Archives of Hydro-Engineering and Environmental Mechanics
Vol. 41 (1994), No. 1–2, pp. 111–131

Elasto-Plastic Interpretation of Oedometric Test

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(Received February 17, 1994; revised May 20, 1994)

Abstract

The paper deals with a new interpretation of oedometric tests that is based on the elasto-plastic approach including such phenomena as compaction and dilation. The analysis starts with a presentation of empirical results obtained from oedometric tests performed on sands with additional measurement of lateral stresses. Three types of sand response were identified during a single cycle of loading and subsequent unloading. During the loading, both reversible and irreversible strains develop in the soil sample. During the unloading one can distinguish two phases. The first one is characterized by a purely elastic response. A characteristic feature of this phase is that the sand deformation takes place in the vertical direction only. The horizontal elastic strain remains constant during the first phase of unloading, and is opposite to the permanent horizontal strain that has developed during the loading (the total lateral strain is zero in oedometric conditions). It is shown that the reversible response of the sand sample obeys the Hooke's linear law for practically important stress levels. The second stage of unloading is characterized by elastic and irreversible (dilation) responses. During that phase the stress deviator remains constant and only the mean pressure is reduced. The second stage of unloading begins when the Failure in Extension Line is attained in the stress space. Elastic constants are determined from the analysis of the first phase of unloading, which is different from the methods of determination of elastic moduli commonly accepted in soil mechanics. A respective discussion on this problem is also presented. The assumption concerning a linear reversible response allows for the extraction of irreversible strains from experimental data. Subsequently the law of compaction is proposed in order to describe irreversible strains developed during the loading. The phenomena of dilation during the second phase of unloading and the "compaction induced lateral stresses" are also discussed in the light of the approach proposed in this paper. Some useful practical formulae enabling a simple analysis of oedometric tests are derived. The general conclusion which follows from the considerations presented in this paper is that the classical oedometric test, with additional measurement of lateral stresses, is a powerful experimental method in soil mechanics investigations.

1. Introduction

This paper is about a new interpretation of oedometric tests that is based on the elasto-plastic approach, including such phenomena as compaction and dilation. It is the Author's impression that a classical interpretation of oedometric tests, presented in numerous soil mechanics textbooks (cf. Chapters 8–12 of Lambe and Whitman; 1969), obscures physical phenomena taking place in the soil during such a test rather than allowing for better understanding of these processes. Consequently, some of the soil parameters, determined from oedometric tests, have a false meaning and their values determined on the basis of classical methods are misleading. For example, the Young modulus and Poisson's ratio define properties of a linear elastic material. These constants have a precise meaning and should be used strictly according to the theory of elasticity. Their values should also be determined precisely if the methods of linear elasticity are to be used in order to describe reversible response of soils. Unfortunately, some strict requirements imposed by the elasticity theory are sometimes violated in geotechnical literature. Some basic shortcomings regarding the classical interpretation of oedometric tests, as well as regarding the determination of soil parameters will be discussed later.

It is realised that the present paper touches on the delicate matter of "well established knowledge". It is not the aim of this paper to provide an extensive criticism of particular publications but, on the other hand, some already existing works should be quoted if the analysis presented in here is to be sufficiently convincing. In order to solve this dilemma the Author has decided to give a single example of a soil mechanics' textbook in which, in his opinion, some of the interpretations of oedometric tests and elastic constants have certain shortcomings. It is the well known and fundamental book of Professors T. W. Lambe and R. V. Whitman (1969; Polish translation by "Arkady", Warsaw 1977). We shall refer to this textbook using the abbreviation L-W. The reason for the choice is that the Authors of this fundamental (and very good indeed!) book are worldwide experts in geotechnical engineering, so some examples presented here will not do any harm to their well established reputations.

The aim of this paper is to present a simple, but rigorous (from the view point of mechanics), analysis of oedometric tests performed on sand, with additional measurement of lateral stresses. In such tests the vertical stress σ_z is controlled, and corresponding lateral stress σ_x and vertical deformation (vertical total strain ε_z) are monitored. The total horizontal strain $\varepsilon_x = 0$ because of the rigid cylinder in which the soil sample is placed, prevents lateral deformation.

Careful examination of extensive experimental data has enabled identification of three phases, during a single cycle of loading and unloading, during which the behaviour of the sand displays different features. During the virgin loading, both reversible and irreversible (compaction) strains develop in the soil. This is a well-known and accepted conclusion. The first stage of unloading is characterized by a purely reversible response. It has been revealed that during this phase the sand behaves as a linear elastic material in the range of practically important stress levels, which is an important conclusion. This observation simplifies the analysis of experimental data essentially and allows for precise determination of elastic constants, i.e. the Young modulus E and Poisson's ratio ν. The values of elastic constants determined on the basis of the new interpretation differ from those suggested in the literature.

The second stage of unloading is characterized by the increase of volume caused by both elastic and plastic (dilation) strains. During that phase the stress deviator remains constant, and only the mean confining pressure is reduced. The phase begins when the stress path attains the Failure in Extension Line (FEL) in the stress space. A gradient of FEL depends on the initial relative density of the sand.

A physically sound, and experimentally supported, assumption about a linear reversible response enables the extraction of irreversible strains from experimental data. Such extracted data enable, in turn, the determination of the compaction law during loading and the analysis of dilation during the second stage of unloading.

It should be mentioned that the theoretical description of phenomena taking place in the soil in oedometric conditions is based on fairly well established concepts of continuum mechanics, such as elasticity and plasticity. No "new models" have been proposed, but rather a "new look" at experimental data. Subsequently, the traditional theoretical tools have been applied in order to describe the behaviour observed. Such an approach has allowed for obtaining simple mathematical formulae describing the behaviour of the sand in oedometric conditions. A discussion of such phenomena as "compaction induced lateral stresses", "hysteresis effect", "failure in extension" has also been possible within the framework of the approach proposed in this paper.

The results reported here indicate that the oedometric test, with additional measurement of lateral stresses, is a powerful method of investigating the behaviour of granular materials. It seems that such possibilities have not been extensively utilized. It is then hoped that the present paper will open the door to some new investigations and interpretations of classical experiments.

2. Linear Representation of Experimental Data

The experimental results presented in this paper were obtained from tests performed in a standard Bishop-type oedometer. In order to enable the measurement of lateral stresses a special cylinder was constructed. Part of the cylinder, half-way up its height, was made thinner on its exterior side and a strain gauge was fixed onto it. A device was calibrated, using a specially constructed chamber, against the water pressure within a range of stresses expected. Various dry sands and some other granular materials (grains) were used as experimental material. In this paper, the experimental results corresponding to the Leighton Buzzard Sand will be shown as typical data. Other sands display similar qualitative behaviour.

