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Deformations due to Gravity in Random Elastic Soil Medium
Part 2. Plane Strain Analysis

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Abstract

The random elasticity theory is applied to statistically homogeneous elastic soil half-plane subjected to gravity. The analysis is performed for the half-space in the plane strain state. Only the elastic modulus is considered to be uncertain and is treated as a two-dimensional random field. On the basis of Green's function approach, the stochastic partial differential equations governing the problem, are converted into stochastic integral equations. Then the perturbation procedure and Adomian's decomposition method are applied. The first one imposes small fluctuation assumptions. A system of input stochastic differential equations with random coefficients is here transformed into a few sets of stochastic differential equations with random forcing terms. There is no small fluctuation assumption in Adomian's decomposition method. This method is essentially the solution of the stochastic Volterra equation by the Neumann series expansion. The methods presented lead to approximated but explicit expressions for the statistical measures of stresses and displacements. Although the numerical calculations were not performed, the approach presented offers the tool which can afford quantitative results to the problem discussed.

1. Introduction

In part 1, a statistically homogeneous, horizontal soil layer subjected only to gravity has been considered. It was assumed that the soil is elastic and only the modulus of elasticity was random, described as a stochastic process. Thus the uniaxial strain state analysis was justified. The governing elasticity equations in the form of stochastic ordinary differential equations were solved using analytical approximate methods.

In this part, the elastic half-plane, subjected also only to gravity, is considered. However, the modulus of elasticity is now treated as a two-dimensional random field. Therefore, the random elasticity equations become a set of two stochastic partial differential equations with random coefficients, usually called a stochastic system. The analysis is carried out in the plane strain conditions and is supposed to be an input to another analysis i.e. two-dimensional schematization of reality. Again the perturbation method and Adomian's decomposition procedure are incorporated. The solutions are based, as in part 1, on Green's function approach, where this functions means the displacements induced by a unit force acting insight elastic half-plane (in a limit case at the surface). Appropriate integration over entire the half-plane leads to expressions determining the displacements due to gravity. Of course, the case of external loadings can also be included, by proper integration over the affected interval of the surface. The knowledge of statistical measures of the displacements due to gravity and external loadings enables determination of statistical measures of stresses.

The analytical, even approximated methods yield insight into some basic relationships. Thus, such methods seem to be almost necessary in the stochastic cases. Most commonly used are perturbation or hierarchy methods, although they essentially limit systems to small fluctuations or truncations and closure approximations are necessary.

The powerful approximated analytical method was considerably expanded by Adomian (1983). Usually, it is called Adomian's decomposition method, although the name Green's stochastic function method can also be found. This method was evolved to achieve statistical separability, avoid truncations and, of course, to omit small fluctuation assumptions. In this method, the solution process for the output of a physical system, is decomposed into additive components, the first being the solution of a simplified linear problem. Each of the other components is then found in terms of a preceding component and, thus, ultimately in terms of the first one. The usual statistical separability problems requiring closure approximations are eliminated with the reasonable assumption of statistical independence of the system input and the system itself.

The aim of this paper is the application of approximated, analytical methods in solving basic boundary problems of random elasticity theory with reference to geotechnical engineering.

2. Basic Equations

Let us consider a soil medium as an elastic half-plane (Fig. 1). Let Young's modulus be a function of position $E(x, y)$ and the only acting forces be those due to gravity.

The fundamental plane strain equations, describing a given problem, are well known in the non-homogeneous theory of elasticity. The equations in displacements can be written in the following form:

$$E_x(-Au_x + Cv_y) + E_y(u_y + v_x) + E(-Au_{xx} + u_{yy} + Dv_{xy}) = 0, \quad \text{(1a)}$$

$$E_y(-Av_y + Cu_x) + E_x(u_y + v_x) + E(-Av_{yy} + v_{xx} + Du_{xy}) = -B \quad \text{(1b)}$$

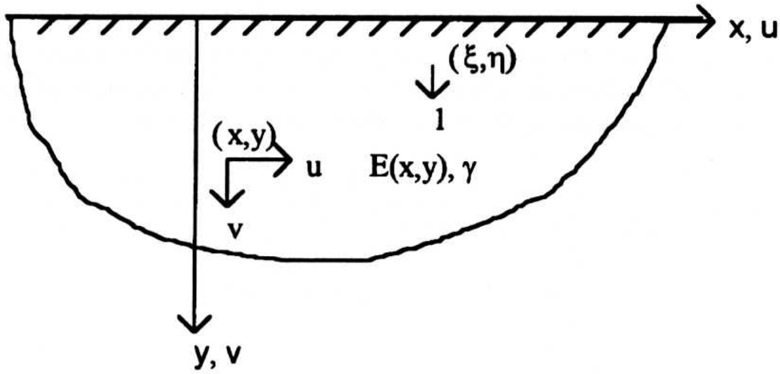

Fig. 1. Elastic half-plane. Basic notations

where:

$$A = -\frac{2(1-\nu)}{1-2\nu}, \quad B = 2\gamma(1+\nu), \quad C = \frac{2\nu}{1-2\nu}, \quad D = \frac{1}{1-2\nu}. \tag{2}$$

In the case of a homogeneous soil medium, the derivatives of Young's modulus E with respect to x and y vanish, and Navier's equations for a homogeneous, elastic material are obtained. Its solution for the half-plane subjected only to gravity is known and can be written in the form:

$$u = 0, \quad v = \frac{(1+\nu)(1-2\nu)}{2E(1-\nu)} \gamma\, y^2 + a + bx \tag{3}$$

where: a and b are the constants of integration and can be treated as an imposed displacement and rotation of a rigid body. Of course, for the half-plane subjected only to gravity, there is no rotation, so $b = 0$.

From the expression (3) it is seen that the state of displacements is not uniquely determined. The displacements in the case of a plane stress analysis would be quite different. However, the stresses are the same in both cases and they are equal to:

$$\sigma_y = \gamma\, y, \quad \sigma_x = \frac{\nu}{1-\nu}\gamma\, y, \quad \tau_{xy} = 0. \tag{4}$$

It is worth emphasizing that the displacement of any particle of the soil medium subjected to its own weight, has a rather abstract meaning. In fact, considering the elastic half-plane, it is obvious that the displacements, due to gravity, have already occurred. However, knowledge of those displacements is necessary in order to determine the state of stresses. In the case of homogeneous material, stresses are given by expression (4).

3. Green's Function

In order to solve the set of two equations (1) the concept of Green's function method and perturbation technique will be applied.

