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Deformations due to Gravity in Random Elastic Soil Medium
Part 1. Uni-axial Strain State

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Abstract

The random elasticity theory is applied to the statistically homogeneous soil layer subjected to its own weight. The modulus of elasticity is assumed to be a stochastic process and the uni-axial strain state is considered. The governing elasticity equations are presented in alternative ways, as a first order stochastic differential equation either with a random coefficient or random forcing function and as a second order stochastic differential equation. The solution is obtained by the approximated analytical methods i.e. perturbation procedure and Adomian's decomposition method. First order perturbation method is used to determine the variance function of the solution and second order to find the average displacement. There is no small fluctuation assumption in Adomian's decomposition method. In a framework of this method the analytical expression defining the displacement as a stochastic process is presented. The second order average solution and the variance of displacement are found, although the explicit expression for the variance is found only for the first order solution.

1. Introduction

The theory of the homegeneous, isotropic elastic half-space has played an extremely important role in the development of foundation engineering.

Usually, the rigidity of soils increases with depth as a consequence of the increasing effective overburden pressure. It has given a rise to a wide literature on the non-homogeneous theory of elasticity, where the elastic parameters were assumed to vary with the location of the point, deterministically (Gibson 1967; Kassir 1972; Carrier 1973; Lomakin 1976; Oner 1990; et al.).

In nature, however, most soils intrinsically involve randomness and uncertainty. Thus, one of the fundamental decisions is whether the model should be deterministic or stochastic. Deterministic models are quite useful, but stochastic models are more realistic. The difficulty has been that the stochastic models are very difficult to handle mathematically. Considerable effort has been devoted recently in improvig the model associated with the material properties of a medium by describing it as a stochastic process, or more generally as a random field. In such a case, the elasticity theory becomes random and is governed by stochastic differential equations. Although such equations have been studied and developed by many researches in different disciplines of Physics and Engineering, only recently attempts have been made to develop the theory of stochastic equations. In fact, the field has not been sufficiently explored, and almost all the results that have been obtained so far refer to very special situations.

Geotechnical problems, based on the random elasticity, do not generally yield to exact analytical solutions. Thus, most of the paper focused on approximate, numerical procedures, mainly on the stochastic finite element method. The assumption of various degrees of severity are introduced (Bucher 1988; Deodatis 1989, 1990; Liu 1986, 1987; Shinozuka 1987; Spanos 1989; Yamazaki 1988; et al.).

Apart from numerical methods, a few approximated analytical approaches are available for solving stochastic differential equations. Most commonly used are perturbation or hierarchy methods, although they essentially limit systems to small fluctuations. Only a few papers deal with the application of these methods in geotechnical engineering. Zeitoun et al. (1988) applied the perturbation method to solve some geotechnical problems governed by the equations of random elasticity. However, they have evaluated the final results numerically.

The powerful approximated analytical method, without small fluctuation assumption, is the decomposition method (called elsewhere the stochastic Green's function method). This approach has been considerably expanded by Adomian (1983) and applied to random elasticity by Eimer (1972).

The aim of this paper is the application of the approximated, analytical methods in solving basic geotechnical problems in a framework of random elasticity. A statistically homogeneous soil medium subjected only to gravity is considered. An external loading will be taken into account in the next stage of the analysis. It is worth emphasizing that the displacement of any particle of the soil medium, subjected only to its own weight, has more of a theoretical meaning. In fact, the displacements due to gravity has already occurred. In the case of uni-axial strain state, the vertical stresses do not depend on elastic parameters, hence they are deterministic. However, knowledge of displacements and their statistical measures for plane strain analysis is necessary in order to determine the geostatical state of stresses which has a stochastic nature.

In part 1, a statistically homogeneous horizontal soil layer and only a uniaxial strain state is considered. The perturbation, as well as the Adomian decomposition methods are applied. The govering elasticity equations are presented in alternative ways, as a first order stochastic differential equation either with random coefficient or random forcing function and as a second order stochastic differential equations.

The second part deals with plane strain conditions, and again the perturbation method and the Adomian decomposition procedure are incorporated.

2. Basic Equations

Let us consider an elastic, horizontal soil layer resting on a rough, rigid base. The soil mass is assumed to be in equilibrium under the influence of its own weight. It is assumed that the soil properties do not vary in a horizontal direction and only Young's modulus E varies with depth and is a homogeneous random function of position $E(z)$. The soil layer is presented in Fig. 1 for two different co-ordinate systems. An analysis performed in this part will be consistent with Fig. 1b, however, some advantages due to the co-ordinate system in Fig. 1a will be visible in a further analysis of the problem.

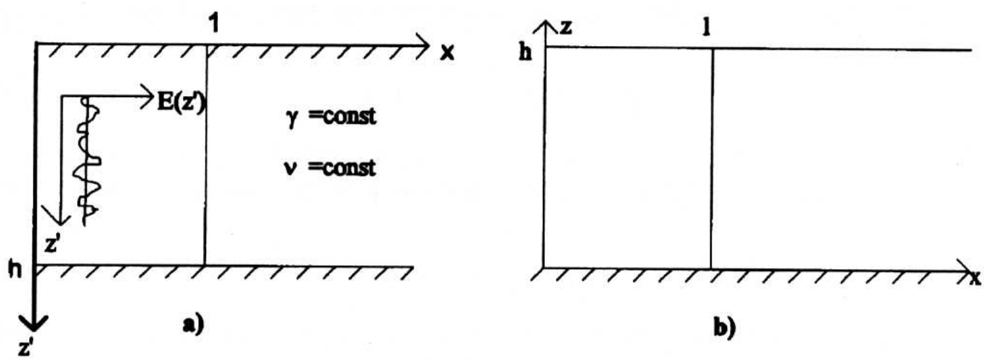

Fig. 1. One-dimensional strain state for a horizontal soil layer

The problem is one-dimensional and is governed by the following elasticity equations: equilibrium equation

$$\frac{d\sigma}{dz} = \gamma, \tag{1}$$

constitutive equation

$$\sigma = EA\varepsilon, \quad \left(A = \frac{(1-\nu)}{(1+\nu)(1-2\nu)}\right), \tag{2}$$

geometric equation

$$\varepsilon = \frac{d\nu}{dz} \tag{3}$$

with boundary conditions:

$$\sigma(0) = 0, \quad v'(0) = 0 \quad v(h) = 0, \qquad \text{(Fig. 1a)} \tag{4a}$$

$$\sigma(h) = 0, \quad v'(h) = 0 \quad v(0) = 0 \qquad \text{(Fig. 1b)} \tag{4b}$$

where: σ, ε and v are vertical stress, strain and displacement respectively and γ is the unit weight of soil.
Substituting (2) and (3) into (1) the following equation is obtained:

$$EA\,\frac{d^2v}{dz^2} = \gamma. \tag{5}$$

Double intergration leads to:

$$EA\,\frac{dv}{dz} = \gamma\; z + c_1, \tag{6a}$$

$$EAv = \frac{1}{2}\gamma\; z^2 + c_1 z + c_2. \tag{6b}$$

In general the value γ/EA can also be considered as a random field.
Taking into account boundary conditions (4b) the final expression for the displacement is a trivial one:

$$EAv = \gamma\left(\frac{1}{2}\,z^2 - hz\right). \tag{7a}$$

In the case of the boundary conditions (4a) the expression for the displacement can simply be obtained by substituting transformation relationship $z = h - z'$ into (7a):

$$EAv = \frac{1}{2}\gamma\;(z'^2 - h^2). \tag{7b}$$

Solution (7) is valid for a homogeneous linear-elastic medium, where E = const.

