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Wave Run-up on Gentle Slopes: a Hybrid Approach

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Abstract

Simplified Eulerian-Langrangian equations are derived to simulate long wave motion in coastal regions. The principal aim of the model is to obtain a better prediction of wave run-up on mild, non necessarily planar slopes. Artificial bottom friction is implemented into the equations as well.

In some very special cases, the equations could be solved analytically.

Selected results of numerical model are attached herein. Special attention is focused on asymmetric shoaling wave profiles. Close agreement between analytical and numerical wave run-up predictions is obtained. Although numerical calaculations concern regular waves, the random ones can be successfully treated as well. The applied Eulerian-Langrangian approach is shown to have many serious advantages (such as e.g. much simpler boundary conditions, no back-flow problems etc.) in comparison with the pure Eulerian one, at least in the considered wave run-up problem.

Notation

A – wave amplitude,
c – long wave celerity,
f – friction coefficient,
g – gravitational acceleration,
h – water depth at position x,
h^{ξ} – water depth at position x^{ξ},
H – wave height, usually at deep water,
H_1 – incoming wave height,
H_2 – reflected wave height,
k – wave number,
l – horizontal distance between origin of co-ordinate system and slope toe,
J_n – Bessel function of n-th order,

R – maximum wave run-up height,
R/H – relative wave run-up height,
t – time,
x – initial position of particle ($t = 0$),
x^{ξ} – position of particle at time t,
Y_n – Weber function of n-th order,
α – slope angle,
ζ – water elevation at position x,
ζ^{ξ} – water elevation at position x^{ξ},
ξ – horizontal displacement of particle,
ρ – water density,
τ^{ξ} – bottom friction at position x^{ξ} according to Voltsinger et al. (1989),
ω – angular frequency.

1. Introduction

Wave run-up on natural beaches is undoubtedly one of the most important phenomena occurring in the surf zone. Estimation of its size is e.g. very important from the point of view of the design of breakwaters and seawalls.

There are many mathematical models simulating wave motion on natural beaches. The deterministic ones are usually expressed in terms of either the Eulerian or the Lagrangian point of view. Worthy of mention here are the works of Kobayashi et al. (1987, 1990, 1992) or Shuto (1967, 1968, 1972, 1974).

However both, i.e. the Eulerian and the Lagrangian approach, applied separately, are not free of some disadvantages. In Eulerian-type models, there are difficulties connected with the description of moving boundaries, whereas the Lagrangian ones are incompatible with the common way of making measurements.

By treating the problem considered with the hybrid approach, many of these inconveniences disappear. In particular there is no problem with the description of moving boundaries. At the same time records of surface elevation changes at fixed points can also be obtained, though it requires some additional, simple calculations.

The model presented herein is based on a one-dimensional description of the fluid motion i.e. in fact depth-averaged velocities are analysed. Using the aforementioned hybrid approach, some relatively simple equations describing the mass and momentum transport could be obtained. They are valid for long waves travelling over a mild-sloping, impermeable (but otherwise optional) bottom. The model allows calculations of wave run-up on slopes for either regular or irregular waves. Some analytical solutions are also given.

2. General Description of the Model

A definition sketch of the hybrid model is shown in Fig. 1. The x-axis is taken (at least at the beginning) on the still water level, the z-axis – vertically, positive downwards. Intersection of the slope with the still water level gives the origin of the co-ordinate system. The bottom is assumed to be motionless.