The vertical and horizontal components of the displacement of the point (x, y) due to the vertical unit force acting at a point (ξ, η) inside the half-plane may be treated as Green's functions. These functions have been derived using complex variable methods and eventually can be presented in the form:

$$U(x, y, \xi, \eta) = \frac{1}{2\pi\mu}\left\{\frac{(x-\xi)(y-\eta)}{4(1-\nu)}\left[\frac{3-4\nu}{r^2}+\frac{1}{r_1^2}\right]+\frac{\eta(x-\xi)y(y+\eta)}{r^4}+\right.$$
$$\left.-(1-2\nu)\tan^{-1}\frac{x-\xi}{y+\eta}\right\}+\omega_0-\nu_0 y, \tag{5a}$$

$$V(x, y, \xi, \eta) = \frac{1}{2\pi\mu}\left\{-\frac{(x-\xi)^2}{4(1-\nu)}\left[\frac{3-4\nu}{r^2}+\frac{1}{r_1^2}\right]+\frac{\eta y}{2(1-\nu)r^2}\times\right.$$
$$\left.\times\left[1-\frac{2(x-\xi)^2}{r^2}\right]-2(1-\nu)\ln r+\frac{(3-4\nu)}{4(1-\nu)}\ln\frac{r}{r_1}\right\}+$$
$$+v_0+\omega_0 x \tag{5b}$$

where:

$$r=\sqrt{(x-\xi)^2+(y+\eta)^2},\quad r_1=\sqrt{(x-\xi)^2+(y-\eta)^2},$$
$$\mu=\frac{E}{2(1+\nu)}.$$

ν_0 and ω_0 are the constants of integration and can be treated as the imposed displacement and rotation of a rigid body.

Finally the total displacements of the point (x, y) due to the soil's own weight can be calculated as double integrals over an area of the half-plane:

$$u(x, y) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\gamma(\xi, \eta)U(x, y, \xi, \eta)d\xi d\eta, \tag{6a}$$

$$v(x, y) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\gamma(\xi, \eta)V(x, y, \xi, \eta)d\xi d\eta. \tag{6b}$$

If the soil is homogeneous the integration of (6) leads to the classical solution (3).

4. Stochastic Description of Soil Medium

It is assumed that only Young's modulus is a function of position $E(x, y)$ and can be considered as a homogeneous, two-dimensional random field. Assuming that the fluctuations are sufficiently small it can be presented in the following form:

$$E=\overline{E}[1+\alpha\ \beta(x, y)] \tag{7}$$

where:

$\bar{E}$ – mean value of the modulus of elasticity,
α – coefficient of variation (small parameter),
$\beta(x,y)$ – homogeneous and normalised two-dimensional random field with expected value $\langle\beta(x,y)\rangle = 0$, variance $Var[\beta(x,y)] = 1$ and given covariance or correlation (in this case both are the same) function $R_\beta(\tau_1, \tau_2)$.

Because the random field of the modulus of elasticity must be differentiable, the following covariance function is assumed:

$$R_\beta(\tau_1, \tau_2) = (1 + \lambda_1\tau_1)(1 + \lambda_2\tau_2)e^{-\lambda_1\tau_1 - \lambda_2\tau_2}, \quad \tau_1, \tau_2 \geq 0. \tag{8}$$

For convenience this covariance function has a rather simple form and separable correlation structure.

Taking into account (7), the derivatives of Young's modulus, appearing in (1), can be presented in the form:

$$E_x = \bar{E}\alpha\beta_x(x,y), \quad E_y = \bar{E}\alpha\beta_y(x,y). \tag{9}$$

It is seen that the derivatives of β, with respect to x and y, are also two-dimensional random fields, with zero mean values and covariance functions given by the following expressions:

$$R_{\beta_x}(\tau_1, \tau_2) = \lambda_1^2(1 - \lambda_1\tau_1)(1 + \lambda_2\tau_2)e^{-\lambda_1\tau_1 - \lambda_2\tau_2}, \tag{10a}$$

$$R_{\beta_y}(\tau_1, \tau_2) = \lambda_2^2(1 - \lambda_2\tau_2)(1 + \lambda_1\tau_1)e^{-\lambda_1\tau_1 - \lambda_2\tau_2}. \tag{10b}$$

There is also a covariance between random fields of the elasticic modulus and its derivatives. For the random field $\beta(x,y)$ the following covariance functions will be used in further analysis:

$$R_{\beta\beta_x}(\tau_1, \tau_2) = \lambda_1^2\tau_1(1 + \lambda_2\tau_2)e^{-\lambda_1\tau_1 - \lambda_2\tau_2}, \tag{11a}$$

$$R_{\beta\beta_y}(\tau_1, \tau_2) = \lambda_2^2\tau_2(1 + \lambda_1\tau_1)e^{-\lambda_1\tau_1 - \lambda_2\tau_2}. \tag{11b}$$

5. Perturbation Method

If the random variations of the elastic modulus are of a low order the perturbation theory may be applied. It was mentioned in part 1 that such theory is valid if the coefficient of variation of the elastic modulus α is less than approximately 0.1 to 0.15. The modulus of elasticity given by (7) is in fact decomposed into two parts: deterministic (mean value) and fluctuated. In the first order perturbation method an unknown function is also decomposed into such parts. It is equivalent with an expansion of the unknown function in the Taylor series, around small parameter

α, and taking into account only the first two terms. In our case unknown are displacements and appearing in (1) their derivatives. We can present them in the following forms:

$$u = u_0 + \alpha \cdot u_1 \quad v = v_0 + \alpha \cdot v_1. \tag{12}$$

Substituting (7), (9) and (12) into (1) a set of the following stochastic differential equations with random coefficients is obtained:

$$\begin{aligned}
&\overline{E}\alpha\beta_x\left[-A(u_{0x} + \alpha u_{1x}) + C(v_{0y} + \alpha v_{1y})\right] + \overline{E}\alpha\beta_y[u_{0y} + \alpha u_{1y} + v_{0x} + \alpha v_{1x}] + \\
&+\overline{E}(1 + \alpha\beta)\left[-A(u_{0xx} + \alpha u_{1xx}) + (u_{0yy} + \alpha u_{1yy}) + D(v_{0xy} + \alpha v_{1xy})\right] = 0, \\
&\overline{E}\alpha\beta_y\left[-A(v_{0y} + \alpha v_{1y}) + C(u_{0x} + \alpha u_{1x})\right] + \overline{E}\alpha\beta_x[u_{0y} + \alpha u_{1y} + v_{0x} + \alpha v_{1x}] + \\
&+\overline{E}(1 + \alpha\beta)\left[-A(u_{0yy} + \alpha v_{1yy}) + (v_{0xx} + \alpha v_{1xx}) + D(u_{0xy} + \alpha u_{1xy})\right] = -B(13)
\end{aligned}$$

Equating the terms with the same power of α the following two sets of two equations are obtained:

$$\begin{aligned}
-Au_{0xx} + u_{0yy} + Dv_{0xy} &= 0, \\
-Av_{0yy} + v_{0xx} + Du_{0xy} &= -B/\overline{E}.
\end{aligned} \tag{14a}$$

$$\begin{aligned}
&\beta_x(-Au_{0x} + Cv_{0y}) + \beta_y(u_{0y} + v_{0x}) + \beta(-Au_{0xx} + u_{0yy} + Dv_{0xy}) + \\
&+ (-Au_{1xx} + u_{1yy} + Dv_{1xy}) = 0, \\
&\beta_y(-Av_{0y} + Cu_{0x}) + \beta_x(u_{0y} + v_{0x}) + \beta(-Av_{0yy} + v_{0xx} + Du_{0xy}) + \\
&+ (-Av_{1yy} + v_{1xx} + Du_{1xy}) = 0.
\end{aligned} \tag{14b}$$