In a case of random elasticity, $E(z)$ is a stochastic process, so the expression for the displacement is quite different. The problem can be described alternatively:

(a) First order equation

Integrating equilibrium equation (1) and taking into account the first boundary condition from (4b), vertical stress is obtained:

$$\sigma = \gamma(z - h). \tag{8}$$

Substituting this stress into (2) and taking into consideration (3), the displacement can be found from the first order differential equation:

$$E(z)A\,\frac{dv}{dz} = \gamma(z - h) \tag{9}$$

with boundary condition $v(0) = 0$.

(b) Second order equation

Having in mind that the modulus of elasticity is a function of z let us differentiate equation (2) with respect to z:

$$\frac{d\sigma}{dz} = A\left(\frac{dE}{dz}\,\varepsilon + E\frac{d\varepsilon}{dz}\right) \tag{10}$$

and next including (3) and substituting (10) into (1) the second order differential equation has the final form:

$$E(z)\frac{d^2v(z)}{dz^2} + \frac{dE(z)}{dz}\frac{dv(z)}{dz} = \frac{\gamma}{A}. \tag{11}$$

Thus, we have obtained the alternative description of the problem either in the form of two equations (8) and (9) or one second order equation (11). Equations (10) and (11) are stochastic differential equations, because $E(z)$ is the random function.

The aim of the paper is to solve these equations i.e. to find some statistical measures of the displacement like a mean value and a covariance function.

3. Characterisation of Uncertainty of the Modulus of Elasticity

It is convenient to present the modulus of elasticity in the following form:

$$E(z) = \overline{E}[1 + \alpha\ \beta(z)] \tag{12}$$

where:

$\overline{E} = \langle E \rangle$	–	mean value of the modulus of elasticity,
α	–	coefficient of variation,
$\beta(z)$	–	one-dimensional, normalised stationary process with expected value $\langle \beta(z) \rangle = 0$, standard deviation $\sigma_\beta = 1$, and a given correlation function $R_\beta(\tau)$, ($\tau = \lvert z_1 - z_2 \rvert$).

The correlation function must be differentiable. In further analysis the following correlation function is assumed:

$$R_\beta(\tau) = (1 + \lambda|\tau|)e^{-\lambda|\tau|} \tag{13}$$

where λ is a correlation distance.

4. The Perturbation Method

If the random variations of the soil properties are sufficiently small so that corrections to the deterministic solution are of a low order, then the perturbation theory is useful in solving the problem. Some numerical analyses show that this type of solution is valid if the coefficient of variation of the modulus of elasticity α is less than approximately 0.1 to 0.15.

In this section the perturbation technique will be applied to solve both the first and second order stochastic equations. The second order perturbation method will be applied in order to find an average solution, and the first order perturbation procedure to find the variance of the solution.

The first order stochastic equation will be presented in an alternative way either as the equation with random coefficient or with random input.

4.1. First Order Stochastic Differential Equation with Random Coefficient

Substituting (12) into (9) one obtains:

$$\bar{E}A[1+\alpha\ \beta(z)]\frac{dv}{dz} = \gamma(z-h). \tag{14}$$

Expanding the unknown function $v(z,\alpha)$ in powers of α, and taking into account only the second order terms, one can write:

$$v(z,\alpha) = v_0(z) + \alpha\ v_1(z) + \alpha^2 v_2(z), \tag{15a}$$

$$\frac{dv(z,\alpha)}{dz} = \frac{dv_0(z)}{dz} + \alpha\ \frac{dv_1(z)}{dz} + \alpha^2\ \frac{dv_2(z)}{dz} \tag{15b}$$

where: $v_0(z) = v(z,0)$, $v_1(z) = \dfrac{\partial v(z,\alpha=0)}{\partial\alpha}$, $v_2(z) = \dfrac{\partial^2 v(z,\alpha=0)}{\partial\alpha^2}$.
Substituting (15b) into (14) leads to the following expression:

$$\bar{E}A(1+\alpha\ \beta)(v_0' + \alpha\ v_1' + \alpha^2 v_2') = \gamma(z-h). \tag{16}$$

Equating coefficients of like powers of α a set of the following equations is obtained:

$$1.\quad \bar{E}Av_0' = \gamma(z-h), \tag{17a}$$

$$2.\quad \bar{E}A(v_1' + \beta\ v_0') = 0, \tag{17b}$$

$$3.\quad \bar{E}A(v_2' + \beta\ v_1') = 0. \tag{17c}$$

Equation (17a) is deterministic and its solution is the same as (7).

In order to solve equation (17b) let us substitute (17a) into it and rewrite in the form:

$$\bar{E}Av_1'(z) = -\gamma(z-h)\beta(z). \tag{18}$$

The unknown component of the displacement v_1 can be determined by integrating both sides of (18):

$$\bar{E}Av_1 = -\gamma \int_0^z (\xi - h)\beta(\xi)d\xi. \tag{19}$$

Substituting expression (19) into (17c) yields:

$$\bar{E}Av_2'(z) = \gamma(z-h)\beta^2(z). \tag{20}$$

Thus, the unknown component v_2, can be presented as a stochastic integral in the following form:

$$\bar{E}Av_2 = \gamma \int_0^z (\xi - h)\beta^2(\xi)d\xi. \tag{21}$$

Substituting (7), (19) and (21) into (15a), the following expression for the displacement is obtained:

$$\overline{E}Av = \gamma\left(\frac{1}{2}z^2 - hz\right) = \alpha\,\gamma\int_0^z (\xi - h)\,\beta(\xi)d\xi + \alpha^2\gamma\int_0^z (\xi - h)\,\beta^2(\xi)d\xi. \quad (22)$$

Now the statistical measures of the approximate displacement v can be determined by standard calculations. In particular the mean or average value is:

$$\langle v\rangle = \frac{\gamma}{\overline{E}A}\left(\frac{1}{2}z^2 - hz\right)(1+\alpha^2). \quad (23)$$

The second order expected value vs depth is presented in Fig. 5 for $\alpha = 0.1$. Also presented is the average solution obtained by the Adomian decomposition method.