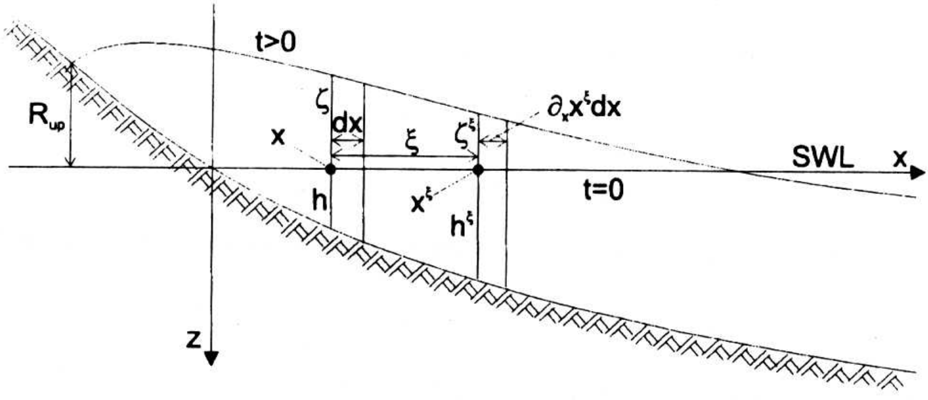

Fig. 1. Definition sketch of the hybrid model

Let x be the initial (corresponding to $t = 0$) position of a selected particle identified with a marked fluid cross-section and x^{ξ} its position at time t. The horizontal displacements of this particle are described by ξ. Water elevation changes, at a given position x, are denoted by ζ. Both h and h^{ξ} denote water depths at x and x^{ξ}, respectively.

The assumptions mentioned above may be summarised by the following set of equations:

$$
\begin{aligned}
&\xi = \xi(x,t),\ x^{\xi} = x + \xi(x,t),\\
&\zeta = \zeta(x,t),\ \zeta^{\xi} = \zeta(x^{\xi},t) = \zeta(x+\xi,t),\\
&h = h(x), \quad h^{\xi} = h(x^{\xi}) = h(x,\xi).
\end{aligned}
\tag{1}
$$

As may easily be noticed, in the presented model, ξ, x^{ξ} and ζ^{ξ} are used in the Lagrangian sense whereas x and ζ represent the Eulerian viewpoint. The wave run-up height R is attained by the water particle which rests at the initial instant on the shore line.

3. Governing Equations

Let us consider the one-dimensional motion of an inviscid fluid. The water pressure is assumed to be hydrostatic. Then the equations of mass and momentum transport are as follows:

$$\rho h dx = \rho(h^\xi + \zeta^\xi)\partial_x x^\xi dx, \tag{2}$$

$$\rho(h^\xi + \zeta^\xi)\partial_x x^\xi dx\, \partial_{tt}\xi + \rho g(h^\xi + \zeta^\xi)\partial_x \zeta dx - \tau^\xi \partial_x x^\xi dx = 0, \tag{3}$$

where τ^ξ denotes the bottom friction at point x^ξ and is assumed to depend mainly on $\partial_t \xi$.

The arrangement of them separately gives

$$\zeta^\xi = (1 + \partial_x \xi)^{-1} h - h^\xi, \tag{4}$$

$$\partial_{tt}\xi = -g(1 + \partial_x \xi)^{-1}\partial_x \zeta^\xi + (1 + \partial_x \xi)\tau^\xi/\rho h. \tag{5}$$

Substitution of Eq. (4) into Eq. (5) yields the following single-ξ equation:

$$\partial_{tt}\xi = g(1 + \partial_x \xi)^{-1}\partial_x \left[h^\xi - (1 + \partial_x \xi)^{-1} h\right] + (1 + \partial_x \xi)\tau^\xi/\rho h. \tag{6}$$

Now introduce the assumptions of mild bottom changes ($|\partial_x h| << 1$) and simultaneously, mild displacement changes, $|\partial_x \xi| << 1$.

The second one is also equivalent to the long wave assumption. In this case the following approximate formulas are valid:

$$h^\xi \approx h + \xi \partial_x h, \tag{7}$$

$$(1 + \partial_x \xi)^{-1} \approx (1 - \partial_x \xi). \tag{8}$$

Now, Eqs. (4) and (6) may be simplified as follows:

$$\zeta^\xi \approx -\partial_x (h\xi), \tag{9}$$

$$\partial_{tt}(h\xi) = gh\partial_{xx}(h\xi) + \tau^\xi/\rho. \tag{10}$$

Combining Eqs. (9) and (10), we obtain the following simplified single-ζ^ξ equation:

$$\partial_{tt}\zeta^\xi = \partial_x(c^2 \partial_x \zeta^\xi) - \partial_x \tau^\xi/\rho, \tag{11}$$

where $c = \sqrt{gh}$ denotes the celerity of the long wave.

4. Some Analytical Solutions

In the following analysis, simplified bottom topography is assumed. The area of motion is divided into a region of constant water depth and one of uniform gentle slope. The connection of the regions is at point $x = l$. A sketch of the model is shown in Fig. 2.