Sand samples were initially pre-loaded with the vertical stress of $\sigma_z^0 = 0.487 \times 10^5$ N/m^2 and then subjected to various histories of loading and unloading, measured from the initial (reference) stress state. During the experiments the vertical stress σ_z was controlled, and the horizontal stress σ_x and the vertical deformation (the total vertical strain ε_z) monitored.

A synthetic picture of oedometric test, performed on a medium dense Leighton Buzzard Sand ($e_0 = 0.6$), is presented in Figure 1. During the virgin loading one

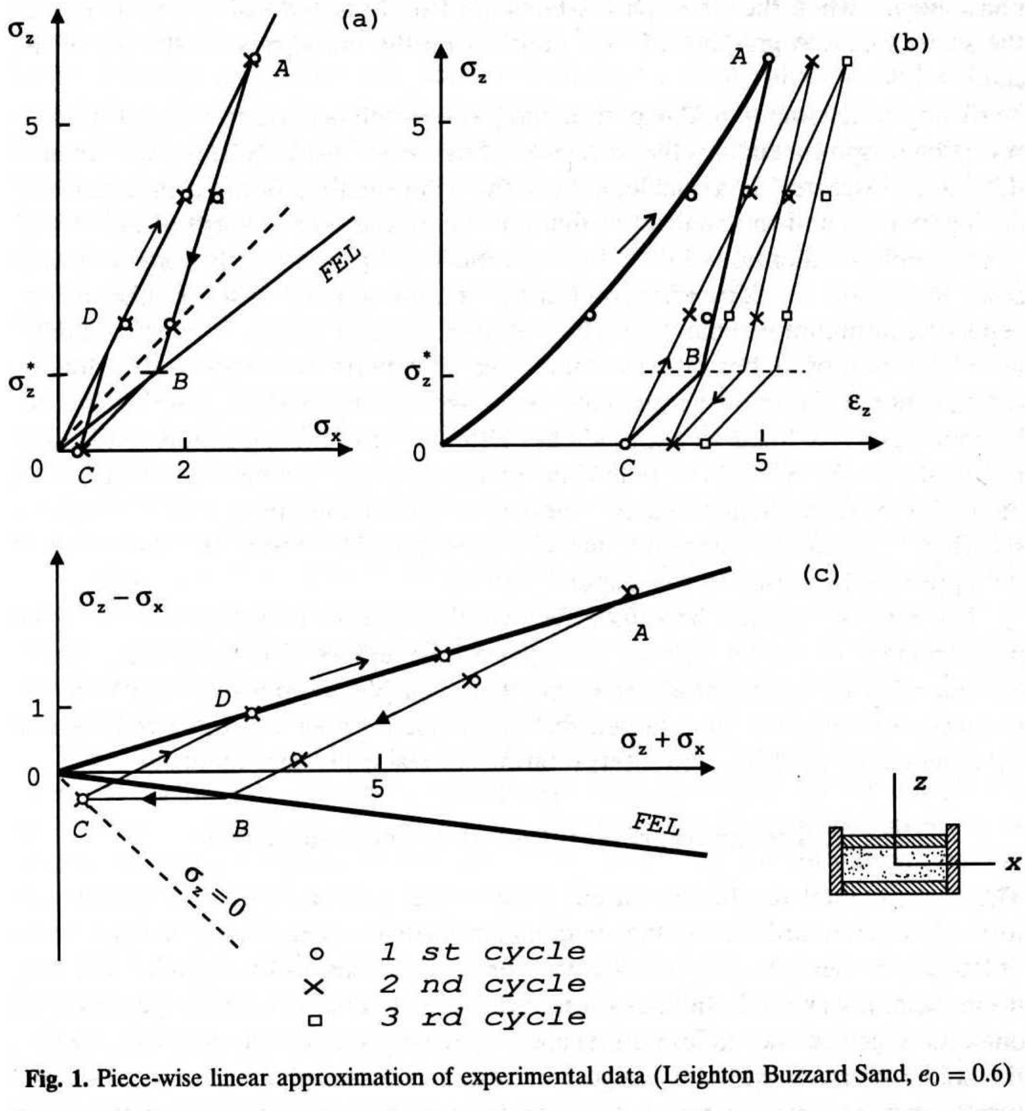


Fig. 1. Piece-wise linear approximation of experimental data (Leighton Buzzard Sand, $e_0 = 0.6$)

follows the straight line 0A in the stress space (Fig. 1a). This is the well known K_0-line from $\sigma_x = K_0\sigma_z$. During the unloading one follows a different path in the stress space, denoted as ABC in Fig. 1a. After unloading, i.e. when the vertical stress is removed, there is a residual lateral stress in the soil (point C in Fig. 1a). It has been discovered that the value of residual lateral stress depends on the initial relative density of the sand. The magnitude of residual lateral stress increases with increasing initial density.

Note that the unloading stress path has been approximated by two linear sectors, denoted as AB and BC in Fig. 1. Such a bi-linear approximation of experimental data, that is really good, plays a key role in the analysis presented in this paper. In the classical literature on soil mechanics (cf. p. 10.4 of L-W; Wroth 1972; Hendron 1963) the unloading paths are approximated by curves in the stress space. Such an "automatic" approximation obscures any physical interpretation of experimental results because it suggests, from the very beginning, a non-linear reversible response of sand in oedometric conditions. A consequence of such a "vision" of experimental data is the development of various non-linear models, introduction of various types of elastic moduli (tangent, secant, ...), etc. For example, it can easily be checked that Hendron's data presented in Fig. 10.12 of L-W can be as nicely approximated by linear sectors as those in Fig. 1. A rather surprising result of Hendron's (1963) experimental work is that he had not observed the residual lateral stresses after the unloading, although the soil samples were subjected to very great stresses.

The stress path shown in Fig. 1a can also be presented in the $\sigma_z + \sigma_x, \sigma_z - \sigma_x$ plane as shown in Fig. 1c. In this paper the variable $\sigma_z + \sigma_x$, instead of the mean pressure $p = (\sigma_z + 2\sigma_x)/3$, will be used for the sake of convenience. An alternative presentation of experimental data reveals a physical meaning of the sector BC in the unloading path. The stress deviator $\sigma_z - \sigma_x$ remains constant during that phase of unloading, which means that the stress increments are equal, i.e. $d\sigma_z = d\sigma_x$. This corresponds to the hydrostatic unloading. The sector BC is horizontal in Fig. 1c, and parallel to the $\sigma_x = \sigma_z$ line in Fig. 1a. Knowing the slopes of sectors AB and BC one can easily determine co-ordinates of point B.

