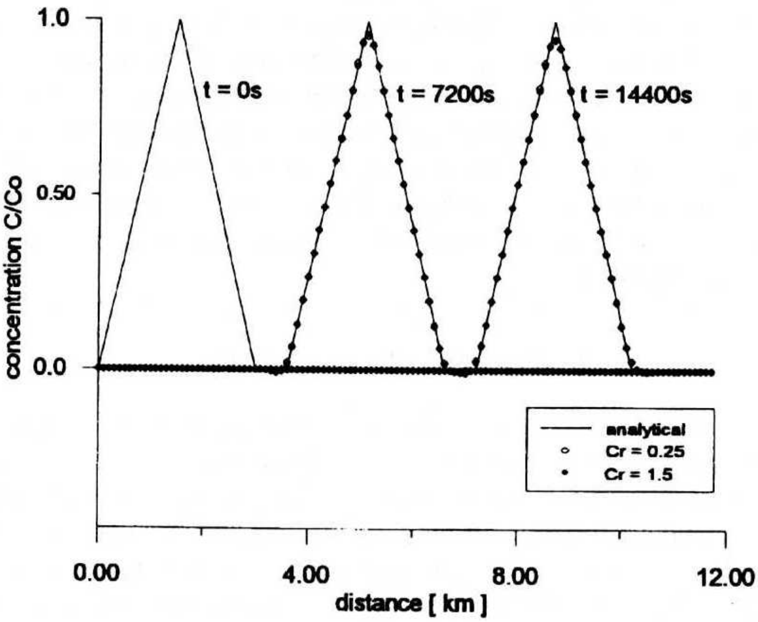

Fig. 4. The pure advective transport of the initial concentration in the form of a triangle

The second example deals with the propagation of the sharp front of concentration caused simultanously by advection and diffusion (without source terms). In the channels network as in the preceding example, the initial concentration is equal to zero, which means $C(x, t = 0) = 0$ for $x \geq 0$. At nodes A and B the accepted boundary conditions are as follows:

$$C(x=0,t)=\begin{cases} 0 \text{ for } t=0 \\ 1 \text{ for } t>0 \end{cases}$$

whereas at the node E $C(x=\infty,t)=0$ is imposed. For the above initial-boundary conditions the advective-diffusive transport equation has an analytical solution (Xue and Xin 1988):

$$C(x,t)=\frac{1}{2}erfc\left(\frac{x-Ut}{\sqrt{4Et}}\right)+\frac{1}{2}\exp\left(\frac{Ux}{E}\right)erfc\left(\frac{x+Ut}{\sqrt{4Et}}\right). \tag{26}$$

In Fig. 5 the comparison between the analytical and numerical solution is presented. This solution is obtained for $\Delta x = 100$ m, $C_a = 0.6$ and $E = 10$ m^2/s. It means that in this case the Peclet number $P_c = 5$ ($P_c = U\Delta x/E$), which indicates that the advection dominates in the transport process. Insignificant differences between the two solutions confirm good properties of the proposed method.

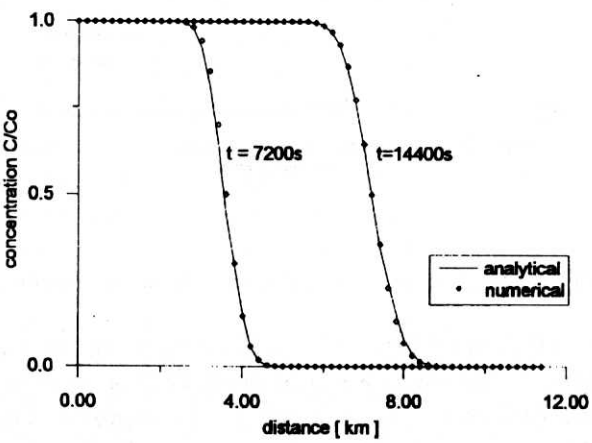

Fig. 5. Solution for the sharp front concentration transport caused by advection and diffusion

The third example deals with the solution of the advection-diffusion transport with the source term, which describes the decay of the transported matter. For the channels network as preceding, similar initial-boundary conditions are imposed. In this case the analytical solution for a steady state is as follows (Elliott and James 1984);

$$C(x)=C_0\exp\left(\frac{U}{2E}(1-m)x\right)\quad (x>0) \tag{27}$$

where:

$$m=\left(1+4\frac{KE}{U^2}\right)^{\frac{1}{2}},$$

C_0 – concentration at the upstream end (node A and B),
K – constant of decay.