It seems to be reasonable to apply the first order perturbation method to determine the second order statistical measures. In such case, only equations (17a) and (17b) can be considered. The general expression for an autocovariance function is as follows:

$$\begin{aligned}
R_v(z_1,z_2) &= \langle[v(z_1)-\overline{v}(z_1)]\,[v(z_2)-\overline{v}(z_2)]\rangle = \\
&= \left(\frac{\alpha\,\gamma}{\overline{E}A}\right)^2\left\langle\int_0^{z_1}(\xi-h)\,\beta(\xi)d\xi\int_0^{z_2}(\xi-h)\,\beta(\xi)d\xi\right\rangle = \\
&= \left(\frac{\alpha\,\gamma}{\overline{E}A}\right)^2\left\langle\int_0^{z_1}\int_0^{z_2}(\xi-h)\,(\eta-h)\,\beta(\xi)\,\beta(\eta)d\xi d\eta\right\rangle = \\
&= \left(\frac{\alpha\,\gamma}{\overline{E}A}\right)^2\int_0^{z_1}\int_0^{z_2}(\xi-h)(\eta-h)\,\langle\beta(\xi)\,\beta(\eta)\rangle d\xi d\eta = \\
&= \left(\frac{\alpha\,\gamma}{\overline{E}A}\right)^2\int_0^{z_1}\int_0^{z_2}(\xi-h)(\eta-h)\,R_\beta(\xi-\eta)d\xi d\eta. \quad (24)
\end{aligned}$$

Substituting the covariance function given by (13) into (24) yields:

$$R_v(z_1,z_2) = (\alpha\,\gamma)^2\int_0^{z_1}\int_0^{z_2}(\xi-h)(\eta-h)(1+\lambda|\xi-\eta|)\,e^{-\lambda|\xi-\eta|}d\xi d\eta. \quad (25)$$

The double integration must be performed over a rectangular area as shown in Fig. 2. The integral for the area where $\xi > \eta$ is as follows:

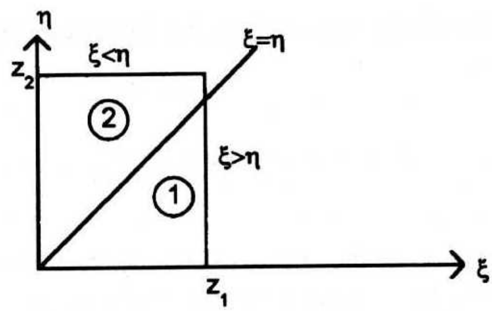

Fig. 2. The rectangular area of integration of (25)

$$I_1 = \int_{\xi=0}^{\xi=z_1} \int_{\eta=0}^{\eta=\xi} (\xi - h)(\eta - h) \; [1 + \lambda(\xi - \eta)] \; e^{-\lambda(\xi-\eta)} d\xi d\eta \tag{26}$$

and for the area $\xi < \eta$ one has:

$$I_2 = \int_{\xi=0}^{\xi=z_1} \int_{\eta=0}^{\eta=z_2} (\xi - h)(\eta - h) \; [1 + \lambda(\eta - \xi)] \; e^{-\lambda(\eta-\xi)} d\xi d\eta. \tag{27}$$

Eventually, the expression for the covariance is as follows:

$$\begin{aligned} R_v(z_1, z_2) = \left(\frac{\gamma\, \alpha}{\overline{E}A}\right)^2 & \left\{ \frac{4}{3\lambda}\, z_1^3 - \frac{4h}{\lambda}\, z_1^2 + \frac{4h^2}{\lambda}\, z_1 + \frac{1}{\lambda^2}\, \left(\frac{5}{\lambda^2} - 3h^2\right) + \right. \\ & + \left[\frac{3h^2}{\lambda^2} - \frac{5}{\lambda^4} - \frac{1}{\lambda}\left(\frac{5}{\lambda^2} + \frac{3h}{\lambda} - h^2\right) z_1 - \frac{1}{\lambda}\left(h + \frac{1}{\lambda}\right) z_1^2\right] e^{-\lambda z_1} + \\ & + \left[\frac{3h^2}{\lambda^2} - \frac{5}{\lambda^4} - \frac{1}{\lambda}\left(\frac{5}{\lambda^2} + \frac{3h}{\lambda} - h^2\right) z_2 - \frac{1}{\lambda}\left(h + \frac{1}{\lambda}\right) z_2^2\right] e^{-\lambda z_2} + \\ & + \left[z_1^2 z_2 - z_1 z_2^2 - \frac{5}{\lambda}\, z_1 z_2 + \left(\frac{5}{\lambda^2} + \frac{3h}{\lambda} - h^2\right) z_2 - \left(\frac{5}{\lambda^2} - \frac{3h}{\lambda} - h^2\right) \times \right. \\ & \left. \times\, z_1 + \; z_1 + \left(\frac{1}{\lambda} - h\right) z_1^2 + \left(\frac{1}{\lambda} + h\right) z_2^2 + \frac{5}{\lambda^3} - \frac{3h^2}{\lambda}\right] \times \\ & \left. \times\, \frac{1}{\lambda}\, e^{-\lambda(z_1 - z_2)} \right\}. \end{aligned} \tag{28}$$

The variance of the displacement can be obtained from (28) for $z = z_1 = z_2$:

$$\begin{aligned} Var[v] = \left(\frac{\alpha\, \gamma^2}{\overline{E}A}\right)^2 & \left\{ \frac{4}{3\lambda}\, z^3 - \left(\frac{3}{\lambda^2} + \frac{4h}{\lambda}\right) z^2 + \left(\frac{6h}{\lambda^2} + \frac{4h^2}{\lambda}\right) z + \right. \\ & + \frac{10}{\lambda^4} - \frac{6h^2}{\lambda^2} 2 \left[\frac{3h^2}{\lambda^2} - \frac{5}{\lambda^4} - \frac{1}{\lambda}\, \left(\frac{5}{\lambda^2} + \frac{3h}{\lambda} - h^2\right) \times \right. \end{aligned}$$

$$\times \quad z - \frac{1}{\lambda}\left(h + \frac{1}{\lambda}\right) z^2\Big] e^{-\lambda z}\Big\}. \tag{29}$$

The variance of the displacement for $h = 10$ m and different values of the correlation distance λ is presented in Fig. 3.

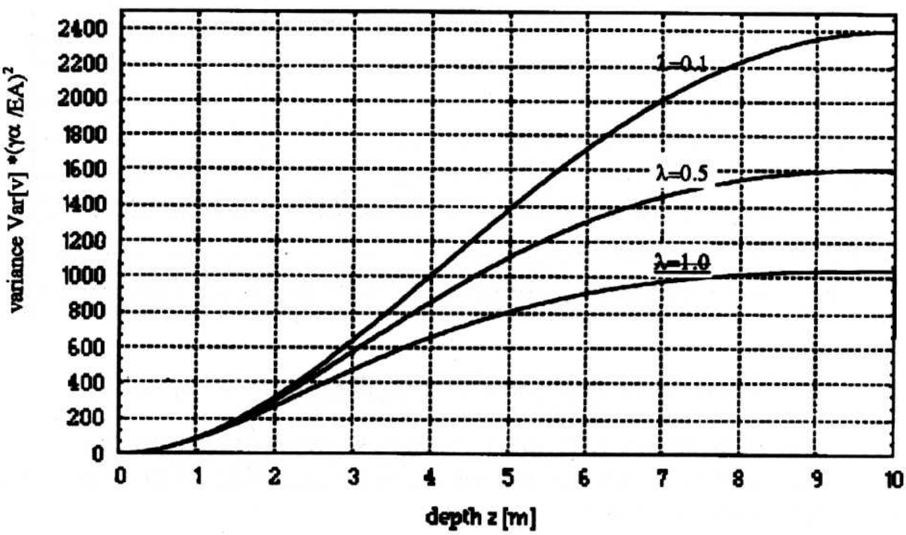

Fig. 3. Variance of displacement vs depth

It is seen from Fig. 3 that the variance of the displacement assumes the highest values for decreasing λ, e.a. increasing correlations. In the limiting case it is maximum for the random variable model (full correlation).