Substituting (14a) into (14b) we can write:

$$\begin{aligned}
-Au_{1xx} + u_{1yy} + Dv_{1xy} &= -\beta_x(-Au_{0x} + Cv_{0y}) - \beta_y(u_{0y} + v_{0x}), \\
-Av_{1yy} + v_{1xx} + Du_{1xy} &= \beta B/E - \beta_y(-Av_{0y} + Cu_{0x}) + \beta_x(u_{0y} + v_{0x}),
\end{aligned} \tag{15}$$

or finally:

$$\begin{aligned}
-Au_{1xx} + u_{1yy} + Dv_{1xy} &= f_1, \\
-Av_{1yy} + v_{1xx} + Du_{1xy} &= f_2,
\end{aligned} \tag{16}$$

where:

$$\begin{aligned}
f_1 &= -\beta_x(-Au_{0x} + Cv_{0y}) - \beta_y(u_{0y} + v_{0x}), \\
f_2 &= B\beta/E - \beta_y(-Av_{0y} + Cu_{0x}) + \beta_x(u_{0y} + v_{0x}).
\end{aligned} \tag{17}$$

Comparing (16) with (14a) it can be seen that we have obtained the same kind of partial differential equations. This means that by small fluctuation assumption, the set of input stochastic differential equations with random coefficients has been transformed into two sets of differential equations, one of which is deterministic (14a), and the second (16) contains the random elements as forcing terms. The solution of the deterministic problem is given by (3) and (4). In order to solve the stochastic problem, Green's function method will be applied.

6. Stochastic Solution

The displacements of any point with co-ordinates x and y due to a unit vertical force acting at point (ξ, η) are given by (5). In the case of the half-plane subjected only to its own weight, the components of the total displacement can be calculated using formulae (6). In the case of application of the first order perturbation method, it is easy to see that the average displacements are equivalent to the deterministic ones. Thus, we can write:

$$u_0 = 0, \quad v_0 = \frac{\gamma}{2\bar{E}A} y^2. \tag{18}$$

Expressions (18) allow for determining the random forcing terms appearing in the set of the stochastic differential equations (16), and defined by (17). Eventually, they can be written in the following form:

$$\begin{aligned} f_1 &= -\beta_x C \frac{\gamma}{\bar{E}A} y, \\ f_2 &= \frac{\gamma}{\bar{E}} [-2(1+\nu)\beta + \beta_y y]. \end{aligned} \tag{19}$$

Now, the displacement's components u_1 and v_1 can be found by substituting f_1 and f_2 instead of γ into expressions (6). This can be written as follows:

$$\begin{aligned} u(x,y) &= \int_0^\infty \int_{-\infty}^\infty f_1(\xi,\eta) U(x,y,\xi,\eta) d\xi d\eta, \\ v(x,y) &= \int_0^\infty \int_{-\infty}^\infty f_2(\xi,\eta) V(x,y,\xi,\eta) d\xi d\eta. \end{aligned} \tag{20}$$

In the above expressions, f_1 and f_2 are random fields because they include β which is a random function of ξ and η.
Substituting (5) and (17) into (20) the fluctuated parts of the displacements can

be written in the following form:

$$u_1(x,y) = \frac{1}{2\pi\nu}\,\frac{-C\gamma}{\overline{E}A}\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty}\left\{\frac{(x-\xi)(y-\eta)}{4(1-\nu)}\left[\frac{3-4\nu}{r^2}+\frac{1}{r_1^2}\right]+\frac{\eta(x-\xi)y(y+\eta)}{r^4}+\right.$$
$$\left.-(1-2\nu)\tan^{-1}\frac{x-\xi}{y+\eta}\right\}\beta_\xi(\xi,\eta)\eta d\xi d\eta, \tag{21a}$$

$$v_1(x,y) = \frac{1}{2\pi\nu}\,\frac{\gamma}{\overline{\overline{E}}}\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty}\left\{-\frac{(x-\xi)^2}{4(1-\nu)}\left[\frac{3-4\nu}{r^2}+\frac{1}{r_1^2}\right]+\frac{\eta y}{2(1-\nu)r^2}\times\right.$$
$$\left.\times\left[1-\frac{2(x-\xi)^2}{r^2}\right]-2(1-\nu)\ln r+\frac{(3-4\nu)}{4(1-\nu)}\ln\frac{r}{r_1}\right\}\times$$
$$\times[-2(1+\nu)\beta(\xi,\eta)+\beta_y(\xi,\eta)\eta]d\xi d\eta. \tag{21b}$$

The mean values of displacements u_1 and v_1 are equal to zero. We have to calculate their variances and a covariance function.

Let us consider first a horizontal displacement u_1. We can write:

$$u_1^*(x,y) = \int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} f_1(\xi_1,\eta_1)U(x,y,\xi_1,\eta_1)d\xi_1 d\eta_1,$$
$$u_1^{**}(x,y) = \int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} f_1(\xi_2,\eta_2)U(x,y,\xi_2,\eta_2)d\xi_2 d\eta_2. \tag{22}$$

The variance of the horizontal displacements is as follows:

$$Var[u_1] = \langle u_1^* u_1^{**}\rangle = \int\limits_0^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty} f_1(\xi_1,\eta_1)f_1(\xi_2,\eta_2)U(x,y,\xi_1,\eta_1)\times$$
$$\times\,U(x,y,\xi_2,\eta_2)d\xi_1 d\xi_2 d\eta_1 d\eta_2. \tag{23}$$

It would be convenient to rewrite (22) in the following form:

$$u_1^* = H\int\limits_0^{\infty}\int\limits_{-\infty}^{\infty}\left[p\frac{(x-\xi_1)(y-\eta_1)\eta_1}{r^2}+q\frac{(x-\xi_1)(y-\eta_1)\eta_1}{r_1^2}+\right.$$
$$\left.+\frac{y\eta_1^2(x-\xi_1)(y+\eta_1)}{r^4}-s\eta_1\tan^{-1}\frac{x-\xi_1}{y+\eta_1}\right]\beta_{\xi_1}(\xi_1,\eta_1)d\xi_1 d\eta_1, \tag{24a}$$

$$u_1^{**} = H\int_0^{\infty}\int_{-\infty}^{\infty}\left[p\frac{(x-\xi_2)(y-\eta_2)\eta_2}{\bar{r}^2} + q\frac{(x-\xi_2)(y-\eta_2)\eta_2}{\bar{r}_1^2} + \right.$$
$$\left. + \frac{y\eta_2^2(x-\xi_2)(y+\eta_2)}{\bar{r}^4} - s\eta_2\tan^{-1}\frac{x-\xi_2}{y+\eta_2}\right]\beta_{\xi_2}(\xi_2,\eta_2)d\xi_2 d\eta_2 \quad (24b)$$

where:

$$H = \frac{1}{2\pi\mu}\frac{-Cy}{\bar{E}A}, \quad p = \frac{3-4\nu}{4(1-\nu)}, \quad q = \frac{1}{4(1-\nu)}, \quad s = 1-2\nu,$$

r, r_1 are functions of ξ_1, η_1 and $\bar{r}, \bar{r_1}$ are functions of ξ_2, η_2.