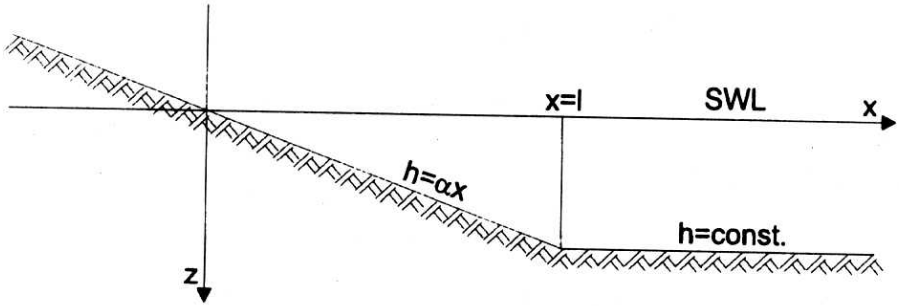

Fig. 2. Definition sketch of the hybrid model with simplified bottom topography

4.1. The Frictionless Case

Dropping the last term on the right-hand side of Eq. (11) one obtains the following equation:

$$\partial_{tt}\zeta^{\xi} = \partial_x(c^2\partial_x\zeta^{\xi}), \tag{12}$$

where $c^2 = gh$.

4.1.1. Region of Constant Water Depth

The assumption of h =const. (and thus c =const.) yields the classical wave equation:

$$\partial_{tt}\zeta^{\xi} = c^2\partial_{xx}\zeta^{\xi}. \tag{13}$$

A general solution of this equation can be given by

$$\zeta^{\xi} = \varphi_1(x+ct) + \varphi_2(x-ct). \tag{14}$$

Assumption of the harmonic motion yields the following solution:

$$\zeta^{\xi} = A_1 \exp\left[i(kx+\omega t)\right] + A_2 \exp\left[-i(kx-\omega t)\right]. \tag{15}$$

4.1.2. Region of Uniform Gentle Slope

With $h = \alpha x\,(\alpha << 1)$, Eq. (12) assumes the form

$$\partial_{tt}\zeta^{\xi} = \partial_x(\alpha g x\,\partial_x\zeta^{\xi}). \tag{16}$$

In order to solve this equation, a new parameter β_r and independent variable s are introduced:

$$\beta_r = \frac{4\omega^2}{g\alpha} = \frac{4hk^2}{\alpha}, \tag{17}$$

$$s^2 = \beta_r x. \tag{18}$$

Now, Eq. (16) can be rewritten as follows:

$$\partial_{tt}\zeta^{\xi} = \frac{\omega^2}{s}\partial_s(s\,\partial_s\zeta^{\xi}). \tag{19}$$

Assuming the solution in the form

$$\zeta^{\xi} = A(s)\exp(i\,\omega t), \tag{20}$$

the following Bessel equation of the zeroth order is obtained:

$$A''(s) + \frac{1}{s}A'(s) + A(s) = 0. \tag{21}$$

A general solution of this equation is given by

$$A(s) = C_1 J_0(s) + C_2 Y_0(s), \tag{22}$$

where J_0 and Y_0 denote, respectively, Bessel and Weber functions of the zeroth order. The boundary condition, $A(0) = R$, yields the following solution

$$A(s) = RJ_0(s), \tag{23}$$

where R denotes the wave run-up height on the slope.

In such a way the solution of Eq. (16) may be written as follows:

$$\zeta^{\xi} = RJ_0(s)\exp(i\,\omega t) = RJ_0\left(\sqrt{\beta_r x}\right)\exp(i\,\omega t). \tag{24}$$

4.1.3. Connection of Both Regions

According to Eqs. (15) and (24) we have:

$$\text{for } x \le l: \ \zeta^{\xi} = \zeta^{\xi}_{-} = RJ_0\left(\sqrt{\beta_r x}\right)\exp(i\,\omega t), \tag{25}$$

$$\text{for } x \ge l: \ \zeta^{\xi} = \zeta^{\xi}_{+} = A_1\exp[i\,(kx+\omega t)] + A_2\exp[-i\,(kx-\omega t)]. \tag{26}$$