Let us now consider the other set of experimental data in which the vertical total strain ε_z is plotted against the vertical stress σ_z as shown in Fig. 1c. During the loading one follows a curvilinear path AB corresponding to the K_0-path in the stress space (sector 0A in Fig. 1a). During the first stage of unloading one follows the linear sector AB, where point B corresponds to σ_z^* which, in turn, can be determined from Figs. 1a or 1c. Point C in Fig. 1b corresponds to the residual vertical strain after the first cycle of loading and unloading.

Note that the unloading path in the σ_z, ε_z space can be approximated nicely by two linear sectors AB and BC. Classical representation of unloading paths is usually non-linear, cf. L-W, Taylor (1967). Perhaps, in some cases, such non-linear approximations are substantiated by experimental evidence, but for practically important stress levels a bi-linear approximation is sufficiently good.

Consider now the subsequent re-loading starting from point C in Fig. 1. At that stage one follows again a linear path in the stress space (sectors CD in Figs. 1a and 1c) that is parallel to the unloading sector AB. During the second stage of re-loading the K_0-line is followed again (sector DA).

During the subsequent unloading one follows the ABC path in the stress space again. It should be mentioned that during the experiments performed on dense sands the position of the point C changes after each loading—unloading cycle. The point C moves along the line $\sigma_z = 0$ depending upon the initial relative density. It may be related to the so-called "compaction induced lateral stresses" (a designation used after Duncan and Seed; 1986). In such cases the CDAB loop also changes its position within each cycle. It is not such an obvious phenomenon in case of loose and medium dense sands. This problem will be discussed later.

An interesting and important experimental observation is that, during the first stage of unloading, within each cycle, one follows a linear path in the σ_z, ε_z plane that is parallel to the sector AB, and then another linear sector, parallel to the path BC, that corresponds to the dilation under a constant stress deviator.

3. Basic Theoretical Concepts

The presentation of experimental data in the manner described in the previous section allows for the determination of three phases during which the soil's behaviour displays different features, see Fig. 2.

(a) **Virgin compaction** takes place during the first loading. In this case one follows the K_0-line in the stress space (or the virgin consolidation line 0A, as marked in Fig. 2), i.e.

$$\sigma_x = K_0 \sigma_z. \tag{1}$$

This result is well known and widely accepted in soil mechanics. During the first loading both the reversible and irreversible strains develop in the soil:

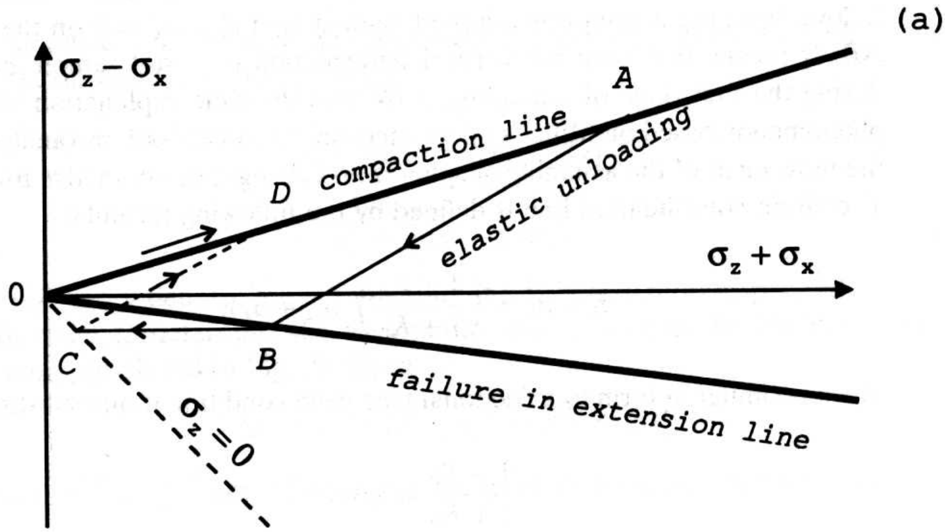
$$\varepsilon_z = \varepsilon_z^e + \varepsilon_z^p, \tag{2}$$

$$\varepsilon_x = \varepsilon_x^e + \varepsilon_x^p, \tag{3}$$

where ε_z and ε_x denote the vertical and horizontal strains respectively. The superscripts "e" and "p" stand for the elastic (reversible) and plastic (irreversible) parts of the total strain. The total horizontal deformation is prevented in oedometric conditions, which is equivalent to the condition $\varepsilon_x = 0$, so Eq. (3) takes the form:

$$\varepsilon_x^e = -\varepsilon_x^p. \tag{4}$$

(b) **Elastic unloading** takes place on the path AB. Most of the experimental data suggest that the path AB can be approximated by a linear sector in the σ_z, ε_z plane. In the case of cyclic oedometric tests the unloading paths are parallel to the initial sector AB, which observation supports the conclusion about the elastic unloading. A similar conclusion can be drawn in the case of the slightly non-linear character of unloading curves. The observation as to the purely elastic response during the first stage of unloading is accepted,



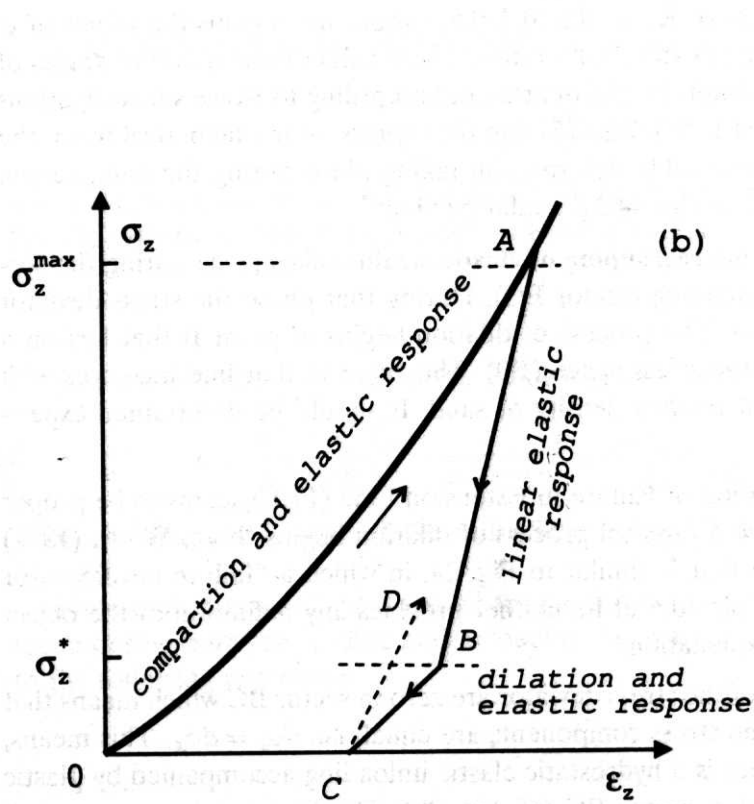


Fig. 2. Illustration of basic concepts

in this paper, as one of the basic assumptions. An important conclusion follows from the assumption adopted, namely that $\varepsilon_x^e = \varepsilon_x^p = 0$ on the path AB. It means that only the vertical deformation of a soil sample occurs during the first stage of unloading. L-W provide some explanation of that phenomenon based on Miller's micro-mechanical considerations (analysis of the movement of the assembly of spheres and changes of interactive forces). The virgin consolidation line is defined by the following formula:

$$\sigma_z - \sigma_x = \left(\frac{1 - K_0}{1 + K_0}\right) (\sigma_z + \sigma_x), \tag{5}$$

which is similar in form to a frictional type yield condition if one substitutes:

$$\frac{1 - K_0}{1 + K_0} = \sin \psi. \tag{6}$$

For typical values of K_0, in the $\langle 0.4, 0.5 \rangle$ range, one obtains the values of ψ belonging to the $\langle 19.45°, 25.4° \rangle$ range. These values correspond to angles of intergranular friction for the quartz sand according to Rowe's investigations (cf. Figs. 6 – 9 of L-W). Eqs. (5) and (6) express, in mathematical form, the fact that the irreversible deformation taking place during the compaction, depends mainly on the intergranular friction.

(c) **Dilation** and some **regrouping of elastic strains** takes place during the second stage of unloading (sector BC). During that phase the stress deviator remains constant. The process of dilation begins at point B that lies on a straight line in the stress space (B0). The slope of that line increases with increasing initial relative density of sand. It should be determined experimentally.

The designation of Failure in Extension Line (FEL) seems to be proper in this case since a physical process of dilation begins there. Wroth (1972) shows a picture that is similar to Fig. 2a, in which a "failure envelope for extension" is depicted, but he neither provides any definition of the object nor gives any explanation.

Increments of the stress deviator are zero in sector BC which means that increments of the stress components are equal, i.e. $d\sigma_z = d\sigma_x$. This means, in turn, that there is a hydrostatic elastic unloading accompanied by plastic strains (dilation) in sector BC (cf. Fig. 2b). The horizontal component of dilation is defined by Eq. (4). The vertical component can be extracted from experimental data if the elastic constants of sand are known.

4. Determination of Elastic Constants

It is assumed that the reversible response of sand is governed by Hooke's law. The elastic strains are the following:

$$\varepsilon_x^e = \frac{1}{E}[(1-\nu)\sigma_x - \nu\sigma_z], \tag{7}$$

$$\varepsilon_z^e = \frac{1}{E}[\sigma_z - 2\nu\sigma_x], \tag{8}$$

where E and ν denote Young's modulus and Poisson's ratio respectively.

In order to determine the elastic constants let us consider the first stage of unloading (path AB in Fig. 2). There is:

$$\Delta\varepsilon_z = \varepsilon_z^A - \varepsilon_z^B = \varepsilon_z^{e,A} - \varepsilon_z^{e,B} = \Delta\varepsilon_z^e, \tag{9}$$

because $\varepsilon_z^{p,A} = \varepsilon_z^{p,B}$ according to the assumptions accepted. There is also:

$$\Delta\varepsilon_x = \varepsilon_x^A - \varepsilon_x^B = 0. \tag{10}$$

It follows from Eqs. (7) and (10) that

$$\frac{1-\nu}{\nu} = \frac{\sigma_z^A - \sigma_z^B}{\sigma_x^A - \sigma_x^B} = a, \tag{11}$$

where a denotes a slope of the unloading path AB in the stress space (cf. Fig. 1a). Eq. (11) allows for the determination of Poisson's ratio:

$$\nu = \frac{1}{1+a}. \tag{12}$$

Eqs. (8), (9) and (12) lead to the following formula for the Young modulus:

$$E = \frac{\sigma_z^A - \sigma_z^B}{\varepsilon_z^A - \varepsilon_z^B}\left[1 - \frac{2}{a(1+a)}\right] = E^*\left[1 - \frac{2}{a(1+a)}\right], \tag{13}$$

where E^* denotes a slope of the unloading sector AB in the σ_z, ε_z space (cf. Fig. 2b).

Note that in a particular case, when one follows the same path in the stress space during both the loading and unloading, i.e. when $a = 1/K_0$, Eq. (12) leads to the following expression

$$\nu = \nu^* = \frac{K_0}{1+K_0} \tag{14}$$

which is sometimes used in soil mechanics in order to determine Poisson's ratio.

Some example data for the Leighton Buzzard Sand are shown in Table 1. The calculated values of elastic constants do not differ very much in the case of loose

Table 1. Leighton Buzzard Sand parameters

	Test 2 $e_0 = 0.68$	Test 13 $e_0 = 0.6$	Test 12 $e_0 = 0.47$
K_0	0.4074	0.5375	0.447
a	3.1075	2.838	3.665
ν (Eq. 12)	0.2435	0.2606	0.2144
ν^* (Eq. 14)	0.29	0.35	0.309
E (Eq. 13), unit 10^8 N/m^2	3.213	3.462	3.616

(Test 2) and dense (Test 12) sands. Average values of these constants are the following: $E = 3.43$ ($\times 10^8$ N/m^2), $\nu = 0.24$. It is clear that Eq. (14) overestimates the value of Poisson's ratio, even by 44% in the case of dense sand. Moreover, Eq. (14) has no sound physical meaning since during the unloading one follows a path in the stress space that is different from the K_0-line.

In the light of the interpretation presented in this paper it is also hard to agree with some comments on Poisson's ratio published in L-W. For example, they admit values of Poisson's ratio greater than 1/2, which contradicts the restriction imposed by the elasticity theory on that coefficient ($0 \leq \nu \leq 1/2$). They also suggest to use different values of Poisson's ratio for different stages of loading, which does not seem necessary. The main source of such misunderstandings is the definition of Poisson's ratio accepted in soil mechanics. According to L-W (p. 12.3.3) it is a ratio of lateral and vertical strains which develop in the sand sample during a tri-axial compression. During such an experiment both the reversible and irreversible strains develop, hence the ratio of lateral and vertical strains may indeed exceed the value of 1/2. But the value of the coefficient so determined is certainly not the value of Poisson's ratio used in the elasticity theory. Such numbers cannot be used in practical applications in which the methods of the theory of elasticity are applied because eventual results will be incorrect.