Taking into account (19) and (24) and having in mind that $\tau_1 = |\xi_1 - \xi_2|$, $\tau_2 = |\eta_1 - \eta_2|$, the variance of horizontal displacement can be calculated from the following expression:

$$Var[u_1] = H^2\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\left[p\frac{(x-\xi_1)(y-\eta_1)\eta_1}{r^2} + q\frac{(x-\xi_1)(y-\eta_1)\eta_1}{r_1^2} + \right.$$
$$\left. + \frac{y\eta_1^2(x-\xi_1)(y+\eta_1)}{r^4} - s\eta_1\tan^{-1}\frac{x-\xi_1}{y+\eta_1}\right] \times \left[p\frac{(x-\xi_2)(y-\eta_2)\eta_2}{\bar{r}^2} + \right.$$
$$\left. + q\frac{(x-\xi_2)(y-\eta_2)\eta_2}{\bar{r}_1^2} + \frac{y\eta_2^2(x-\xi_2)(y+\eta_2)}{\bar{r}^4} - s\eta_2\tan^{-1}\frac{x-\xi_2}{y+\eta_2}\right] \times$$
$$\times \lambda_1^2(1-\lambda_1|\xi_1-\xi_2|)(1+\lambda_2|\eta_1-\eta_2|)e^{-\lambda_1|\xi_1-\xi_2|}e^{-\lambda_2|\eta_1-\eta_2|} \times$$
$$\times d\xi_1 d\xi_2 d\eta_1 d\eta_2. \quad (25)$$

According to Fig. 2 the following cases should be considered:

1. $\xi_1 - \xi_2 > 0, \quad \eta_1 - \eta_2 > 0,$
2. $\xi_1 - \xi_2 > 0, \quad \eta_1 - \eta_2 < 0,$
3. $\xi_1 - \xi_2 < 0, \quad \eta_1 - \eta_2 > 0,$
4. $\xi_1 - \xi_2 < 0, \quad \eta_1 - \eta_2 < 0.$

The integration of (25) is rather cumbersome. In spite of the necessity for four-folded integration, which can be performed only numerically, there exist a few kinds of singularities. Let us consider, for example, the areas of integration 1 and 3. The variance of the displacement, defined as a four-folded integral over those areas, after some rearrangements can be presented as follows:

$$I_{1-3} = H^2 \int\limits_{\xi_1=0}^{\xi_1=a} \int\limits_{\xi_2=\xi_1}^{\xi_2=a} \int\limits_{\eta_1=0}^{\eta_2=b} \int\limits_{\eta_2=\eta_1}^{\eta_2=b} \Big[p^2 \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{r^2\bar{r}^2} +$$

$$+ pq \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{r^2\bar{r}_1^2} +$$

$$+ p \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y+\eta_2)\eta_1\eta_2^2 y}{r^2 r^4} +$$

$$- ps \frac{(x-\xi_1)(y-\eta_1)\eta_1\eta_2}{r^2} \tan^{-1}\frac{x-\xi_2}{y+\eta_2} +$$

$$+ pq \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{r_1^2\bar{r}^2} +$$

$$+ q^2 \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{r_1^2\bar{r}_1^2} +$$

$$+ q \frac{(x-\xi_1)(y-\eta_1)(x-\xi_2)(y+\eta_2)\eta_1\eta_2^2 y}{r_1^2\bar{r}^4} +$$

$$- qs \frac{(x-\xi_1)(y-\eta_1)\eta_1\eta_2}{r_1^2} \tan^{-1}\frac{x-\xi_2}{y+\eta_2} +$$

$$+ p \frac{(x-\xi_1)(y+\eta_1)(x-\xi_2)(y-\eta_2)\eta_1^2\eta_2 y}{r^4\bar{r}^2} +$$

$$+ q \frac{(x-\xi_1)(y+\eta_1)(x-\xi_2)(y-\eta_2)\eta_1^2\eta_2 y}{r^4\bar{r}_1^2} +$$

$$+ \frac{(x-\xi_1)(y+\eta_1)(x-\xi_2)(y+\eta_2)\eta_1^2\eta_2^2 y^2}{r^4\bar{r}^4} +$$

$$- s \frac{(x-\xi_1)(y+\eta_1)\eta_1^2\eta_2 y}{r^4} \tan^{-1}\frac{x-\xi_2}{y+\eta_2} +$$

$$-ps \frac{(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{\bar{r}^2} \tan^{-1}\frac{x-\xi_1}{y+\eta_1} +$$

$$- qs \frac{(x-\xi_2)(y-\eta_2)\eta_1\eta_2}{\bar{r}_1^2} \tan^{-1}\frac{x-\xi_1}{y+\eta_1} +$$

$$-s \frac{(x-\xi_2)(y+\eta_2)\eta_1\eta_2^2 y}{\bar{r}^4} \tan^{-1}\frac{x-\xi_1}{y+\eta_1} +$$

$$+ s^2\eta_1\eta_2 \tan^{-1}\frac{x-\xi_1}{y+\eta_1} \tan^{-1}\frac{x-\xi_2}{y+\eta_2} \Big] [1 + \lambda_2(\eta_1-\eta_2) - \lambda_1(\xi_1-\xi_2) +$$

$$- \lambda_1\lambda_2(\xi_1-\xi_2)(\eta_1-\eta_2)]\lambda_1^2 e^{-\lambda_1(\xi_1-\xi_2)} e^{-\lambda_2(\eta_1-\eta_2)} d\xi_1 d\xi_2 d\eta_1 d\eta_2. \qquad (26)$$

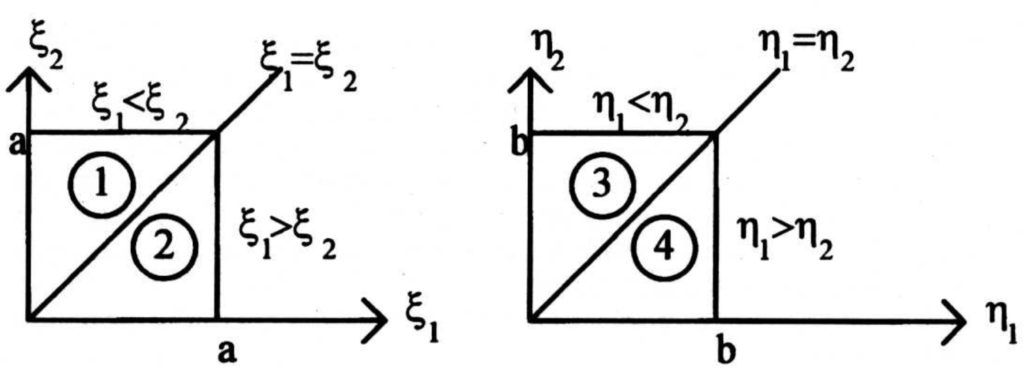

Fig. 2. Area of integration of (25)

The integration of (26) may be performed over the limited area without losing accuracy. It is so, because the exponent terms present under the integral vanish much faster than the other terms. That means that only randomness of the soil medium in the vicinity (at a distance less than a or b) has a meaningful influence on the calculated variance. Of course the size of the integration area, i.e. values a and b depend on the parameters of the covariance function of the elastic modulus. In the case of full correlation which corresponds to a deterministic problem, integration must be performed over a whole half-plane. If the correlation is considerably small, only the closest vicinity may be taken into account.