Now the following conditions at matching point, $x = l$, are applied:

$$\begin{cases} \zeta^{\xi}_{-}(x=l) = \zeta^{\xi}_{+}(x=l) \\ \partial_x\zeta^{\xi}_{-}|_{x=l} = \partial_x\zeta^{\xi}_{+}|_{x=l}. \end{cases} \tag{27}$$

Then

$$\begin{cases} RJ_0\left(\sqrt{\beta_r l}\right)\exp(i\,\omega t) = A_1\exp\left(ik(l+ct)\right) + A_2\exp\left(-ik(l-ct)\right) \\ i\,RJ_l\left(\sqrt{\beta_r l}\right)\exp(i\,\omega t) = [A_1\exp(ikl) - A_2\exp(-ikl)]\exp(i\,\omega t), \end{cases} \tag{28}$$

and

$$\begin{cases} 2A_1 = R\left[J_0\left(\sqrt{\beta_r l}\right) + iJ_1\left(\sqrt{\beta_r l}\right)\exp(-ikl)\right] \\ 2A_2 = R\left[J_0\left(\sqrt{\beta_r l}\right) - iJ_1\left(\sqrt{\beta_r l}\right)\exp(ikl)\right]. \end{cases} \tag{29}$$

The assumption of a standing wave gives

$$\begin{cases} |2A_1| = H_1 \\ |2A_2| = H_2, \end{cases} \quad \text{and } H_1 = H_2 = H \tag{30}$$

where

H_1 – height of the incoming wave,
H_2 – height of the reflected wave.

Hence, the applied matching condition leads to the following expression:

$$R/H = \left|J_0\left(\sqrt{\beta_r l}\right) + iJ_1\left(\sqrt{\beta_r l}\right)\right|^{-1} = \left(J_0^2\left(\sqrt{\beta_r l}\right) + J_1^2\left(\sqrt{\beta_r l}\right)\right)^{-0.5} \tag{31}$$

in which R/H denotes the relative wave run-up height on the slope considered.

For large values of the argument $\sqrt{\beta_r l}$, the following, very simple relation may be obtained:

$$R/H = \pi\sqrt{2l/L}. \tag{32}$$

Expression (31) is formally equivalent to the well-known solution obtained by Keller & Keller (1964) by means of the Eulerian point of view. However the interpretation of Eq. (31) is now quite different, because ζ^{ξ} denotes the surface elevation at the point $x + \xi$.

4.2. Solution with Bottom Friction

Voltsinger et al. (1989) introduced the following friction assumption for reasons of simplicity:

$$\tau^{\xi} = -f\rho h \partial_t \xi. \tag{33}$$

This formula is assumed to be valid for long waves under the condition of $f << \omega$ where the dimension of factor f is $1/s$. Rewriting Eq. (11) by using Eqs. (9) and (33), the following equation can be obtained:

$$\partial_{tt}\zeta^{\xi} + f\partial_t\zeta^{\xi} = \partial_x(c^2\partial_x\zeta^{\xi}). \tag{34}$$

4.2.1. Region of Uniform Water Depth

In the case of a constant bottom, h = const., Eq. (34) takes the following form:

$$\partial_{tt}\zeta^{\xi} + f\partial_t\zeta^{\xi} = c^2\partial_{xx}\zeta^{\xi}. \tag{35}$$

4.2.2. Region of Uniform Gentle Slope

For $h = \alpha x (\alpha << 1)$ we have

$$\partial_{tt}\zeta^{\xi} + f\partial_t\zeta^{\xi} = \partial_x(\alpha g x \partial_x \zeta^{\xi}). \quad (36)$$

A solution of Eq. (36) is similar to the solution of Eq. (16), provided that a complex parameter β, is now introduced:

$$\beta = \beta_r + i\beta_i = 4\omega^2/g\alpha + i(-4f\omega/g\alpha) = 4\tilde{\omega}^2/g\alpha. \quad (37)$$

The independent variable s is now in the following relation with x:

$$s^2 = \beta x. \quad (38)$$

As a result of the above conditions, Eq. (36) now takes the form

$$\partial_{tt}\zeta^{\xi} + f\partial_t\zeta^{\xi} = \frac{\tilde{\omega}^2}{s}\partial_s(s\partial_s\zeta^{\xi}). \quad (39)$$