Another common misunderstanding has its source in the interpretation of the loading curve 0A in the σ_z, ε_z space. The non-linear character of the loading curve is caused mainly by irreversible strains. The slope of such a curve increases with increasing stress level. A commonly accepted interpretation of this obvious observation is that the elastic modulus is a function of confining pressure (cf. Eq. 12–15 of L-W). It seems, in the light of the analysis presented in this paper, that such an interpretation should be used rather carefully, and perhaps for higher stress levels, since the reversible response of sands can be well approximated by the Hooke's law, at least for practically important stress levels!

The other consequence which follows from the classical interpretation of experimental data is that the Young modulus depends greatly on the initial relative density. L-W (Table 12.4) quote Chen's results which indicate that the Young modulus for dense sand is almost twice as high as the modulus for the same, loose, material. The ratio of the Young moduli for dense and loose sands is not so high, cf. Table 1.

5. Extraction of Irreversible Strains from Experimental Data

Knowing the elastic constants one can easily determine the elastic strains that develop in the sand during loading and unloading. The plastic strains can then be determined from Eqs. (2) and (4) because the total vertical strain is known. For the sake of convenience the following **units** will be used throughout this paper: **stress unit** 10^5 N/m^2, **strain unit** 10^{-3} and **modulus unit** 10^8 N/m^2.

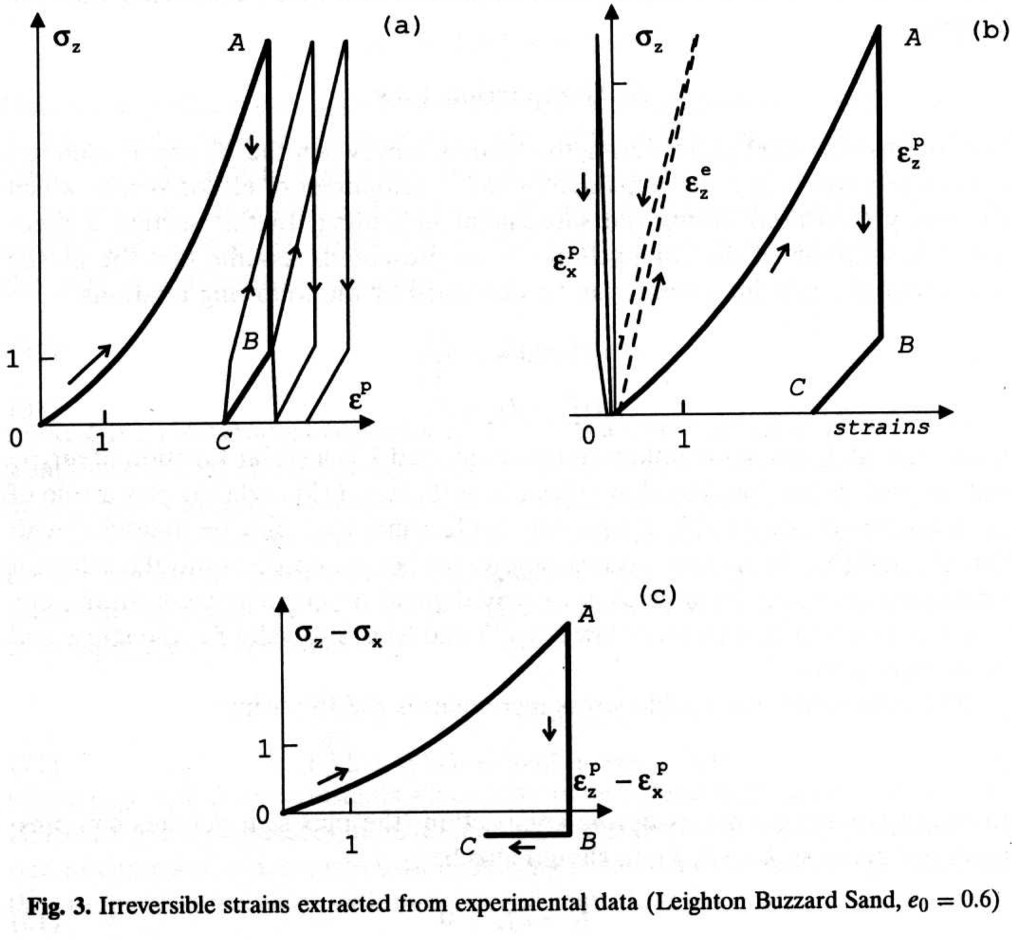


Fig. 3. Irreversible strains extracted from experimental data (Leighton Buzzard Sand, $e_0 = 0.6$)

Fig. 3 illustrates the plastic strains which developed in the Leighton Buzzard Sand (Test 13, $e_0 = 0.6$) during the first cycle of loading and unloading. Plastic volumetric strains developed during the first three cycles are additionally shown in Fig. 3a. The maximum elastic strains, corresponding to $\sigma_z^{max} = 5.842$, are the following: $\varepsilon_z^e = 1.215$, $\varepsilon_x^e = 0.231$. The respective plastic strains are: $\varepsilon_z^p = 3.99$, $\varepsilon_x^p = -0.231$.

During the first stage of unloading (AB) only the vertical elastic strain changes and the other strain components remain constant, as shown by vertical sectors AB in Fig. 3. The second stage of unloading is characterized by the dilation and regrouping of elastic strains. After the unloading (point C) the following residual strains remain in the sand: $\varepsilon_z^e = -0.05$, $\varepsilon_x^e = 0.075$, $\varepsilon_z^p = 2.97$, $\varepsilon_x^p = -0.075$.

The above example shows that the residual elastic strains can be neglected in comparison with the residual plastic vertical strain as an engineering approximation. An important conclusion which follows from the analysis already presented is that we are able to extract both types of irreversible strains from experimental data, i.e. those corresponding to the compaction and those connected with the dilation.

6. Compaction Law

The compaction takes place during the loading only, when the K_0-line is followed in the stress space. It is accompanied by the development of elastic strains which are nearly recovered during the subsequent unloading. In this section a theoretical description of the compaction will be presented. Assume that the plastic (compaction) strain increments can be described by the following relations:

$$d\varepsilon_x^p = d\lambda \times f_x, \tag{15}$$

$$d\varepsilon_z^p = d\lambda \times f_z, \tag{16}$$

where f_x and f_z are some unknown functions, and λ is a scalar function of a type that appears in the plasticity flow rules ($d\lambda \geq 0$). Eqs. (15) and (16) play a role of the "compaction flow rule". Respective "yield condition" may be identified with Eqs. (5) and (6). At present, no restrictions are being imposed onto the shape of functions $f_i, i = x, z$. In general, they may depend on current stress, strain, etc. It is also assumed that Hooke's law (Eqs. 7 and 8) is also valid for the stress and strain increments.