In Fig. 6 the analytical and numerical solutions are presented. They were obtained for $C_a = 1$, $K = 0.0004$ s^{-1}, $E = 50$ m^{2}s^{-1}, $\Delta x = 100$ m. It can be seen that also in this case the good conformity between both solutions is obtained.

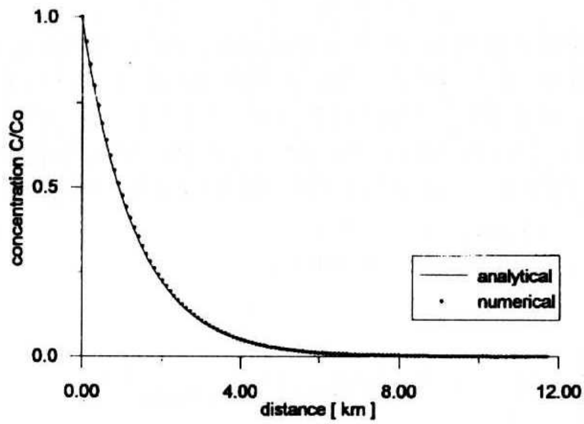

Fig. 6. Solution of the advection-diffusion transport with source term for a steady state

To illustrate the solution of Eqs. (1) and (2) describing the transport of *BOD* and *DO* the discharge of the waste water at point *D* (Fig. 3) is assumed. Initially, *BOD* $(x, t = 0)$ and *DO* $(x, t = 0) = 5$ mg l^{-1} is imposed. The time constant results in the *BOD* increasing to 30 mg l^{-1} at point *D*. For the network of the same channels, the hydraulic data and for of the $K_1 = 1.2 \cdot 10^{-4}$ s^{-1}, $K_2 = 9 \cdot 10^{-4}$ s^{-1}, $K_3 = 5 \cdot 10^{-6}$ s^{-1}, $C_s = 5$ mg l^{-1}, $E = 5$ m^2 s^{-1}, $\Delta t = 100$ s, $\Delta x = 100$ m the obtained results of calculations are presented in Fig. 7. The computed shape of *BOD*(x) and *DO*(x) is typical downstream of the point discharging the waste water of the time constant load.

The last example deals with the modelling of *BOD* and *DO* for a real river, the Reda 46 km long, which flows into the Gulf of Gdańsk. At three control stations S_1, S_2 and S_3 located along the river with distance 6300 m between S_1 and S_2 and 2000 m between S_2 and S_3, two series of observations were collected over a period 24 h. The measurements of the *BOD*, *DO*, water temperature and flow velocities were recorded at time intervals of 1 h. The stream flow in this time was about 3 m^3 s^{-1}, and wetted cross-sectional areas varied from 8 to 10 m^2. The average river flow velocities in two cases were 0.34–0.44 m/s. The