Fig. 4 presents a coefficient of variation of the displacement for $\alpha = 0.1$ and different values of the correlation distance λ. This coefficient is calculated for the second order expected value (23) and it approaches zero for $z = 0$ whereas it reaches maximum for z close to zero.

From (28) and (29) it is seen that the random function of the displacement is not homogeneous.

4.2. First Order Stochastic Differential Equation with Random Input

Let us again consider equation (9). Dividing both sides by E it can be rewritten in the following form:

$$\frac{dv}{dz} = \frac{1}{E}\,\frac{\gamma}{A}\,(z - h). \tag{30}$$

Substituting (12) into (32) one gets:

$$\frac{dv}{dz} = \frac{1}{\overline{E}(1 + \alpha\,\beta)}\,\frac{\gamma}{A}\,(z - h). \tag{31}$$

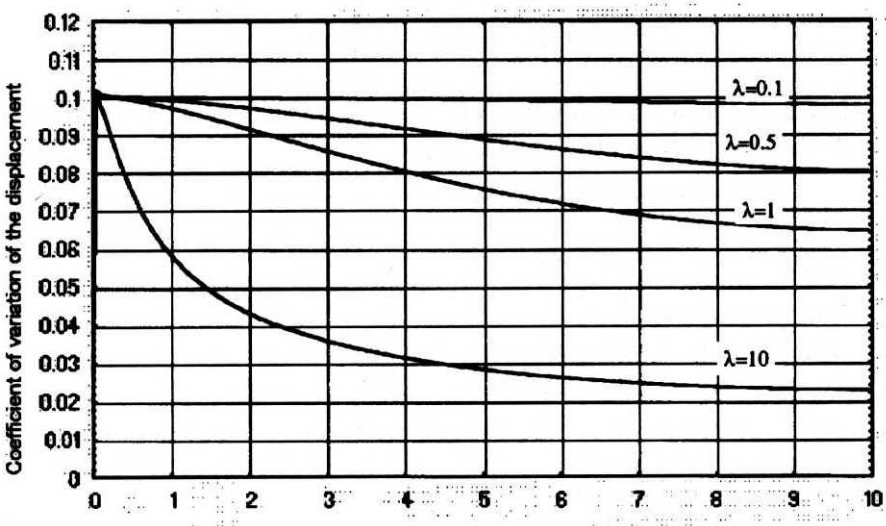

Fig. 4. Coefficient of variation of the displacement vs depth

It is seen that in equation (31) the randomness, represented by β, occurs only in the forcing function, thus the equation can be classified as the random input stochastic differential equation.

Now, let us introduce a new random variable w defined as follows:

$$w = \frac{1}{\overline{E}(1 + \alpha\ \beta)}. \tag{32}$$

In fact, w is a random function or stochastic process $w(z)$ because $\beta(z)$ is also a stochastic process. We are going to determine its statistical parameters, in particular a covariance function. Unfortunately, an exact determination seems to be impossible, as the formula (32) presents a non-linear transformation of random functions. However, a simple solution can be obtained in a framework of second order correlation analysis. Instead of random function $\beta(z)$ we will consider two random variables β_1 and β_2 characterised by the bivariate normal probability density function:

$$f(\beta_1, \beta_2) = \frac{1}{2\pi\sigma_{\beta_1}\sigma_{\beta_2}\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\left(\frac{\beta_1 - \overline{\beta_1}}{\sigma_{\beta_1}}\right)^2 + \right.\right.$$
$$\left.\left. + \left(\frac{\beta_2 - \overline{\beta}_2}{\sigma_{\beta_2}}\right)^2 - 2r\frac{(\beta_1 - \overline{\beta_1})(\beta_2 - \overline{\beta_2})}{\sigma_{\beta_1}\sigma_{\beta_2}}\right]\right\}. \tag{33}$$

The random variables β_1 and β_2 correspond to the states of stochastic process for two locations z_1 and z_2, i.e. $\beta_1 = \beta(z_1)$ and $\beta_2 = \beta(z_2)$. Of course the expected values and variances of those two variables are identical and the covariance between them can be determined by substitution of the given values of z_1 and z_2 into the covariance function.

For the covariance function (13) assumed in the paper we have:

$$Cov[\beta_1, \beta_2] = (1 + \lambda|z_1 - z_2|)\, e^{-\lambda|z_1 - z_2|} \tag{34}$$

and the probability density function is as follows:

$$f(\beta_1, \beta_2) = \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\left(\frac{\beta_1}{\sigma}\right)^2 - 2r\frac{\beta_1\beta_2}{\sigma^2} + \left(\frac{\beta_2}{\sigma}\right)^2\right]\right\} \tag{35}$$

where: $\sigma = \sigma_{\beta_1} = \sigma_{\beta_2} = 1$, $\overline{\beta}_1 = \overline{\beta}_2 = 0$, $r = (1 + \lambda|z_1 - z_2|)\, e^{-\lambda|z_1 - z_2|}$.
Now we want to find the probability density function of the variables w_1 and w_2. This is a problem of two functions of two random variables:

$$w_1 = \frac{1}{\overline{E}(1 + \alpha\, \beta_1)},\ w_2 = \frac{1}{\overline{E}(1 + \alpha\, \beta_2)}. \tag{36}$$

The Jakobian of the transformation is as follows:

$$J = \begin{vmatrix} \frac{\partial w_1}{\partial \beta_1} & \frac{\partial w_1}{\partial \beta_2} \\ \frac{\partial w_2}{\partial \beta_1} & \frac{\partial w_2}{\partial \beta_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{\overline{E}} \frac{-\alpha}{(1 + \alpha\, \beta_1)^2} & 0 \\ 0 & \frac{1}{\overline{E}} \frac{-\alpha}{(1 + \alpha\, \beta_2)^2} \end{vmatrix} =$$

$$= \frac{1}{\overline{E}^2} \frac{\alpha^2}{(1 + \alpha\, \beta_1)^2\, (1 + \alpha\, \beta_2)^2}. \tag{37}$$

The probability density function of new variables (36) is defined as:

$$f(w_1, w_2) = \frac{1}{|J|}\, f(\beta_1, \beta_2). \tag{38}$$

Substituting (37) into (38) one obtains:

$$f(w_1, w_2) = \overline{E}^2(1 + \alpha\, \beta_1)^2(1 + \alpha\, \beta_2)^2\, \frac{1}{2\pi\sqrt{1-r^2}} \times$$
$$\times \exp\left\{-\frac{1}{2(1-r^2)}\, [(\beta_1)^2 - 2r\beta_1\beta_2 + (\beta_2)^2]\right\}. \tag{39}$$

Finding from (36) the inverse relationships:

$$\beta_1 = \frac{1}{\alpha}\left(\frac{1}{\overline{E}w_1} - 1\right),\quad \beta_2 = \frac{1}{\alpha}\left(\frac{1}{\overline{E}w_2} - 1\right), \tag{40}$$

and substituting them into (39), the following expression for probability density function is finally obtained:

$$f(w_1, w_2) = \frac{1}{2\pi w_1 w_2 \sqrt{1-r^2}} \exp\left\{-\frac{1}{\alpha^2}\left[\left(\frac{1}{\overline{\overline{E}}w_1} - 1\right)^2 + \right.\right.$$

$$\left.\left. - 2r\left(\frac{1}{\overline{\overline{E}}w_1} - 1\right)\left(\frac{1}{\overline{\overline{E}}w_2} - 1\right) + \left(\frac{1}{\overline{\overline{E}}w_2} - 1\right)^2\right]\right\}. \quad (41)$$