The volumetric irreversible stress increment is the following:

$$d\varepsilon^p = d\varepsilon_z^p + 2d\varepsilon_x^p = d\lambda(f_z + 2f_x). \tag{17}$$

Since the volume decreases during compaction (the plus sign denotes compression) and because $d\lambda \geq 0$, there should also be

$$f_z + 2f_x \geq 0. \tag{18}$$

The are three functions, namely $d\lambda$, f_x, f_z, which should be determined in order to describe irreversible strains. Eqs. (4), (7) and (15) allow for the elimination of one of these functions, namely:

$$d\lambda = \frac{1}{Ef_x}[\nu - (1-\nu)K_0]d\sigma_z. \tag{19}$$

Substitution of Eq. (19) into Eq. (17) gives:

$$d\varepsilon^p = \frac{1}{E}[\nu - (1-\nu)K_0](2+f)d\sigma_z, \tag{20}$$

or

$$\frac{d\varepsilon^p}{d\sigma_z} = \frac{1}{E}[\nu - (1-\nu)K_0](2+f), \tag{20'}$$

where $f = f_z/f_x$ is an unknown function which should be determined from experimental data. Having determined that function one can easily determine the vertical plastic strain increments, since

$$d\varepsilon_z^p = f d\varepsilon_x^p = -f d\varepsilon_x^e. \tag{21}$$

Hence the problem has been reduced to the determination of a single function f. This can be done in different ways. One of them is a purely empirical "curve fitting" exercise which, however, is not of interest to us.

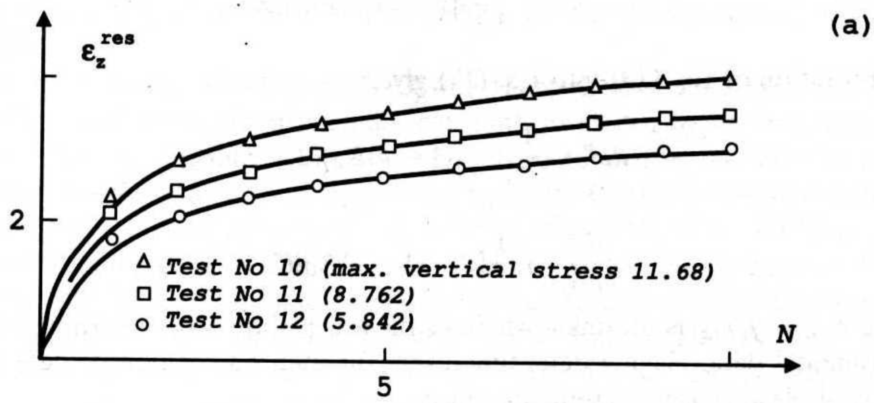
In order to determine a function f we shall adopt some concepts proposed by Morland and Sawicki (1983, 1985). They have proposed the compaction law in the form of a differential equation:

$$\frac{d\varepsilon^p}{d\xi} = F(J, \varepsilon^p), \tag{22}$$

where ξ is an increasing loading parameter, F is a constitutive function, J denotes some invariant of either the strain or the stress tensor. An extensive discussion of Eq. (22) is presented elsewhere (cf. Morland and Sawicki 1983, 1985, see also Morland 1993). The incremental form of the constitutive equation (22) will allow for the analysis of irreversible strains which develop during complex loading histories. In this paper we shall adopt a simple form of Eq. (22) that is consistent with experimental data, namely:

$$\frac{d\varepsilon^p}{d\xi} = C_1\sqrt{\sigma_z - \sigma_x}\exp(-C_2\varepsilon^p), \tag{23}$$

where $C_i, i = 1, 2$, are constants which have to be determined experimentally. The RHS of Eq. (23) expresses, in mathematical form, empirical observations that the rate of compaction decreases with increasing permanent volume change, and that the compaction increases with increasing deviatoric stress level.



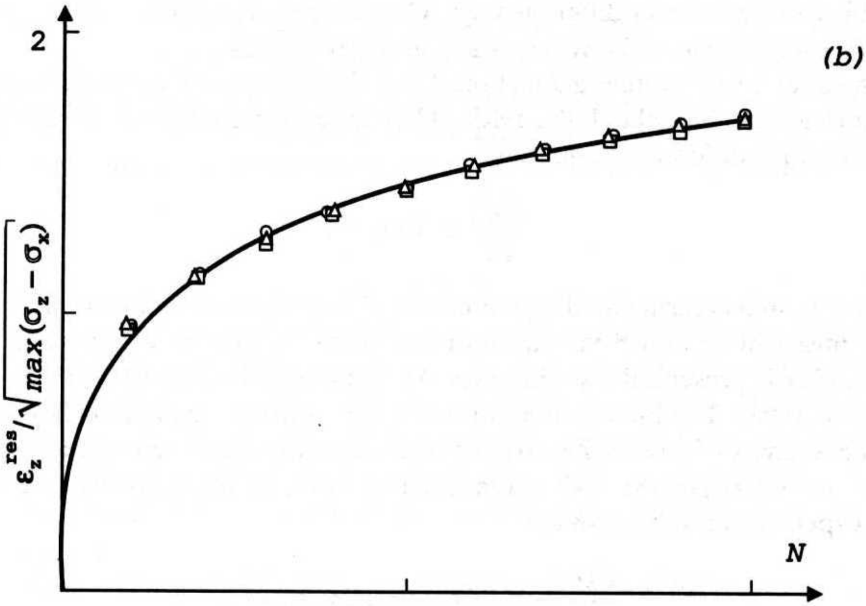


Fig. 4. Residual vertical strains after unloading against number of cycles N (Leighton Buzzard Sand, $e_0 = 0.47$).
(a) Compaction curves for the tests performed at various vertical stress amplitudes.
(b) Common compaction curve after re-normalization

The experimental results shown in Fig. 4 support the latter conclusion. Fig. 4a presents the residual vertical strains as functions of the number of loading cycles N. Each curve corresponds to a single experiment performed on a dense Leighton Buzzard Sand ($e_0 = 0.47$), but at different maximum vertical stress levels σ_z^{max}. Fig. 4 shows the same data, but on the vertical axis, instead of the residual vertical strain, there is a ratio of that quantity and the square root from a maximum stress deviator within each cycle. After such a re-normalization a common curve is obtained for all three experiments. The above observation suggests that $\sqrt{\sigma_z - \sigma_x}$ on the RHS of Eq. (23) is the proper choice. Note that $\sqrt{\sigma_z - \sigma_x} \propto J^{1/4}$, where J is the second invariant of the stress deviator. We shall also assume that:

$$d\xi = d(\sigma_z - \sigma_x). \tag{24}$$

In the case of virgin compaction Eq. (23) takes the following form:

$$\frac{d\varepsilon^p}{d\sigma_z} = C_3\sqrt{\sigma_z}\exp(-C_2\varepsilon^p), \tag{25}$$

where $C_3 = C_1(1 - K_0)^{3/2}$.