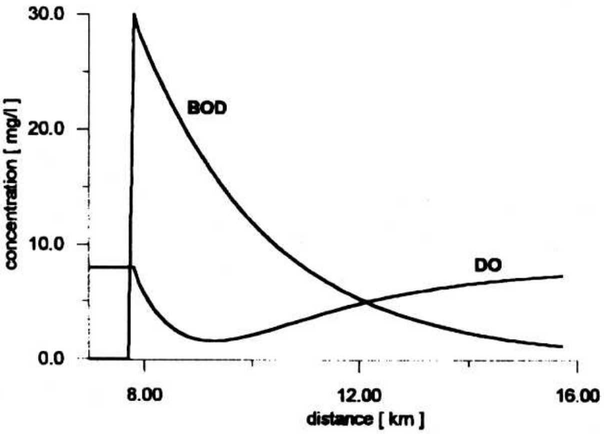

Fig. 7. Distribution of BOD and DO downstream of the point of entrance of pollutants

time translation of the traser between stations S_1 and S_2 was about 7 h. The observations recorded at station S_2 were imposed as the boundary conditions for transport problems. The following values of parameters were accepted: Δx = 100 m, Δt = 240 s, $E = 0.8\ \mathrm{m^2s^{-1}}$ and $B = 0$. The calculations indicated that the influence of the parameters mentioned above is insignificant. Coefficients K_1 and K_2 have an outstanding influence on the results. Using the trial and error method the values of these parameters were defined, to obtain better agreement between computations and observations at control stations S_2 and S_3. Basing on data from 29.06.1994 the values of $K_1 = 8 \cdot 10^{-6}\ \mathrm{s^{-1}}$ and $K_2 = 7 \cdot 10^{-8}\ \mathrm{s^{-1}}$ were obtained. The results of calculations and observations are presented in Figs. 8 and 9. For the same values of parameters, calculations for a second set of data from 25.10.1994 were carried out. In this case, as is shown in Figs. 10 and 11, the agreement obtained is similar to that for the first example. This means that the parameters K_1 and K_2 do not change very much and they are similar in both summer and autumn. The estimated values of these parameters for the Reda river conform with those proposed for rivers by others authors (James 1986).

6. Conclusions

The presented method of solving the advective-diffusive transport equations of BOD and DO in a network of open channels gives satisfactory results. The technique of splitting the transport equation into two parts enables the advection equation by the method of characteristics to be solved. The interpolation using the spline function of third order ensures significant accuracy of solution and requires a calculation effort comperable to the one exigent by the implicit schemes. The method is absolutely stable and does not require any limitation of time step. The comparison of the computational results with analytical solutions and real