It can be seen from (26) that after multiplication there are 48 four-folded integrals in this expression. Thus, in order to calculate the variance of the horizontal displacement, almost two hundred such integrals must be computed. Unfortunately, the expression (26) has not yet been numerically integrated, because computational facilities were not available. It is suggested that the Monte Carlo method of integration would be of great profit. This is rather a technical problem, although the singularities appearing here must be carefully considered. In fact, we deal with a four-dimensional space $\xi_1, \xi_2, \eta_1, \eta_2$ and three kinds of singularities:

- double singularity if all $\xi_1 = x$, $\xi_2 = x$, $\eta_1 = y$, $\eta_2 = y$,
- singular hypersurface if $\xi_1 = x$ and $\eta_1 = y$ or $\xi_2 = x$ and $\eta_2 = y$,
- singularity due to the infinite endpoints of the interval of integration.

The first two kinds of singularities have been computed and the values of integrals at single points are found to be equal to zero. The third kind of singularity may be dealt with by some approximate procedures or numerically.

It is worth noting that putting $x = 0$ into (26) will not change the generality of this expression but it would only make it a little bit simpler.

The variance of the vertical displacements can be determined in an analogous way.

7. Adomian's Decomposition Procedure

The basic feature of the perturbation technique is an ssumption of a small parameter i.e. that the coefficient of variation of the elastic modulus is small enough. The set of stochastic differential equations with random parameters is here transformed into a few sets of stochastic differential equations with random forcing terms. There is no limitation of small fluctuations in Adomian's decomposition procedure, although some convergence criteria must be satisfied. The Adomian's decomposition method is essentially the solution of the stochastic Volterra integral equation by Neumann series expansion.

Taking into account expressions (7) and (9), equations (1) can be written in the following form:

$$\alpha\beta_x(-Au_x + Cv_y) + \alpha\beta_y(u_y + v_x) + (1+\alpha\beta)(-Au_{xx} + u_{yy} + Dv_{xy}) = 0, \quad (27a)$$

$$\alpha\beta_y(-Av_y + Cu_x) + \alpha\beta_x(u_y + v_x) + (1+\alpha\beta)(-Av_{yy} + v_{xx} + Du_{xy}) = \\ = -B/\overline{E}. \quad (27b)$$

Let us rewrite the equation (27a) in the following alternative way:

$$Au_{xx} = u_{yy} + Dv_{xy} + \alpha\beta(-Au_{xx} + u_{yy} + Dv_{xy}) + \\ + \alpha\beta_x(-Au_x + Cv_y) + \alpha\beta_y(u_y + v_x)$$

or

$$u_{yy} = Au_{xx} - Dv_{xy} + \alpha\beta(Au_{xx} - u_{yy} - Dv_{xy}) + \alpha\beta_x(Au_x - Cv_y) + \\ - \alpha\beta_y(u_y + v_x) \quad (28a)$$

and respectively equation (27b):

$$v_{xx} = Av_{yy} - Du_{xy} + \alpha\beta(Av_{yy} - v_{xx} - Du_{xy}) + \alpha\beta_y(Av_y - Cu_x) + \\ - \alpha\beta_x(u_y + v_x) - B/\overline{E}$$

or

$$Av_{yy} = v_{xx} + Du_{xy} + \alpha\beta(-Av_{yy} + v_{xx} + Du_{xy}) + \alpha\beta_y(-Av_y + Cu_x) + \\ - \alpha\beta_x(u_y + v_x) + B/\overline{E}. \quad (28b)$$

Now, let us first consider the equation (18a), which in general can be written as follows:

$$u_{xx} = f(x, y, \overline{E}, \beta, \beta_x, \beta_y, u, v, u_x, v_x, \ldots). \quad (29)$$

It is convenient to present (29a) as an operator equation in the form:

$$L_{xx}u = f \tag{30}$$

where: $L_{xx} = \frac{d^2}{dx^2}$ is a linear differential operator.
The solution of (30) can be given as the sum:

$$u = u_0 + L_{xx}^{-1} f = u_0 + \int G(x, \xi) f(\xi) d\xi \tag{31}$$

where: u_0 is the general solution of the associated homogeneous differential equation, $G(x, \xi)$ is Green's function for the operator L_{xx} and for given boundary conditions.
The second term of the sum in (31) is of course a particular solution of (30).

The equation (28a) in the operator form can be presented as follows:

$$\begin{aligned} Au = u_0^x + L_{xx}^{-1} u_{yy} + DL_{xx}^{-1} v_{xy} + \alpha L_{xx}^{-1} [\beta(-Au_{xx} + u_{yy} + Dv_{xy})] + \\ + \alpha L_{xx}^{-1} [\beta_x(-Au_x + Cv_y)] + \alpha L_{xx}^{-1} [\beta_y(u_y + v_x)] \end{aligned} \tag{32a}$$

or

$$\begin{aligned} u = u_0^y + AL_{yy}^{-1} u_{xx} - DL_{yy}^{-1} v_{xy} + \alpha L_{yy}^{-1} [\beta(Au_{xx} - u_{yy} - Dv_{xy})] + \\ + \alpha L_{yy}^{-1} [\beta_x(Au_x - Cv_y)] - \alpha L_{yy}^{-1} [\beta_y(u_y + v_x)]. \end{aligned} \tag{32b}$$

Taking into account (31), the expressions for horizontal displacements can be written in the following form:

$$\begin{aligned} Au = u_0^x + \int G(x, \xi)(u_{yy} + Dv_{xy}) d\xi + \alpha \int G(x, \xi) \beta(-Au_{xx} + u_{yy} + Dv_{xy}) d\xi + \\ + \alpha \int G(x, \xi) \beta_x(-Au_x + Cv_y) d\xi + \alpha \int G(x, \xi) \beta_y(u_y + v_x) d\xi \end{aligned} \tag{33a}$$

or

$$\begin{aligned} u = u_0^y + A \int G(y, \eta)(u_{xx} - v_{xy}) d\eta + \alpha \int G(y, \eta) \beta(Au_{xx} - u_{yy} - Dv_{xy}) d\eta + \\ + \alpha \int G(y, \eta) \beta_x(Au_x - Cv_y) d\eta - \alpha \int G(y, \eta) \beta_y(u_y + v_x) d\eta. \end{aligned} \tag{33b}$$