Now, the statistical measures can be computed from the following formulas:

$$\overline{w}_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_1 f(w_1, w_2)\, dw_1 dw_2, \quad (42a)$$

$$\overline{w}_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2 f(w_1, w_2)\, dw_1 dw_2, \quad (42b)$$

$$Cov[w_1, w_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (w_1 - \overline{w}_1)(w_2 - \overline{w}_2) f(w_1, w_2)\, dw_1 dw_2. \quad (42c)$$

The covariance between variables w_1 and w_2 can be computed in an easy way by the linearization i.e. first order approximation method. Application of a well known approximating formula:

$$Cov[w_1, w_2] = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial w_1}{\partial \beta_i}\frac{\partial w_2}{\partial \beta_j}\, Cov[\beta_i, \beta_j] \quad (43)$$

leads to the final expression for the covariance in the form:

$$Cov[w_1, w_2] = \frac{\alpha^2}{\overline{\overline{E}}^2}\, R_\beta(z_1 - z_2). \quad (44)$$

Now let us go back to the basic random input stochastic differential equation (31). It can be rewritten in the form:

$$\frac{dv}{dz} = \frac{\gamma}{A}\,(z - h)w(z) \quad (45)$$

where the covariance of $w(z)$ is given by (44) and the first order approximation of its average value is:

$$\overline{w} = \frac{1}{\overline{\overline{E}}}. \quad (46)$$

The solution of (45) can be obtained directly in the form of a stochastic integral:

$$v = \frac{\gamma}{A} \int_0^z (\xi - h)\ w(\xi)\ d\xi. \tag{47}$$

The average value of the displacement is as follows:

$$\overline{v} = \left\langle \frac{\gamma}{A} \int_0^z (\xi - h) w(\xi)\ d\xi \right\rangle = \frac{\gamma}{A} \int_0^z \frac{\xi - h}{\overline{E}}\ d\xi = \frac{\gamma}{\overline{E}A} \left(\frac{1}{2}\ z^2 - hz \right) \tag{48}$$

and is of course the same as the one computed before (19).

The covariance between the displacements at two locations is:

$$\begin{aligned}
&R_v(z_1, z_2) = Cov[v_1, v_2] = \langle (v_1 - \overline{v}_1)(v_2 - \overline{v}_2) \rangle = \\
&= \left\langle \left[\frac{\gamma}{A} \int_0^{z_1} (\xi - h) w(\xi)\ d\xi \frac{\gamma}{\overline{E}A} \left(\frac{1}{2}\ z_1^2 - hz_1 \right) \right] \times \right. \\
&\times \left. \left[\frac{\gamma}{A} \int_0^{z_2} (\eta - h) w(\eta) d\eta - \frac{\gamma}{\overline{E}A} \left(\frac{1}{2}\ z_2^2 - hz_2 \right) \right] \right\rangle = \\
&= \left(\frac{\gamma}{A} \right)^2 \int_0^{z_1} \int_0^{z_2} (\xi - h)(\eta - h) \langle w(\xi) w(\eta) \rangle d\xi d\eta - \left(\frac{\gamma}{\overline{E}A} \right)^2 \left(\frac{1}{2} z_1^2 - hz_1 \right) \times \\
&\times \left(\frac{1}{2} z_2^2 - hz_2 \right) = \left(\frac{\gamma}{A} \right)^2 \int_0^{z_1} \int_0^{z_2} (\xi - h)(\eta - h) \{ Cov[w_1, w_2] + \\
&- \overline{w}_1 \overline{w}_2 \} - \left(\frac{\gamma}{\overline{E}A} \right)^2 \left(\frac{1}{2} z_1^2 - hz_1 \right) \left(\frac{1}{2} z_2^2 - hz_2 \right) = \\
&= \left(\frac{\gamma\ \alpha}{\overline{E}A} \right)^2 \int_0^{z_1} \int_0^{z_2} (\xi - h)(\eta - h) \langle \beta(\xi) \beta(\eta) \rangle d\xi d\eta.
\end{aligned} \tag{49}$$

Thus, we have obtained the same formula for the covariance as (24).

4.3. Second Order Stochastic Differential Equation

Now let us consider the equation (11) with boundary conditions (4b). Substituting (12) into (11) one obtains:

$$[1 + \alpha\ \beta(z)] \frac{d^2 v}{dz^2} + \frac{d}{dz} [1 + \alpha\ \beta(z)] \frac{dv}{dz} = \frac{\gamma}{\overline{E}A} \tag{50a}$$

or

$$\frac{d^2v}{dz^2} + \alpha\ \beta(z)\frac{d^2v}{dz^2} + \alpha\ \frac{d\beta(z)}{dz}\frac{dv}{dz} = \frac{\gamma}{\overline{\overline{E}}A}. \tag{50b}$$

Now, it would be convenient to use an operator notation. Defining the deterministic operator:

$$L = \frac{d^2}{dz^2} \tag{51}$$

and the stochastic one:

$$\mathfrak{R} = \alpha\ \beta(z)\frac{d^2}{dz^2} + \alpha\ \frac{d\beta(z)}{dz}\frac{d}{dz}. \tag{52}$$

The equation (50b) can be written in the operator form as follows:

$$Lv = \frac{\gamma}{\overline{\overline{E}}A} - \mathfrak{R}v. \tag{53}$$

Limiting our consideration only to the first order perturbation method the unknown displacement can be presented as before, i.e. (15a). Substituting it into (53) leads to:

$$Lv_0 + \alpha\ Lv_1 = \frac{\gamma}{\overline{\overline{E}}A} - \mathfrak{R}v_0 - \alpha\ \mathfrak{R}v_1. \tag{54}$$

Equating coefficients of like powers of α a set of two recurrent equations is obtained:

$$1. \quad Lv_0 = \frac{\gamma}{\overline{\overline{E}}A}, \tag{55}$$

$$2. \quad Lv_1 = -\frac{1}{\alpha}\ \mathfrak{R}v_0. \tag{56}$$

For the general differential operator L, its inverse L^{-1} is an integral operator with a kernel $G(z, \xi)$, so that the solution of equation (55) can be written in the form:

$$v_0 = \theta + \int_0^h G(z, \xi)\ \frac{\gamma}{\overline{\overline{E}}A}\ d\xi \tag{57}$$

where: θ is the solution of the homogeneous equation $Lv_0 = 0$, and for the boundary conditions (4b) $\theta = 0$, $G(z, \xi)$ is Green's function.
For the deterministic differential operator (51) and given boundary conditions (4b) Green's function may be presented in the form:

$$G(z, \xi) = \begin{cases} -\xi & \text{for} \quad 0 \le \xi < z, \\ -z & \text{for} \quad z < \xi \le h. \end{cases} \tag{58}$$

Substituting (58) into (57) one obtains:

$$v_0 = -\int_0^z \xi \frac{\gamma}{\overline{EA}} d\xi - \int_z^h z \frac{\gamma}{\overline{EA}}\, d\xi = \frac{\gamma}{\overline{EA}}\left(\frac{1}{2}z^2 - hz\right). \tag{59}$$

Again, it is seen that so determined displacement is identical as (7) or (19).