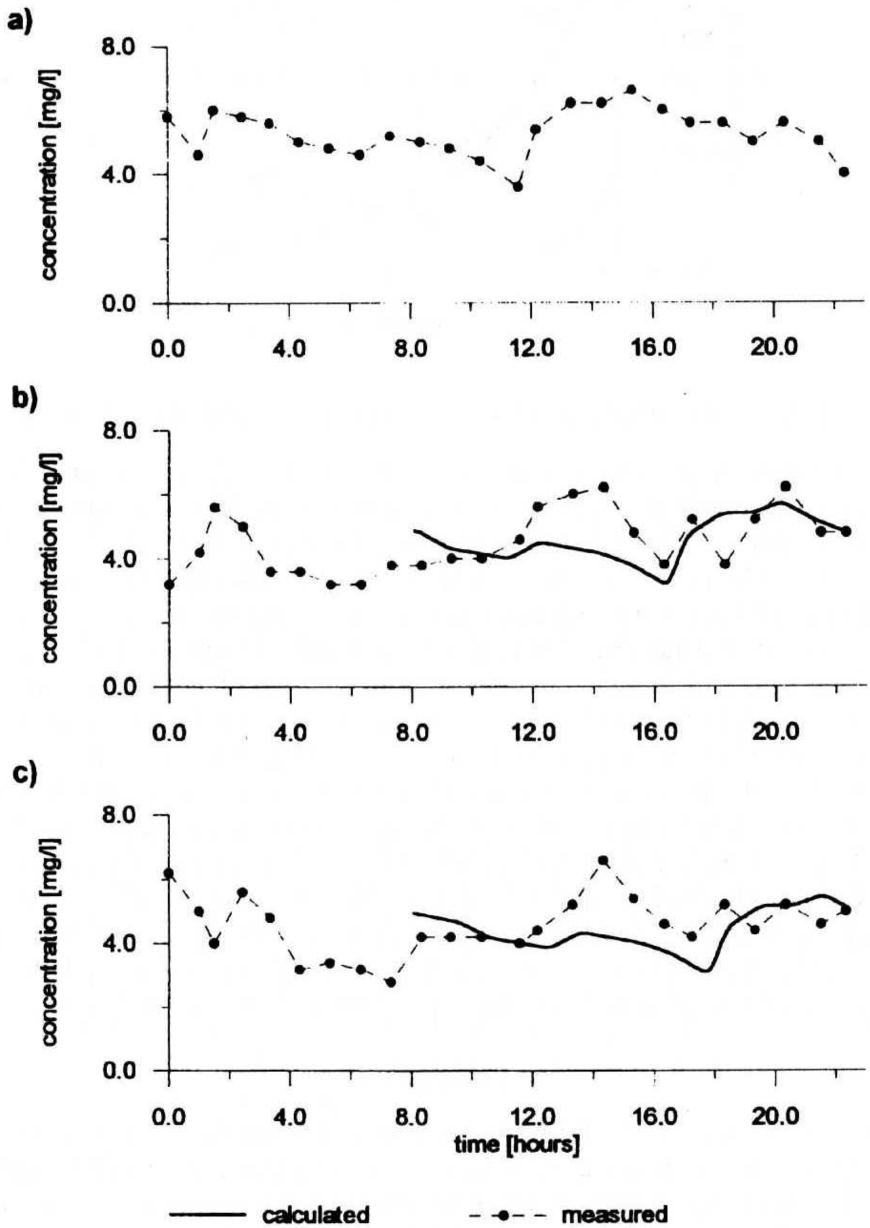

Fig. 8. Predicted and observed *BOD* concentrations at stations S_1 (a), S_2 (b) and S_3 (c) on 29.06.1994