The expressions for the vertical displacements may be derived in the same way. Eventually, they take the form:

$$\begin{aligned} v = v_0^x + A \int G(x, \xi)(v_{yy} - Du_{xy}) d\xi + \alpha \int G(x, \xi) \beta(Av_{yy} - v_{xx} - Du_{xy}) d\xi + \\ + \alpha \int G(x, \xi) \beta_y(Av_y - Cu_x) d\xi - \alpha \int G(x, \xi) \beta_x(u_y + v_x) d\xi + \\ - B/\overline{E} \int G(x, \xi) d\xi, \end{aligned} \tag{34a}$$

or

$$Av = v_0^y + \int G(y,\eta)(v_{xx} + Du_{xy})d\eta + \alpha \int G(y,\eta)\beta(-Av_{yy} + v_{xx} + Du_{xy})d\eta + \\ + \alpha \int G(y,\eta)\beta_y(-Av_y - Cu_x)d\eta + \alpha \int G(y,\eta)\beta_x(u_y + v_x)d\eta + \\ -B/\overline{E} \int G(y,\eta)d\eta. \tag{34b}$$

It must be emphasised that, according to (31), all functions appearing in the integrands of the expressions (33a), (34a) are functions of ξ and (33b), (34b) are functions of η.

Both horizontal and vertical components of the displacement are presented in alternative forms. A linear combination of these two forms is necessary. Let us multiply the second equation of (33) by A, add to the first one and then divide by two. Eventually we obtain:

$$2Au = u_0^x + Au_0^y + \int G(x,\xi)(u_{yy} + Dv_{xy})d\xi + A^2 \int G(y,\eta)(u_{xx} - v_{xy})d\eta + \\ + \alpha \int G(x,\xi)\beta(-Au_{xx} + u_{yy} + Dv_{xy})d\xi + \alpha A \int G(y,\eta) \times \\ \times \beta(Au_{xx} - u_{xy} - Dv_{xy})d\eta + \alpha \int G(x,\xi)\beta_x(-Au_x + Cv_y)d\xi + \\ + \alpha A \int G(y,\eta)\beta_x(Au_x - Cv_y)d\eta + \alpha \int G(x,\xi)\beta_y(u_y + v_x)d\xi + \\ - \alpha A \int G(y,\eta)\beta_y(u_y + v_x)d\eta. \tag{35}$$

The vertical displacement can be presented in the same way:

$$2Av = Av_0^x + v_0^y + A^2 \int G(x,\xi)(v_{yy} - Du_{xy})d\xi + \int G(y,\eta)(v_{xx} + Du_{xy})d\eta + \\ + \alpha A \int G(x,\xi)\beta(Av_{yy} - v_{xx} - Du_{xy})d\xi + \alpha \int G(y,\eta) \times \\ \times \beta(-Av_{yy} + v_{xx} + Du_{xy})d\eta + \alpha A \int G(x,\xi)\beta_y(Av_y - Cu_x)d\xi + \\ + \alpha \int G(y,\eta)\beta_y(-Av_y + Cu_x)d\eta - \alpha A \int G(x,\xi)\beta_x(u_y + v_x)d\xi + \\ - \alpha \int G(y,\eta)\beta_x(u_y + v_x)d\eta - AB/\overline{E} \int G(x,\xi)d\xi + \\ + B/\overline{E} \int G(y,\eta)d\eta. \tag{36}$$

For the elastic half-plane considered subjected only to its own weight and for the notation given in Fig. 1, the general solutions of the associated homogeneous differential equations are equal to:

$$u_0^x = 0, \quad u_0^y = 0, \quad v_0^x = \frac{B}{2\overline{E}A} y^2 + a, \quad v_0^y = c. \tag{37}$$

Some comments should be given to the solution v_0^x. It is the homogeneous solution which corresponds to the non-homogeneous boundary conditions and it represents the vertical displacements along every vertical line. In fact, this is also a deterministic solution of our problem. Thus, in the deterministic analysis the solution is the same as the boundary conditions, and the problem is a trivial one. However, it is important in Adomian's decomposition procedure, where half-plane is limited by two vertical lines, and so the boundary conditions should be known on these lines.

Green's function $G(y, \eta)$ for the second order differential operator L_{yy} for given boundary conditions (elastic half-plane subjected only to gravity) can be considered as a the so-called one-sided Green's function and has the form:

$$G(y, \eta) = y - \eta. \tag{38}$$

In the case of $G(x, \xi)$ some limitations on boundary conditions must be imposed. This is so, because in the case of singular boundary conditions i.e. if x approaches infinity, Green's function, for the considered operator, ceases to exist. It seems to be justified to assume that the horizontal displacements for $x = 0$ and $x = c =$ const and for all $y > 0$ are equal to zero. In this case Green's function takes the form:

$$G(x, \xi) = \begin{cases} \dfrac{\xi}{d}(x - d) & \text{for } 0 \le \xi < x \\ \dfrac{\xi - d}{d} x & \text{for } x < \xi \le c \end{cases}. \tag{39}$$

Now, according to Adomian's method, the unknown functions and their derivatives with respect to x and y, are decomposed into a sum of undefined functions, as follows:

$$\begin{aligned} u(x, y, \beta) &= u_0(x, y) + u_1(x, y, \beta) + u_2(x, y, \beta) + \ldots + \\ &\quad + u_n(x, y, \beta) = \sum_{i=0}^{n} u_i, \\ v(x, y, \beta) &= v_0(x, y) + v_1(x, y, \beta) + v_2(x, y, \beta) + \ldots + \\ &\quad + v_n(x, y, \beta) = \sum_{i=0}^{n} v_i. \end{aligned} \tag{40}$$

Substituting (37), (38), (39) into (35), the following expression for the horizontal component of the displacement is obtained:

$$2A(u_0 + u_1 + \ldots + u_n) = \int_0^x \frac{\xi}{d}(x-d)[u_{0yy} + \ldots + u_{nyy} + D(v_{0xy} + \ldots + v_{nxy})] \times$$

$$\times\, d\xi + \int_x^d \frac{\xi - d}{d} x \left[u_{0yy} + \ldots + u_{nyy} + D(v_{0xy} + \ldots + v_{nxy})\right] d\xi +$$

$$+\, \alpha \int_0^x \frac{\xi}{d}(x-d)\beta[-A(u_{0xx} + \ldots + u_{nxx}) + u_{0yy} + \ldots + u_{nyy} +$$

$$+\, D(v_{0xy} + \ldots + v_{nxy})]d\xi + \alpha \int_x^d \frac{\xi - d}{d} x\beta \times$$

$$\times\, [-A(u_{0xx} + \ldots + u_{nxx}) + u_{0yy} + \ldots + u_{nyy} + D(v_{0xy} + \ldots + v_{nxy})]d\xi +$$

$$+\, \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_x[-A(u_{0x} + \ldots + u_{nx}) + C(v_{0y} + \ldots + v_{ny})]d\xi +$$

$$+\, \alpha \int_x^d \frac{\xi - d}{d} x\beta_x[-A(u_{0x} + \ldots + u_{nx}) + C(v_{0y} + \ldots + v_{ny})]d\xi +$$

$$+\, \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_y(u_{0y} + \ldots + u_{ny} + v_{0x} + \ldots + v_{nx})d\xi +$$