The solution of equation (56) can be written as follows:

$$v_1 = -\frac{1}{\alpha}\, L^{-1}\Re v_0 = -\int_0^h G(z,\xi)\left[\beta(\xi)\frac{d^2 v_0}{d\xi^2} + \beta'(\xi)\frac{dv_0}{d\xi}\right] d\xi. \tag{60}$$

Differentiating (59) with respect to z and substituting $z = \xi$ we have:

$$\frac{dv_0}{d\xi} = \frac{\gamma}{\overline{EA}}(\xi - h), \quad \frac{d^2 v_0}{d\xi^2} = \frac{\gamma}{\overline{EA}}. \tag{61}$$

Substituting (61) and (58) into (60) the final expression for the second component of the displacement v_1 takes the form:

$$v_1 = \frac{\gamma}{\overline{EA}}\left\{\int_0^z [\xi\beta(\xi) + \beta'(\xi)\xi(\xi - h)]d\xi + \int_z^h [z\beta(\xi) + \beta'(\xi)z(\xi - h)]d\xi\right\}. \tag{62}$$

It is easy to find that the average value of $\bar{v}_1$ is equal to zero, thus the average displacement is equal to its deterministic value i.e. $\bar{v} = v_0$. In the case of the second order perturbation procedure, the average displacement is the same as the one obtained before (23).

In order to determine the covariance between the displacements at two locations z_1 and z_2, let us rewrite (62):

$$v_1(z_1) = \frac{\gamma}{\overline{EA}}\left\{\int_0^{z_1} [\xi\beta(\xi) + \beta'(\xi)\xi(\xi - h)]d\xi + \right.$$
$$\left. + \int_{z_1}^h [z_1\beta(\xi) + \beta'(\xi)z_1(\xi - h)]d\xi\right\}, \tag{63a}$$

$$v_1(z_2) = \frac{\gamma}{\overline{EA}}\left\{\int_0^{z_2} [\xi\beta(\xi) + \beta'(\xi)\xi(\xi - h)]d\xi + \right.$$
$$\left. + \int_{z_2}^h [z_2\beta(\xi) + \beta'(\xi)z_2(\xi - h)]d\xi\right\}. \tag{63b}$$

The general expression for the covariance can be presented in the form of the sum of the double integrals as follows:

$$
\begin{aligned}
&Cov[v(z_1), v(z_2)] = \langle[v(z_1) - \bar{v}(z_1)][v(z_2) - \bar{v}(z_2)]\rangle = \alpha^2\langle v_1(z_1)v_1(z_2)\rangle = \\
&= \left(\frac{\gamma\alpha}{\overline{E}A}\right)^2 \Bigg[\int_{\xi=0}^{\xi=z_1}\int_{\eta=0}^{\eta=z_2} \xi\eta\langle\beta(\xi)\beta(\eta)\rangle d\xi d\eta + \int_{\xi=0}^{\xi=z_1}\int_{\eta=0}^{\eta=z_2} \xi\eta(\eta-h)\langle\beta(\xi)\beta'(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=0}^{\xi=z_1}\int_{\eta=z_2}^{\eta=h} z_2\xi(\eta-h)\langle\beta(\xi)\beta'(\eta)\rangle d\xi d\eta + \int_{\xi=0}^{\xi=z_1}\int_{\eta=0}^{\eta=z_2} \xi\eta(\xi-h)\langle\beta'(\xi)\beta(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=0}^{\xi=z_1}\int_{\eta=0}^{\eta=z_2} \xi\eta(\eta-h)\langle\beta'(\xi)\beta'(\eta)\rangle d\xi d\eta + \int_{\xi=0}^{\xi=z_1}\int_{\eta=z_2}^{\eta=h} z_2\xi(\xi-h)\langle\beta'(\xi)\beta(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=0}^{\xi=z_1}\int_{\eta=z_2}^{\eta=h} z_2\xi(\xi-h)(\eta-h)\langle\beta'(\xi)\beta(\eta)\rangle d\xi d\eta + \int_{\xi=0}^{\xi=z_1}\int_{\eta=z_2}^{\eta=h} z_2\xi\langle\beta(\xi)\beta(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=z_1}^{\xi=h}\int_{\eta=z_2}^{\eta=h} z_1z_2(\xi-h)\langle\beta'(\xi)\beta(\eta)\rangle d\xi d\eta + \int_{\xi=z_1}^{\xi=h}\int_{\eta=z_2}^{\eta=h} z_1z_2(\eta-h)\langle\beta(\xi)\beta'(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=z_1}^{\xi=h}\int_{\eta=0}^{\eta=z_2} z_1\eta(\eta-h)\langle\beta(\xi)\beta'(\eta)\rangle d\xi d\eta + \int_{\xi=z_1}^{\xi=h}\int_{\eta=0}^{\eta=z_2} z_1\eta(\xi-h)\langle\beta'(\xi)\beta(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=z_1}^{\xi=h}\int_{\eta=0}^{\eta=z_2} z_1\eta(\xi-h)(\eta-h)\langle\beta'(\xi)\beta'(\eta)\rangle d\xi d\eta + \int_{\xi=z_1}^{\xi=h}\int_{\eta=z_2}^{\eta=h} z_1z_2\langle\beta(\xi)\beta(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=z_1}^{\xi=h}\int_{\eta=z_2}^{\eta=h} z_1z_2(\xi-h)(\eta-h)\langle\beta'(\xi)\beta'(\eta)\rangle d\xi d\eta + \\
&+ \int_{\xi=z_1}^{\xi=h}\int_{\eta=0}^{\eta=z_2} z_1\eta\langle\beta(\xi)\beta(\eta)\rangle d\xi d\eta\Bigg]. \qquad (64)
\end{aligned}
$$

Appearing in (64) average values, according to an assumed notation, are just the covariance functions of stochastic process $\beta(z)$ and its derivative $\beta'(z)$. For the covariance function defined by (13), the particular covariances are found to be:

$$\langle \beta(\xi)\beta'(\eta)\rangle = R_{\beta\beta'}(\tau) = -\frac{d}{d\tau}[R_\beta(\tau)] = \lambda^2 \tau e^{-\lambda\tau}, \tag{65a}$$

$$\langle \beta'(\xi)\beta'(\eta)\rangle = R_{\beta'}(\tau) = \lambda^2(1-\lambda\tau)e^{-\lambda\tau} \tag{65b}$$

where: $\tau = |\xi - \eta|$.
All integrals appearing in (64) can be easily determined either numerically or analytically, although the second way is time consuming. Nevertheless, it seems that the final result i.e. the covariance function should be the same as the one calculated for the first order stochastic differential equation (28).

5. Adomian's Decomposition Method

The compilation of statistical soil data shows that the coefficient of variation of the elastic modulus may be even as high as 0.5. Evidently therefore, the small fluctuation approximation does not provide a solution which is valid in the entire range of practical interest. There is no assumption concerning the size of the randomness in the Adomian's decomposition method. Adomian has proposed expressing unknown function $v(z)$ in the stochastic Volterra equation and then assuming the solution can be decomposed into components to be determined in a manner to be discussed. We shall see that the method is very convenient computationally and offers some significant advantages.