Comparison of the RH sides of Eqs. (20') and (25) allows for the determination of the unknown function

$$f = \frac{EC_3\sqrt{\sigma_z}\exp(-C_2\varepsilon^p)}{[\nu - (1 - \nu)K_0]} - 2. \tag{26}$$

Eqs. (21) and (26) define the compaction strains.

7. Determination of Compaction Coefficients

Integration of Eq. (25) gives the following formula for the compaction volumetric strain:

$$\varepsilon^p = D_1 \ln(1 + D_2\sigma_z^{3/2}), \tag{27}$$

where $D_1 = 1/C_2$, $D_2 = \frac{2}{3}C_2C_3$. These coefficients can be determined from experimental data presented in the ε^p, σ_z plane.

For example, for a medium dense Leighton Buzzard Sand ($e_0 = 0.6$) the respective coefficients are the following: $D_1 = 1.05$, $D_2 = 1.37$. One must remember that the stress and strain units have been introduced in this paper. A conformity of theory and experiment is displayed in Fig. 5. Solid curves correspond to the theoretical prediction, assuming the elastic constants from Table 1 (Test 13) and above quoted compaction coefficients. The behaviour of sand during Test 22 (Leighton Buzzard Sand, $e_0 = 0.6$) was predicted assuming values of respective coefficients determined from Test 13.

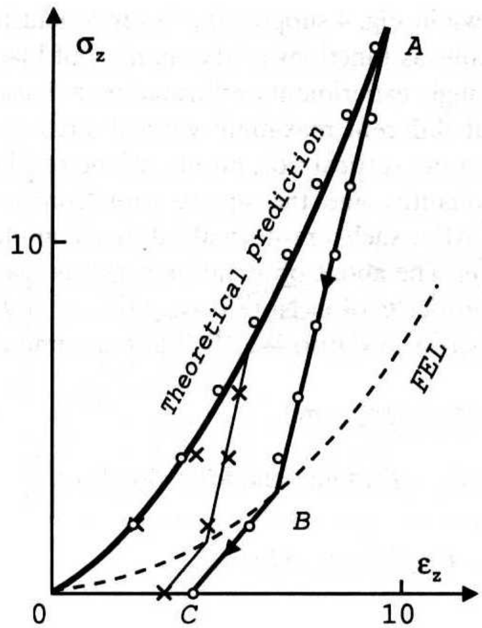


Fig. 5. Vertical stress-total strain curves and Failure in Extension Line for Leighton Buzzard Sand ($e_0 = 0.6$). Theory against experimental data

8. Failure in Extension and Residual Lateral Stresses

The failure in extension begins when point B on the unloading path is attained. Experimental data suggest that point B lies on the Failure in Extension Line (FEL) defined by the following equation:

$$\sigma_z = A\sigma_z \text{ or } (\sigma_z - \sigma_x) = \alpha(\sigma_z + \sigma_x), \ \alpha = \frac{A-1}{A+1}, \tag{28}$$

where A denotes a slope of FEL in the stress space.

The co-ordinates of point B are the following:

$$\sigma_x^* = M\sigma_z^{\max}, \ \sigma_z^* = AM\sigma_z^{\max}, \ M = \frac{1 - aK_0}{A - a}. \tag{29}$$

The vertical total strain component, corresponding to Eqs. (29), is the following:

$$\varepsilon_z^* = \varepsilon_z^{\max} + \frac{1}{E^*}(\sigma_z^* - \sigma_z^{\max}), \tag{30}$$

where

$$\varepsilon_z^{\max} = \varepsilon_z^{e,\max} + \varepsilon_z^{p,\max} = Q\sigma_z^{\max} + D_1 \ln(1 + D_2(\sigma_z^{\max})^{3/2}), \tag{31}$$

where $Q = \frac{1}{E}[2(1-\nu)K_0 - 2\nu(1+K_0) + 1]$.

Substitution of $\sigma_z^{\max}$, determined from the second of Eqs. (29), into Eq. (30) allows for elimination of this parameter and obtaining of the following equation, which is equivalent to FEL in the σ_z, ε_z plane:

$$\varepsilon_z = D_1 \ln[1 + D_3(\sigma_z)^{3/2}] + \left[\frac{Q}{AM} + \frac{1}{E^*} - \frac{1}{E^* AM}\right]\sigma_z, \tag{32}$$

where $D_3 = D_2(AM)^{-3/2}$.

The above result is illustrated in Fig. 5 for the data corresponding to medium dense Leighton Buzzard Sand. The dotted line in Fig. 5 is described by the following equation: $\varepsilon_z = 1.05 \ln[1 + 16.674(\sigma_z)^{3/2}] + 0.507\sigma_z$. Note that for the sake of convenience the asterisks, like those appearing in Eqs. (29), have been omitted in Eq. (32).

Eq. (32) is an important theoretical result, which follows from the concept of FEL. Classical interpretations of oedometric tests data are not based on the concepts proposed in this paper. Usually, the unloading paths, in the σ_z, ε_z plane, are represented by curves, which suggest a general non-linear behaviour of granular materials. Subsequently, various non-linear, and physically rather artificial, models of soil behaviour are proposed.

As already mentioned, the position of FEL should be determined experimentally. This means that the slope of FEL may be interpreted as the parameter of the material similar to the coefficient of earth pressure K_0. Available experimental data show that the value of A depends on the initial relative density of sand. There is insufficient experimental data to enable establishing of the quantitative relationships between A and D_r. Available data suggest that for loose sands a good approximation is $A = 1$. A respective approximation for dense sands could be $A = K_0$.

An important consideration that is connected with the analysis presented in this paper is the problem of so-called "compaction induced lateral stresses". It has already been realized that an additional measurement of lateral stresses in oedometric tests provides much more information than classical experiments. Surprisingly, relatively little attention to this problem has been devoted in soil mechanics investigations. After the pioneering work of Hendron (1963) not many papers on this subject have been published, cf. Duncan and Seed (1986), Seed and Duncan (1986), Ingold (1987), Peck and Mesri (1987). These authors also provide the state-of-the art in the field of investigations.