$$+\, \alpha \int_x^d \frac{\xi - d}{d} x\beta_y(u_{0y} + \ldots + u_{ny} + v_{0x} + \ldots + v_{nx})d\xi +$$

$$+\, A^2 \int_0^y (y-\eta)(u_{0xx} + \ldots + u_{nxx} - v_{0xy} - \ldots - v_{nxy})d\eta + \alpha A \int_0^y (y-\eta)\beta \times$$

$$\times\, [A(u_{0xx} + \ldots + u_{nxx}) - u_{0yy} - \ldots - u_{nyy} - D(v_{0xy} + \ldots + v_{nxy})]d\eta +$$

$$+\, \alpha A \int_0^y (y-\eta)\beta_x[A(u_{0x} + \ldots + u_{nx}) - C(v_{0y} + \ldots + v_{ny})]d\eta +$$

$$-\, \alpha A \int_0^y (y-\eta)\beta_y(u_{0y} + \ldots + u_{ny} + v_{0x} + \ldots + v_{nx})d\eta. \tag{41}$$

Since the functions $u_i(x, y, \beta)$, $v_i(x, y, \beta)$ for $i = 0, 1, 2, \ldots, n$ are still not specified, it is possible to make the following identification:

$$2Au_0 = 0, \tag{42a}$$

$$\begin{aligned}
2Au_1 = & \int_0^x \frac{\xi}{d}(x-d)(u_{0yy} + Dv_{0xy})\,d\xi + \int_x^d \frac{\xi - d}{d} x(u_{0yy} + Dv_{0xy})\,d\xi + \\
& + \alpha \int_0^x (x - d) \frac{\xi}{d}\beta(-Au_{0xx} + u_{0yy} + Dv_{0xy})\,d\xi + \alpha \int_x^d \frac{\xi - d}{d} x\beta \times \\
& \times (-Au_{0xx} + u_{0yy} + Dv_{0xy})\,d\xi + \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_x \times \\
& \times (-Au_{0x} + Cv_{0y})\,d\xi + \alpha \int_x^d \frac{\xi - d}{d} x\beta_x(-Au_{0x} + Cv_{0y})d\xi + \\
& + \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_y(u_{0y} + v_{0x})\,d\xi + \alpha \int_x^d \frac{\xi - d}{d} x\beta_y(u_{0y} + v_{0x})d\xi + \\
& + A^2 \int_0^y (y-\eta)(u_{0xx} - v_{0xy})\,d\eta + \alpha A \int_0^y (y-\eta)\beta \times \\
& \times (Au_{0xx} - u_{0yy} - Dv_{0xy})d\eta + \alpha A \int_0^y (y-\eta)\beta_x(Au_{0x} - Cv_{0y})\,d\eta + \\
& - \alpha A \int_0^y (y-\eta)\beta_y(u_{0y} + v_{0x}d\eta),
\end{aligned} \tag{42b}$$

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$$2Au_n = \int_0^x \frac{\xi}{d}(x-d)(u_{n-1,yy} + Dv_{n-1,xy})d\xi + \int_x^d \frac{\xi - d}{d} x(u_{n-1,yy} + Dv_{n-1,xy})d\xi +$$

$$+ \alpha \int_0^x \frac{\xi}{d}(x-d)\beta(-Au_{n-1,xx} + u_{n-1,yy} + Dv_{n-1,xy})d\xi +$$

$$+ \alpha \int_x^d \frac{\xi - d}{d} x\beta(-Au_{n-1,xx} + u_{n-1,yy} + Dv_{n-1,xy})d\xi +$$

$$+ \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_x(-Au_{n-1,x} + Cv_{n-1,y})d\xi +$$

$$+ \alpha \int_x^d \frac{\xi - d}{d} x\beta_x(-Au_{n-1,x} + Cv_{n-1,y})d\xi +$$

$$+ \alpha \int_0^x \frac{\xi}{d}(x-d)\beta_y(u_{n-1,y} + v_{n-1,x})d\xi +$$

$$+ \alpha \int_x^d \frac{\xi - d}{d} x\beta_y(u_{n-1,y} + v_{n-1,x})d\xi +$$

$$+ A^2 \int_0^y (y-\eta)(u_{n-1,xx} - v_{n-1,xy})d\eta +$$

$$+ \alpha A \int_0^y (y-\eta)\beta(Au_{n-1,xx} - u_{n-1,yy} - Dv_{n-1,xy})d\eta +$$

$$+ \alpha A \int_0^y (y-\eta)\beta_x(Au_{n-1,x} - Cv_{n-1,y})d\eta +$$

$$- \alpha A \int_x^y (y-\eta)\beta_y(u_{n-1,y} + v_{n-1,x})d\eta. \tag{42n}$$

Similarly, the proper terms of the vertical displacements have been derived:

$$2Av_0 = 2Aa + \frac{B}{2\overline{E}}y^2 - AB/\overline{E}\left[\int_0^x \frac{\xi}{d}(x-d)d\xi + \int_x^d \frac{\xi - d}{d} x d\xi\right] +$$

$$+ B/\overline{E} \int_0^y (y-\eta)d\eta, \tag{43a}$$

$$2Av_1 = A^2 \int_0^x \frac{\xi}{d}(x-d)(v_{0yy} - Du_{0xy})d\xi + A^2 \int_x^d \frac{\xi - d}{d} x (v_{0yy} - Du_{0xy})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta(Av_{0yy} + v_{0xx} + Du_{0xy})d\xi +$$

$$+ \alpha A \int_x^d \frac{\xi - d}{d} x\beta(Av_{0yy} + v_{0xx} + Du_{0xy})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta_x(u_{0y} + v_{0x})d\xi + \alpha A \int_x^d \frac{\xi - d}{d} x\beta_x(u_{0y} + v_{0x})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta_y(Av_{0y} - Cu_{0x})d\xi +$$

$$+ \alpha A \int_x^d \frac{\xi - d}{d} x\beta_y(Av_{0y} - Cu_{0x})d\xi + \int_0^y (y-\eta)(v_{0xx} + Du_{0xy})d\eta +$$

$$+ \alpha \int_0^y (y-\eta)\beta(-Av_{0yy} + v_{0xx} + Du_{0xy})d\eta +$$

$$+ \alpha \int_0^y (y-\eta)\beta_y(-Av_{0y} + Cu_{0x})d\eta + \alpha \int_0^y (y-\eta)\beta_x(u_{0y} + v_{0x})d\eta \quad (43b)$$