The stochastic differential equation (50b) can be changed into the stochastic Volterra integral equation with a kernel $G(z,\xi)$:

$$\begin{aligned} v(z) &= L^{-1}Lv(z) = v_0 + L^{-1}\frac{\gamma}{\overline{E}A} - L^{-1}\Re v(z) = \\ &= v_0 + \int G(z,\xi)\frac{\gamma}{\overline{E}A}d\xi - \alpha \int G(z,\xi)\Re v(\xi)d\xi \end{aligned} \tag{66}$$

where: v_0 is the solution of the homogeneous equation $Lv = 0$ and has been found to be equal to zero.
Substituting the expression defining stochastic differential operator (53) into (66) leads to:

$$v(z) = \frac{\gamma}{\overline{E}A}\int G(z,\xi)d\xi - \alpha \int G(z,\xi)[\beta(\xi)v''(\xi) + \beta'(\xi)v'(\xi)]d\xi. \tag{67}$$

The double integration of (69) by parts yields:

$$\begin{aligned} v(z) = \frac{\gamma}{\overline{E}A}\int G(z,\xi)d\xi - \alpha\Big[G(z,\xi)\beta(\xi)v'(\xi) - G'(z,\xi)\beta(\xi)v(\xi) + \\ + \int G''(z,\xi)\beta(\xi)v(\xi)d\xi + \int G'(z,\xi)\beta'(\xi)v(\xi)d\xi\Big]. \end{aligned} \tag{68}$$

The lower and upper limits of integration are 0 and h respectively, and the derivatives appearing in integrands are with respect to ξ. The first derivative of Green's function defined by (58) is as follows:

$$G'(z,\xi) = \frac{dG(z,\xi)}{d\xi} = \begin{cases} -1 & \text{for} \quad 0 \le \xi < z, \\ 0 & \text{for} \quad z < \xi \le h. \end{cases} \tag{69}$$

It is seen from the above expression that for $\xi = h$, the derivative of Green's function $G'(z,h) = 0$. According to the boundary conditions (4b), we have $v'(h) = 0$ and $v(0) = 0$, so $G(z,0) = 0$. The second derivative $G''(z,\xi)$ is, of course, equal to zero. Thus, some terms appearing in (68) vanish. Taking into account the above considerations, substituting (58) and (69) into (68), and integrating its first term yields:

$$v(z) = \frac{\gamma}{\overline{E}A}\left(\frac{1}{2}z^2 - hz\right) + \alpha \int_0^z \beta'(\xi) v(\xi) d\xi. \tag{70}$$

Now, let us assume that the unknown function $v(z)$ can be represented as the sum of undefined functions as follows:

$$v(z) = v_0(z) + \sum_{i=1}^{\infty} v_i(z,\omega) \tag{71}$$

where: ω is a realisation parameter, characterising randomness,
$v_0(z)$ is a deterministic solution (7a).

Since the functions $v_i(z,\omega)$ are still not specified, it is possible to make the following identification:

$$v_1(z,\omega) = \alpha \int_0^z \beta'(\xi) v_0(\xi) d\xi,$$

$$v_2(z,\omega) = \alpha \int_0^z \beta'(\xi) v_1(\xi,\omega) d\xi,$$

....

$$v_n(z,\omega) = \alpha \int_0^z \beta'(\xi) v_{n-1}(\xi,\omega) d\xi. \tag{72}$$

The system of integral equations (72) possesses a number of very attractive properties. Each element of this system is given in terms of lower-order terms only, so the procedure does not require a closure approximation. The derivation of equations (72) requires no assumption about the nature of the random function $E(z)$. The series representation of the solution (71) is meaningful only if it converges. Adomian (1983) as well as Zeitoun and Baker (1990) have proved that the expansion converges (in the mean square sense) if the coefficient of variation of the elastic modulus is less than 0.5.

Back substitution of the term v_i in equations (72) yields an explicit representation of these in the form of multiple integrals as follows:

$$v_1(z,\omega) = \frac{\gamma\alpha}{\overline{E}A}\int_0^z \beta'(\xi)\left(\frac{1}{2}\xi^2 - h\xi\right)d\xi,$$

$$v_2(z,\omega) = \frac{\gamma\alpha^2}{\overline{E}A}\int_0^z\int_0^\xi \beta'(\xi)\beta'(\eta)\left(\frac{1}{2}\eta^2 - h\eta\right)d\xi d\eta,$$

....

$$v_n(z,\omega) = \frac{\gamma\alpha^n}{\overline{E}A}\int_0^z\int_0^\xi\ldots\int_0^\delta \beta'(\xi)\beta'(\eta)\ldots\beta'(\delta)\left(\frac{1}{2}\delta^2 - h\delta\right)d\xi d\eta\ldots d\delta. \quad (73)$$

It is seen that high-order terms in the expansion for $v(z)$, become increasingly complex and are quite difficult to evaluate even numerically. A real test of the usefulness of the Adomian procedure is therefore whether or not low-order terms of the expansion provide significant improvement over the other methods. Thus, some error analysis must be performed. This is however not in the range of the present analysis.

The average solution

In the paper, only the first three terms of the expansion are taken into account. Formally, it is called the second order solution because the first terms is given as the deterministic solution.

Substituting (73) into (71) yields:

$$v(z,\omega) = \frac{\gamma}{\overline{E}A}\left(\frac{1}{2}z^2 - hz\right) + \frac{\gamma\alpha}{\overline{E}A}\int_0^z \beta'(\xi)\left(\frac{1}{2}\xi^2 - h\xi\right)d\xi +$$

$$+ \frac{\gamma\alpha^2}{\overline{E}A}\int_0^z\int_0^\xi \beta'(\xi)\beta'(\eta)\left(\frac{1}{2}\eta^2 - h\eta\right)d\xi d\eta. \quad (74)$$

Taking the expectation of (74), we obtain:

$$\overline{v}(z) = \langle v(z)\rangle = \frac{\gamma}{\overline{E}A}\left[\left(\frac{1}{2}z^2 - hz\right) + \alpha^2\int_{\eta=0}^{\eta=z}\int_{\xi=0}^{\xi=\eta}\langle\beta'(\xi)\beta'(\eta)\rangle \times\right.$$

$$\times \left(\frac{1}{2}\xi^2 - h\xi\right) d\xi d\eta \Bigg]. \tag{75}$$

The expectation appearing in the integrand of (75) is just the correlation function described by (65b). After substituting (65b) into (75) and integrating the expression for the second order average solution finally assuming the form:

$$\bar{v}(z) = \frac{\gamma}{\overline{E}A}\left(\frac{1}{2}z^2 - hz\right) + \frac{\gamma\alpha^2}{\overline{E}A}K(z) \tag{76}$$

where:

$$K(z) = \frac{1}{2}z^2 + \left(\frac{2}{\lambda} - h\right)z + \frac{4}{\lambda^2} - \frac{2h}{\lambda} + \left[\left(\frac{2}{\lambda} - h\right)z + \frac{2h}{\lambda} - \frac{h}{\lambda^2}\right]e^{-\lambda z}. \tag{77}$$

The second order average solution for $\alpha = 0.1$ and different values of the correlation distance λ is presented in Fig. 5. Also shown are solutions obtained by the first (deterministic) and second order perturbation method.