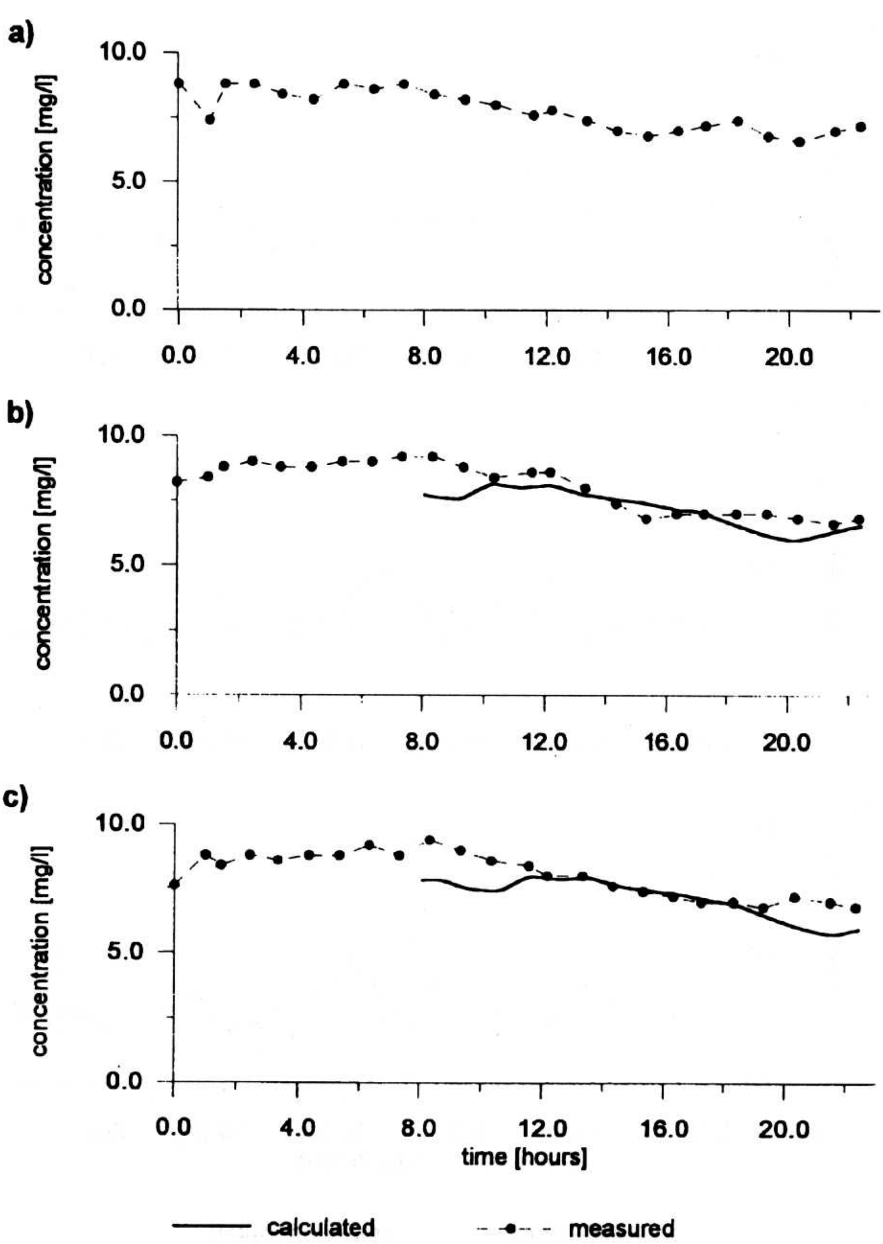

Fig. 9. Predicted and observed *DO* concentrations at stations S_1 (a), S_2 (b) and S_3 (c) on 29.06.1994