The theoretical approach presented in this paper enables the determination of residual lateral stresses that are "generated" during the oedometric test. Some simple algebraic manipulations, connected with the analysis of Figs. 1 and 2, enable the derivation of analytical formula that express the residual lateral stresses generated within a single cycle of loading and unloading:

$$\sigma_x^{res} = \frac{(aK_0 - 1)(1 - A)}{a - A}\sigma_z^{\max}. \tag{33}$$

The formula (33) has been supported by available experimental results. It shows that the level of residual lateral stresses depends on the maximum stress to which the soil sample has been subjected. There is no direct relationship between the compaction and the "compaction induced lateral stresses" although one may formally substitute $\sigma_z^{\max}$ appearing in Eq. (33) by ε^p (cf. Eq. 27). After some formal manipulations one obtains:

$$\sigma_z^{\max} = \left\{ \frac{1}{D_2} [\exp(\varepsilon^{p,\max}/D_1) - 1] \right\}^{2/3} \tag{34}$$

which directly follows from Eq. (27). By the substitution of Eq. (34) into Eq. (33) one can easily relate σ_x^{res} to the compaction $\varepsilon^{p,\max}$.

Eq. (33) relates σ_x^{res} to some characteristics of sand such as K_0, a, A. If one follows the same path during the loading and unloading, i.e. when $1/a = K_0$, then $\sigma_x^{res} = 0$ as it should be. For loose sands ($A \to 1$) there is $\sigma_x^{res} \to 0$ as it should be from the point of view of the theory proposed and experimental data.

9. Dilation Law

Figs. 1, 3 and 5 show that the dilation (plastic flow connected with the increase in volume) has a linear character. Recall that during the dilation (paths BC) the stress deviator remains constant. Some simple algebraic manipulations allow for the determination of respective finite increments of the plastic strain deviator and the plastic volumetric strain, on path BC. There is:

$$\Delta\hat{\varepsilon}^p = \hat{\varepsilon}^{p,C} - \hat{\varepsilon}^{p,B} = \varepsilon_z^{res} - \varepsilon_z^*, \tag{35}$$

where $\hat{\varepsilon}^{p,C} = \varepsilon_z^{p,C} - \varepsilon_x^{p,C}$, etc.,
and

$$\Delta\varepsilon^p = \varepsilon^{p,C} - \varepsilon^{p,B} = \varepsilon_z^{res} - \varepsilon^{e,res} - D_1 \ln[1 + D_2(\sigma_z^{\max})^{3/2}], \tag{36}$$

where $\varepsilon^{e,res} = \dfrac{2(1-2\nu)}{E}\sigma_x^{res}$.

The simplest dilation law is proposed in order to describe the observed phenomena, namely:

$$\frac{\Delta\varepsilon^p}{\Delta\hat{\varepsilon}^p} = \beta = \text{const.} \tag{37}$$

Eqs. (35) – (37), after some re-arrangements, lead to the following formula for the residual vertical strain:

$$\varepsilon_z^{res} = \varepsilon^{p,\max} + \frac{1}{1-\beta}\left[\varepsilon^{e,res} - \beta\left(Q + \frac{AM}{E^*} - \frac{1}{E^*}\right)\sigma_z^{\max}\right], \tag{38}$$

where $\varepsilon^{p,\max}$ is given by Eq. (27) for the maximum value of vertical stress.

The value of $\beta = 0.593$ has been determined from Test 13. Substitution of this value, as well as of other soil parameters determined previously, enables determination of the residual vertical strain in Test 22. This gives $\varepsilon_z^{res} = 3.44$ compared with the value of 3.79 obtained from the test. The difference between the measured and predicted values of the residual strain is less that 10% which is acceptable. The parameter β plays the role of constant for the material.

10. Conclusions

The most important results obtained in this paper can be summarized as follows:

(a) The original interpretation of oedometric tests, with additional measurement of lateral stresses, enables identification of three phases during which the behaviour of the sand displays different features. It has been shown that during the unloading this behaviour can be approximated by two linear sectors in the σ_x, σ_z and σ_z, ε_z planes, contrary to the commonly accepted non-linear approximation which obscures physical phenomena taking place during the unloading.

(b) Analysis of the first stage of unloading enables precise determination of the elastic constants (Eqs. 12 and 13). The observation of the linear reversible response of sand, at least within the range of practically important stress levels, indicates that there is no necessity to create, physically artificial, non-linear models describing reversible response of sands. The possibilities of most simple linear models should be checked before introducing more complex descriptions. An important practical conclusion which follows from the analysis presented is that the values of Young's modulus and Poisson's ratio quoted in geotechnical textbooks are incorrect.

(c) The physically sound determination of elastic constants enables extraction of the irreversible strains from experimental data, including lateral strains, which are usually ignored in analyses of oedometric tests. Two types of irreversible strains, compaction and dilation, have been identified. Respective conditions, such as the compaction and failure in extension lines, have been formulated in order to define particular phases of the oedometric test. The analytical representation of FEL in the σ_z, ε_z plane (Eq. 32) is an original result.

(d) A simple compaction law has been proposed in the incremental form (Eq. 25). This equation has been integrated for a monotonic loading giving a useful algebraic formula for the compaction volumetric strain (Eq. 27). The proposed dilation law (Eq. 37) also conforms with the experimental data.

(e) The problem of so-called "compaction induced lateral stresses" has been discussed and the analytical formula defining these stresses derived (Eq. 33). This is also an original feature of this paper.

It is believed that the results presented will open the way for some new research possibilities in soil mechanics. In the light of the approach proposed, it seems that a number of classical experimental results should be re-interpreted and some new experiments designed. A short catalogue of some problems is presented below:

(A) Determination of new values of elastic constants for various sands, and further experimental verification of validity of the observation concerning the linear reversible response. Determination of the stress range in which the reversible response of sands is linear. This should also include re-analysis of empirical data obtained from the "wave propagation" experiments.

(B) Experimental and theoretical investigations on the Failure in Extension Line. In particular, relationship between the initial relative density of sand and slope of FEL should be determined. Research on experimental determination of the coefficient β is also suggested.

(C) Investigation of behaviour of granular materials in oedometric conditions for various loading paths and cyclic loads. Such investigations should also include saturated materials. Simultaneous measurement of lateral stresses and pore pressures will provide valuable information about behaviour of materials.

Acknowledgements

Experimental results reported in this paper were obtained in the Soil Mechanics Laboratory of Birmingham University, while the author was a Research Fellow, in the academic year 1990/91. The experiments were performed, under the author's supervision, by Mr. J. Fernandez-Hernandez as part of his MEng. project. The author is grateful to Dr. J. Billam for creating a pleasant atmosphere and for discussions on the experimental aspects of this work. The help of Dr. G. Ghataora in preparation of experiments is also acknowledged.

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