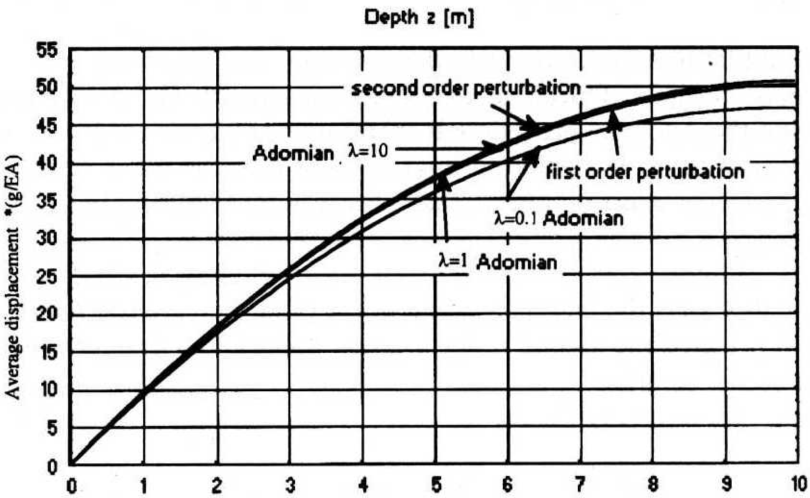

Fig. 5. The average displacment vs depth

The average displacements, calculated by Adomian decomposition procedure, both for a small ($\lambda = 10$) as well as high correlation ($\lambda = 0.1$) converage to the perturbation solution. Some discrepancies can be seen for $\lambda = 1$.

Variance of the solution

The variance function of the displacement is defined as the expectation of the squared difference between displacement and its average value. Taking into account only the first two terms of expression (74) and the first one of (75), the first order variance function can be obtained. It can be presented in the following form:

$$Var[v(z)] = \left(\frac{\gamma\alpha}{\overline{E}A}\right)^2 \int\limits_{\eta=0}^{\eta=z}\int\limits_{\xi=0}^{\xi=\eta} \langle\beta'(\xi)\beta'(\eta)\rangle \left(\frac{1}{2}\xi^2 - h\xi\right)\left(\frac{1}{2}\eta^2 - h\eta\right) d\xi d\eta =$$

$$= 2\left(\frac{\gamma\alpha}{\overline{E}A}\right)^2 H(z) \tag{78}$$

where:

$$H(z) = \frac{1}{8}z^4 - \left(\frac{1}{3\lambda} + \frac{h}{2}\right)z^3 + \left(\frac{h}{\lambda} + \frac{h^2}{2}\right)z^2 + \left(\frac{5}{\lambda^4} + \frac{h}{\lambda^3} - \frac{5h^2}{2\lambda^2}\right) +$$
$$+ \left[-\left(\frac{1}{2\lambda} + \frac{h}{2}\right)z^3 + \left(\frac{-5}{2\lambda^2} - \frac{h}{2\lambda} + h^2\right)z^2 + \left(-\frac{5}{\lambda^3} - \frac{h}{\lambda^2} + \frac{5h^2}{2\lambda}\right)z +\right.$$
$$\left. + \left(-\frac{5}{\lambda^4} - \frac{h}{\lambda^3} + \frac{5h^2}{2\lambda^2}\right)\right] e^{-\lambda z}\Bigg\}. \tag{79}$$

The first order variance, obtained by Adomian decomposition procedure, turned out to be too high (two to four times higher) as compared with the one obtained by perturbation method. Thus second order variance must be considered. Taking into account (74) and (75), the variance function, after some transformation, can be presented in the following form:

$$Var[v(z)] = \left(\frac{\gamma\alpha}{\overline{E}A}\right)^2 \Bigg[K(z) - \alpha^2 H(z) + \alpha^2 \int\limits_{\xi=0}^{\xi=z}\int\limits_{\eta=0}^{\eta=\xi}\int\limits_{\delta=0}^{\delta=z}\int\limits_{\varepsilon=0}^{\varepsilon=\delta} \times$$
$$\times \langle\beta'(\xi)\beta'(\eta)\beta'(\delta)\beta'(\varepsilon)\rangle \left(\frac{1}{2}\eta^2 - h\eta\right)\left(\frac{1}{2}\varepsilon^2 - h\varepsilon\right) d\xi d\eta d\delta d\varepsilon\Bigg] \tag{80}$$

In the derivation of the above expression, it was assumed that the probability density function of the elastic modulus is normal. In such a case, an expectation appearing in the integrand of the expression (80), for the Gaussian case, can be given in the form:

$$\langle\beta'(\xi)\beta'(\eta)\beta'(\delta)\beta'(\varepsilon)\rangle = \langle\beta'(\xi)\beta'(\eta)\rangle\langle\beta'(\delta)\beta'(\varepsilon)\rangle +$$
$$+\langle\beta'(\xi)\beta'(\delta)\rangle\langle\beta'(\eta)\beta'(\varepsilon)\rangle + \langle\beta'(\xi)\beta'(\varepsilon)\rangle\langle\beta'(\eta)\beta'(\delta)\rangle. \tag{81}$$

Taking into account (81), where the general expression for covariance is given by (65b), the four-fold integral in (80), can be determined even in an analytical way. Unfortunately, the framework for the evaluation of the variance of the solution was derived without the presentation of numerical results. It is, however, worth noting, that the variance of displacement caluclated by (80), neglecting the term involving four-folded integral, was found to be at most twice the one obtained by perturbation procedure. Thus, the standard deviation of the displacement is not higher than 50%.

6. Conclusions

The random elasticity theory is applied to the statistically homogeneous soil layer subjected to its own weight. In this part, only the modulus of elasticity is assumed to be a stochastic process and the uni-axial strain state is considered. So, the problem is one-dimensional and is described by a stochastic ordinary differential equation. The governing elasticity equations are presented in alternative ways, as a first order stochastic differential equation either with a random coefficient or random forcing function and as a second order stochastic differential equation.

The solution is obtained by approximated analytical methods i.e. perturbation procedure and Adomian's decomposition method. Under the assumption of small fluctuation (coefficient of variation of elastic modulus less than 0.15), first order perturbation method is used to determine the variance function of the solution and second order to find the average displacement.

There is no small fluctuation assumption in Adomian's decomposition method and a solution is valid for the coefficient of variation up to 0.5. In the framework of this method the analytical expression defining the displacement as a stochastic process is presented. The second order (first three components of the decomposition are taken into account) the average value is found. Also second order solution for the variance of the displacement is determined, although the explicit expression is found only for the first order solution.

For a given boundary problem and considered values of parameters characterising uncertainties, the average displacements obtained by both methods as similar and converge to the deterministic solution. In the case of variance function, the first order Adomian's decomposition procedure significantly overestimates the results and the variance of displacement is up to four times higher than the one obtained by perturbation procedure.

The paper may be treated as the introduction to the application of approximated analytical methods into the geotechnical analysis, where the subsoil and acting loadings are, in general, random. In particular, it aims for a fast and rough estimation of soil behaviour in different conditions and one or two-dimensional schematization of reality. The Adomian's decomposition approach presented in the paper seems to be a powerful and convenient tool for such analysis.

References

See: Part 2.