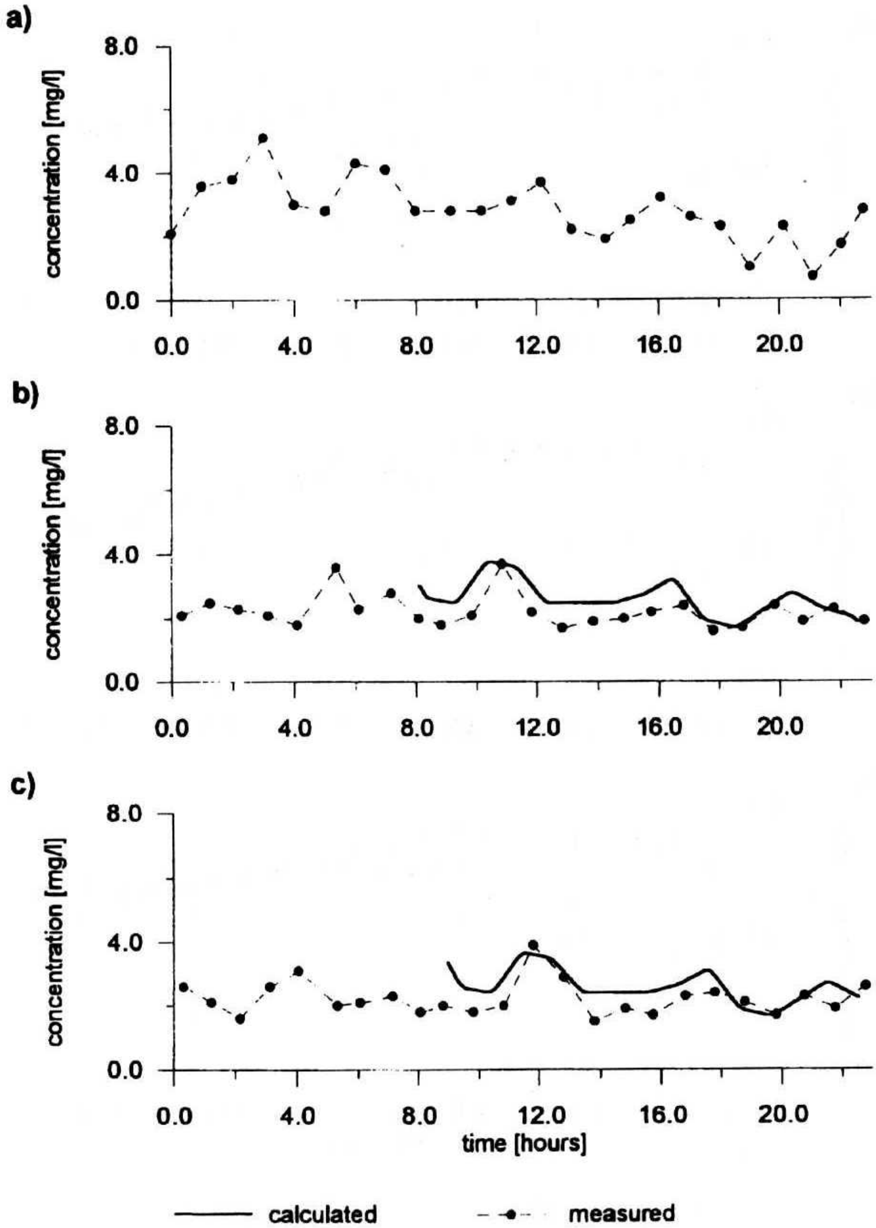

Fig. 10. Predicted and observed *BOD* concentrations at stations S_1 (a), S_2 (b) and S_3 (c) on 25.10.1994