Assuming (as earlier) the solution in the form of

$$\zeta^{\xi} = A(s)\exp(i\omega t), \quad (40)$$

the following Bessel equation can be obtained

$$A''(s) + \frac{1}{s}A' + A(s) = 0. \quad (41)$$

For a boundary condition, $A(s = 0) = R$, its solution is

$$A(s) + RJ_0(s). \quad (42)$$

Finally, the solution of Eq. (36) takes the following form

$$\zeta^{\xi} = RJ_0\left(\sqrt{\beta x}\right)\exp(i\omega t) \cong R\left[J_0\left(\sqrt{\beta_r x}\right) - i(\beta_i/4)J_1\left(\sqrt{\beta_r x}\right)\right]\exp(i\omega t), \quad (43)$$

where J_1 is the Bessel function of the first order.

4.2.3. Connection of Both Regions

Applying the conditions at the connection, $x = 1$, and assuming large values of $\sqrt{\beta l}$, Voltzinger et al. (1989) derived the following approximate relation

$$R/H \approx \pi\sqrt{2l/L}\exp\left(-f\sqrt{l/g\alpha}\right) = \pi\sqrt{2l/L}\exp(-fl/c). \quad (44)$$

5. Selected Results of Numerical Computations

Eq. (11) has been solved numerically for the frictionless case. Some of the results obtained are presented in this chapter.

Progressive waves in the shallow water are affected by the bottom topography. As the wave is propagated shoreward its steepness H/L increases. The water surface grows steeper at the crest and flatter at the trough. An example of the regular wave train coming towards the shore with the close-up is shown in Figs. 3. and 4. In the Eulerian-type models, the obtaining of such a wave profile is not possible for the first order of approximation. The trajectories of particles for several cross-sections are drawn in Fig. 5.

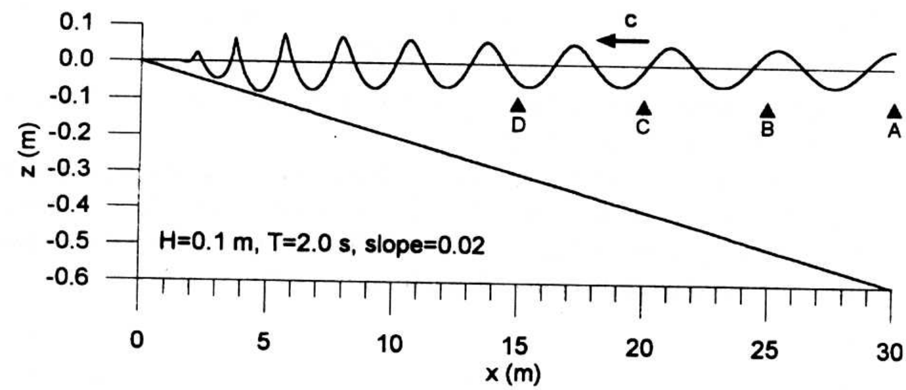

Fig. 3. Profile of the regular wave train in the shoaling water

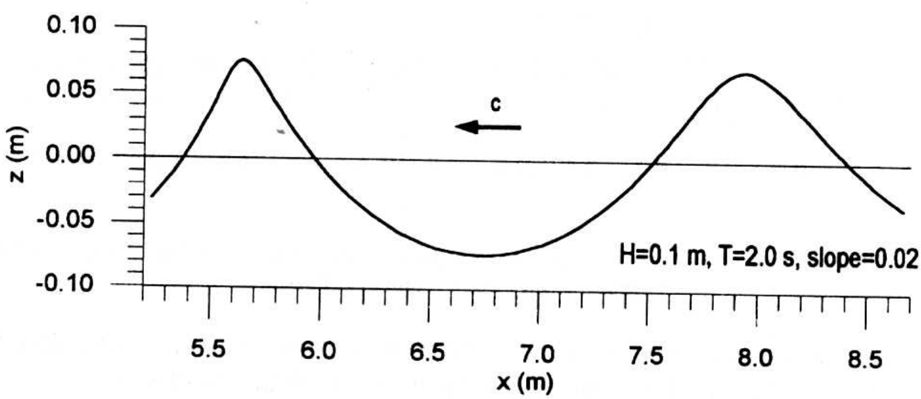

Fig. 4. Close-up view of the wave profile near the shoreline

Keller & Keller proposed the analytical solution of relative run-up height R/H of non-breaking waves (Eq. 31). This formula is valid for the case of a uniform slope connected to the bottom of constant depth. Numerical computations were