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$$2Av_n = A^2 \int_0^x \frac{\xi}{d}(x-d)(v_{n-1,yy} - Du_{n-1,xy})d\xi +$$

$$+ A^2 \int_x^d \frac{\xi - d}{d} x(v_{n-1,yy} - Du_{n-1,xy})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta(Av_{n-1,yy} + v_{n-1,xx} + Du_{n-1,xy})d\xi +$$

$$+ \alpha A \int_x^d \frac{\xi - d}{d} x\beta(Av_{n-1,yy} + v_{n-1,xx}) + Du_{n-1,xy})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta_x(u_{n-1,y} + v_{n-1,x})d\xi +$$

$$+ \alpha A \int_x^d \frac{\xi - d}{d} x\beta_x(u_{n-1,y} + v_{n-1,x})d\xi +$$

$$+ \alpha A \int_0^x \frac{\xi}{d}(x-d)\beta_y(Av_{n-1,y} - Cu_{n-1,x})d\xi + \alpha A \int_x^d \frac{\xi - d}{d} x\beta_y \times$$

$$\times (Av_{n-1,y} - Cu_{n-1,x})d\xi +$$

$$+ \int_0^y (y-\eta)(v_{n-1,xx} + Du_{n-1,xy})d\eta + \alpha \int_0^y (y-\eta) \times$$

$$\times \beta(-Av_{n-1,yy} + v_{n-1,xx} + Du_{n-1,xy})d\eta +$$

$$+\alpha \int_0^y (y-\eta)\beta_y(-Av_{n-1,y} + Cu_{n-1,x})d\eta +$$

$$+ \alpha \int_0^y (y-\eta)\beta_x(u_{n-1,y} + v_{n-1,x})d\eta. \tag{43n}$$

Thus, we have obtained two systems of the recurrent integral equations. Back substitution of the terms in equations (42) yields an explicit representation of these terms in the form of multiple integrals.

It should be emphasised that the series representation of the solution (40) is meaningful only if it converges. According to Adomian's remarks (1983), one can expect that in the case of the normal probability density function of β, the sufficient condition for the convergence of the series depends mainly on its coefficient of variation. Baker and Zeitoun (1990) analysed the "mean square convergence" and presented a solution which is valid up to $\alpha < 0.5$.

The system of integral equations (42) and (43) possesses a number of very attractive properties. Each element of this system is given in terms of lower-order terms only, and the assumption of the small fluctuation is not required. Instead of the stochastic differential equation there is a set of stochastic integrals to be solved.

The deterministic solution

It is obvious that the problem is deterministic if the coefficient of variation of the elastic modulus is equal to zero. Thus substituting $\alpha = 0$ into (27), Navier's well-known equations are obtained:

$$\begin{aligned} -Au_{xx} + u_{yy} + Dv_{xy} &= 0, \\ -Av_{yy} + v_{xx} + Du_{xy} &= -B/\overline{E}. \end{aligned} \tag{44}$$

From (42a) and (43a) we have:

$$\begin{aligned} u_0 &= 0, \\ v_0 &= a + \frac{1}{2}\frac{B}{\overline{E}A}y^2 - \frac{1}{4}\frac{B}{\overline{E}}x(x-d). \end{aligned} \tag{45a}$$

Substitution of (45a) and its derivatives with respect to x and y into (42b) and (43b) leads to:

$$\begin{aligned} u_1 &= 0, \\ v_1 &= \frac{1}{8}\frac{B}{\overline{E}A}y^2 - \frac{1}{4}\frac{B}{\overline{E}}x(x-d). \end{aligned} \tag{45b}$$

Repeating the above procedure gives all $u_i = 0$ and the following sequence of expressions determining v_i:

$$v_2 = -\frac{1}{8}\frac{B}{\overline{E}}y^2 + \frac{1}{16}\frac{B}{\overline{E}A}x(x-d), \tag{45c}$$

$$v_3 = -\frac{1}{32}\frac{B}{\overline{E}}y^2 - \frac{1}{16}\frac{B}{\overline{E}A}x(x-d), \tag{45d}$$

$$v_4 = \frac{1}{32}\frac{B}{\overline{E}}y^2 - \frac{1}{64}\frac{B}{\overline{E}A}x(x-d), \tag{45e}$$

$$v_5 = \frac{1}{128}\frac{B}{\overline{E}}y^2 + \frac{1}{64}\frac{B}{\overline{E}A}x(x-d). \tag{45f}$$

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Substituting (45) into (40) gives the following expression for the vertical displacements:

$$v = a + \frac{B}{\overline{E}A}y^2\left(\frac{1}{2}+\frac{1}{8}-\frac{1}{8}-\frac{1}{32}+\frac{1}{32}-\frac{1}{128}+\dots\right)+$$
$$+\frac{B}{\overline{E}}x(x-d)\left(-\frac{1}{4}+\frac{1}{4}+\frac{1}{16}-\frac{1}{16}-\frac{1}{64}+\frac{1}{64}+\dots\right)=a+\frac{B}{2\overline{E}A}y^2. \quad (46)$$

Of course, the so derived expression for the vertical (also horizontal) displacement is the same as presented earlier i.e. the classical one (3).

Statistical measures

The final result, of the stochastic problem formulated in this paper, should be some information concerning the statistical measures of the displacements and eventually stresses. So, one is interested in finding expressions determining these measures in terms of the corresponding statistical measures of input variables. Statistical measures such as the mean and the correlation can be obtained in the same manner as in the one-dimensional case. A framework for evaluation of the average value and the variance of the vertical displacements was derived in part 1.

To obtain the solution for the means $\langle u\rangle$ and $\langle v\rangle$ the solution processes u and v must be averaged over an appropriate probability space. Each term from (42) and (43) can be calculated and ensemble averaged, without closure approximations.

Knowing statistical measures of displacements, statistics of stresses can be determined, based on random function calculus and appropriate elasticity theory relationships.

Unfortunately, the numerical calculations have not been carried out yet. It is planned to perform more detailed analysis of the problem in the future.

8. Remarks and Conclusions

The random elasticity theory is applied to the statistically homogeneous elastic soil half-plane subjected to gravity. The analysis is performed in the plane strain state. Only the elastic modulus is considered to be uncertain and is treated as a two-dimensional random field.

On the basis of Green's function approach, the stochastic partial differential equations governing the problem are converted into stochastic integral equations. Then the perturbation procedure and Adomian's decomposition method are applied. The first one imposes a small fluctuation assumption. The system of input stochastic differential equations with random coefficients is here transformed into a few sets of stochastic differential equations with random forcing terms. The average displacements obtained by the first order perturbation method are the same as the deterministic ones. The second order statistics, as a variance for example, are presented in the form of the four-fold integrals. These integrals can be computed only numerically, and the Monte Carlo method of integration is recommended.

There is no small fluctuation assumption in Adomian's decomposition method. In this approach the solution is valid for the coefficient of variation of elastic modulus of up to 0.5. Adomian's decomposition method is essentially the solution of the stochastic Volterra equation by Neumann series expansion. The statistical separatibility is a major advantage of this method, so the closure approximations or truncations are unnecessary here. The Adomian's decomposition procedure enables the finding of the explicit expressions for the statistical measures of stresses and displacements, for example their average values and variances. Unfortunately, in this method, high-order terms in expansions for u and v become increasingly complex. Thus, the real test of usefulness of the method is whether or not low-order terms provide significant improvement over the small fluctuation solution.

Although the numerical calculations were not performed, the presented approach offers the tool which can give quantitative results to the problem discussed. It is planned to carry out more detailed analysis of the problem, generally based on the numerical calculus.

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