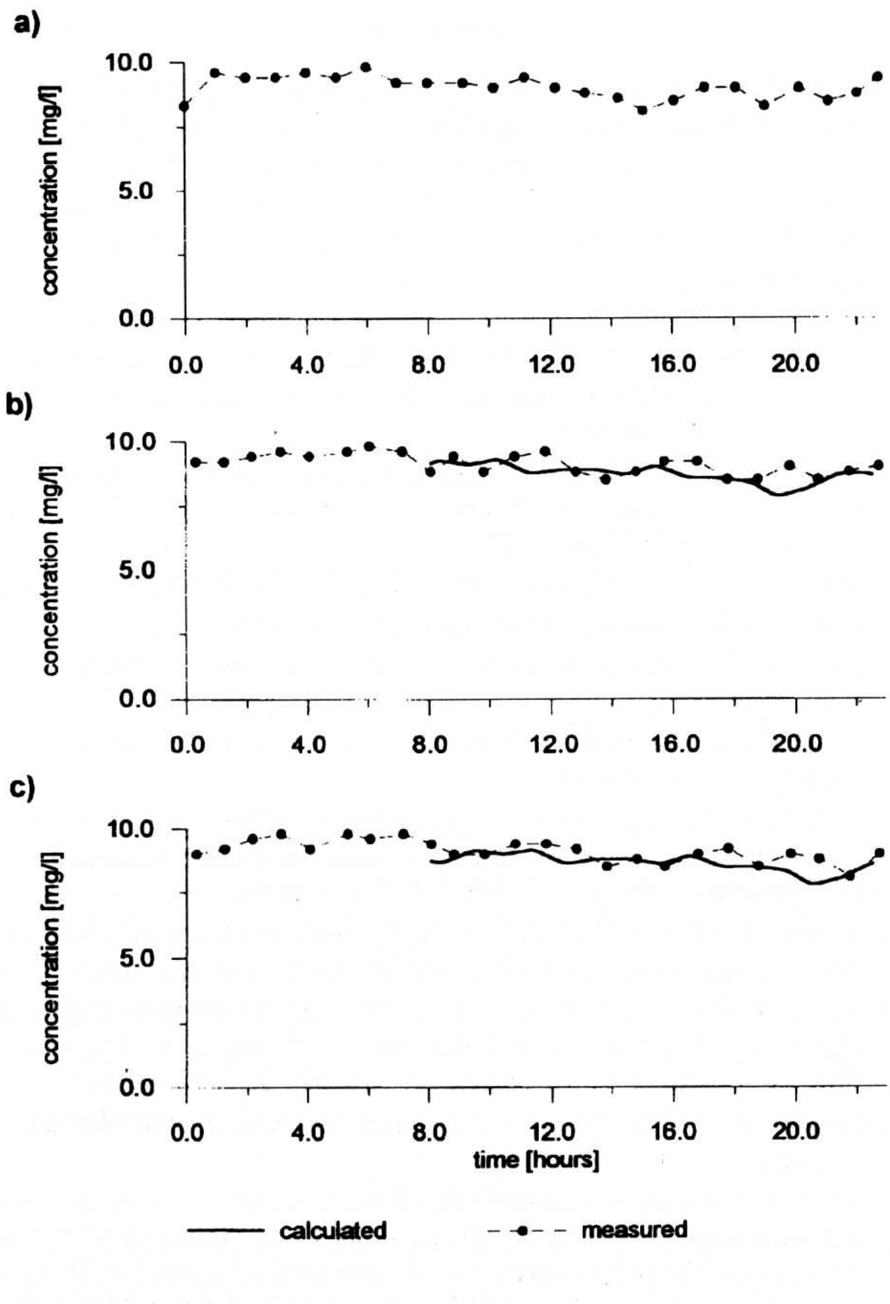

Fig. 11. Predicted and observed *DO* concentrations at stations S_1 (a), S_2 (b) and S_3 (c) on 25.10.1994

observation indicates that the proposed method can be useful for environmental engineers.

References

Cunge J., Holly Jr F. M., Verwey A. (1980), *Practical Aspects of Computational River Hydraulics*, Pitman, London.

Elliot D. J., James A. (1984), Models of Water Quality in Rivers, In: A. James (Editor), *An Introduction to Water Quality Modelling*, J. Wiley, New York.

Fletcher C. A. J. (1991), *Computational Techniques for Fluid Dynamics*, Vol. 1, Springer-Verlag.

Holly F. Jr, Preissmann A. (1977), Accurate Calculation of Transport in Two Dimensions, *Journal of Hydraulic Engineering ASCE*, vol. 103, HY11.

James A. (1986), *Mathematical models in water pollution control*, Arkady, Warsaw (Polish edition).

Neumann S. P. (1984), Adaptive Eulerian-Langrangian Finite Element Method for Advection-Dispersion, *International Journal for Numerical Methods in Engineering*, Vol. 20, 321–327.

Patankar S. V. (1980), *Numerical Heat Transfer and Fluid Flow*, Hemisphere Publishing Corporation Washington, McGrow-Hill.

Stoer J. (1979), *Einfürung in die Numerische Mathematik I*, PWN (Polish edition), Warsaw.

Stoer J., Bulirsch R. (1980), *Einfürung in die Numerische Mathematik II*, PWN (Polish edition), Warsaw.

Szymkiewicz R. (1993), Solution of the advection-diffusion equation using the spline function and finite elements, *Communications in Numerical Methods in Engineering*, Vol. 9, 197–206, J. Willey & Sons, Ltd.

Szymkiewicz R. (1995), Method to solve 1-D unsteady transport and flow equations, *Journal of Hydraulic Engineering ASCE*, Vol. 121, No. 5, 396–403.

Yeh G. T., Chang J. R., Short T. E. (1992), An Exact Peak Capturing and Oscillation-Free Scheme to Solve Advection-Dispersion Transport Equations, *Water Resources Research*, Vol. 28, No. 11, 2937–2951.

Zienkiewicz O. (1972), *The Finite Element Method*, Arkady Warsaw (Polish edition).

Xue Y., Xin C. (1988), A characteristic alternating direction implicit scheme for the advection-dispersion, In: M. A. Celia et al. (Editors), *Computational Methods in Water Resources*, Vol. 2 *Numerical Methods for Transport and Hydrologic Processes*, Proc. VII International Conference, MIT, USA, June, Elsevier S. P., Amsterdam.