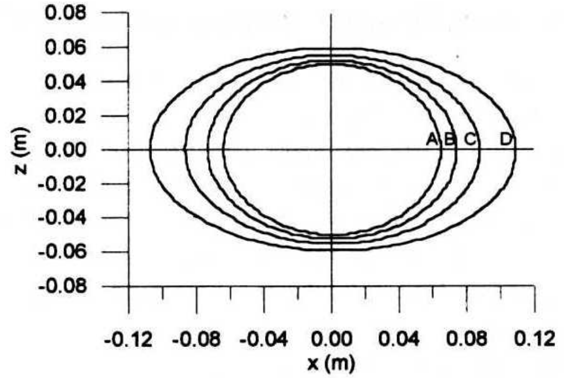

Fig. 5. Trajectories of water particles at cross-sections of 15, 20, 25 and 30 m from the shoreline (The smallest ellipse is for the farthest cross-section)

carried out for the following conditions: $T = 2.0$ s, $H = 0.05 \div 0.2$ m, $h = 0.6 \div 1.0$ m and the slope between 0.1 and 0.2.

Fig. 6. shows the comparison of the relative run-up height where the solid curve is the result given by the Keller & Keller formula and the black circles are from numerical computations. However the results obtained exceed the analytical ones a little, the agreement is quite satisfactory.

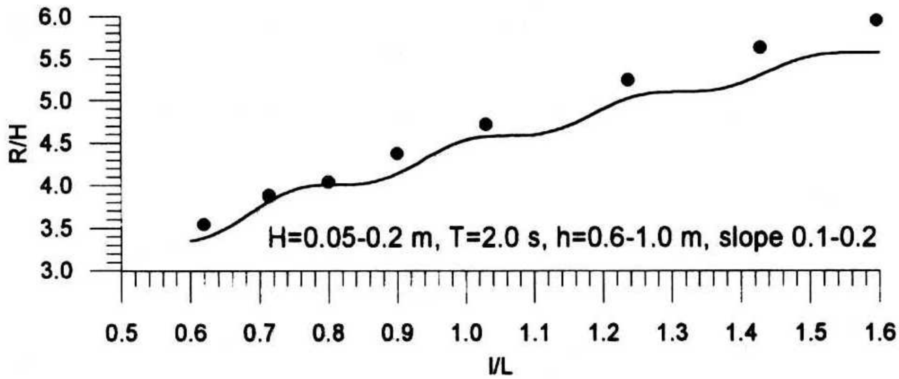

Fig. 6. Comparison of the numerical results with the analytical ones (solid line – Keller & Keller formula, black circles – numerical results)

Fig. 7 shows an example of the wave rush on slope. For the frictionless condition the maximum run-up has to be equal to maximum run-down.

6. Summary

In this paper, the hybrid (Eulerian-Lagrangian) approach has been presented. The equations derived are valid for long waves travelling over a mild-sloping bottom. The run-up height can also be calculated precisely. Some of the computational

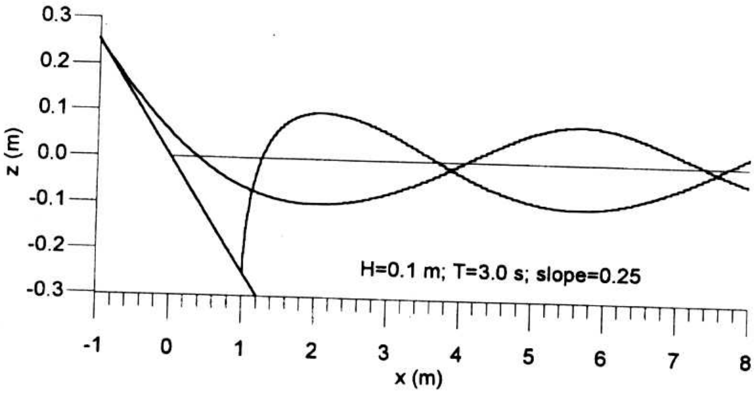

Fig. 7. Example of the maximum run-up and maximum run-down for the frictionless case

results for the frictionless case have been attached. Numerical simulation of the irregular wave motion and implementation of bottom friction is also possible.

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