Archives of Hydro-Engineering and Environmental Mechanics
Vol. 41 (1994), No. 1–2, pp. 101–110

Stochastic FEM in Soil Mechanics
Part II. Aspects of Application

Grzegorz Różyński, Władysław Knabe

Institute of Hydro-Engineering, 80-953 Gdańsk, ul. Kościerska 7, Poland

(Received March 16, 1993; revised February 8, 1994)

Abstract

The paper is devoted to some aspects of the Monte Carlo technique in stochastic FEM for soil mechanics. Sample size, "total cutting" in covariance matrix, problems of symmetry, grouping of elements and the comparison between "point" and local average discretization are analyzed.

1. Introduction

The first part of this work was focused on problems of discretization of two dimensional continuous random fields in stochastic FEM for soil mechanics. The theory of local averages was incorporated to map the initial continuous field into discrete, finite random variables of soil property (Różyński, Knabe 1993). This part concentrates on problems encountered in the application of the Monte Carlo technique in stochastic FEM.

1.1. Random Fields

The investigation was carried out for a gaussian first order Markovian, homogeneous and isotropic random field representing the modulus of elasticity of a soil stratum for a plane strain condition. The stratum is subjected to deterministic load of one or three vertical forces. Together, the stratum and forces form a symmetrical problem. The field $\underline{E}(x, y)$ has the following parameters:

(I) mean $\overline{E}$

(II) variance σ_p^2

(III) exponential correlation function between two points "1" and "2" of the field:

$$\rho(d) = \exp(-\beta d) \tag{1a}$$

where:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2^2 - y_1^2)} \tag{1b}$$

and β is the parameter of correlation decay.

In numerical calculations $\overline{E}$, σ_p^2 and deterministic Poisson's ratio ν were constant and equal to:

$$\overline{E} = 30 \text{ MPa},$$
$$\sigma_p^2 = 56.25 \text{ MPa}^2, \text{ thus } \sigma_p = 7.5 \text{ MPa},$$
$$\nu = 0.25.$$

The values of $\overline{E}$ and ν are characteristic for non-cohesive soils depending on their densities or cohesive soils for which the moisture content is close to their plastic limit. The value of the standard deviation σ_p indicates that the field can vary greatly throughout the stratum, which is typical of soils.

The values of β cover a wide range of variability in order to simulate different random fields which are either highly, poorly or medium-correlated, and equal to 0.1, 0.2, 0.3, 0.4, 0.5, 1.0, 2.0, and 4.0.

1.2. Soil Stratum Geometry, Loads and Meshes of Elements

The stratum is $H = 20$ m thick. As the analysis concerns mainly the displacements of loaded nodes the length in the analysis was assumed to be only 1.41 times longer ($L = 28.28$ m) than H. Its extention improves little the results in the vicinity of the vertical symmetry line no matter what mesh is examined.

The basic load variant is one vertical force equal to 1 MN placed symmetrically on the surface of the layer. Some other calculations were also performed for a variant of three vertical forces of 1 MN each, which also form a symmetrical issue. This variant was utilized to establish the sample size of the Monte Carlo technique that was adapted to the stochastic FEM.

Two homogeneous meshes of practically square elements were used in order to obtain and compare the results:

(I) 10×7 elements, $l = 2.828$ m, $t = 2.857$ m, known as mesh A (large squares). The calculation results for both load variants were examined for this mesh.

(II) 14×10 elements, $l = 2.02$ m, $t = 2.0$ m, known as mesh B (small squares).

Both meshes, load variants and boundary conditions are shown in Fig. 1a, b, c. It is easy to see that the top node lying on the vertical symmetry line is common for both meshes and can thus be used for comparison of the results. Deterministic calculation results of vertical displacements of this node are:

- mesh A, one force load $v = -0.069298$ m,
- mesh A, three force load $v = -0.1236$ m,
- mesh B, one force load $v = -0.07612$ m.

In additionally, vertical displacements of neighbouring and loaded nodes for three force load and mesh A were equal to -0.1130 m.

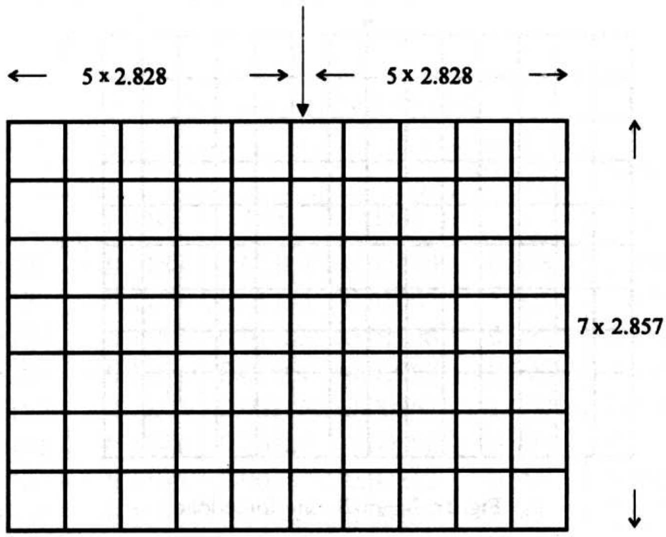

Fig. 1a. Mesh A, one force load

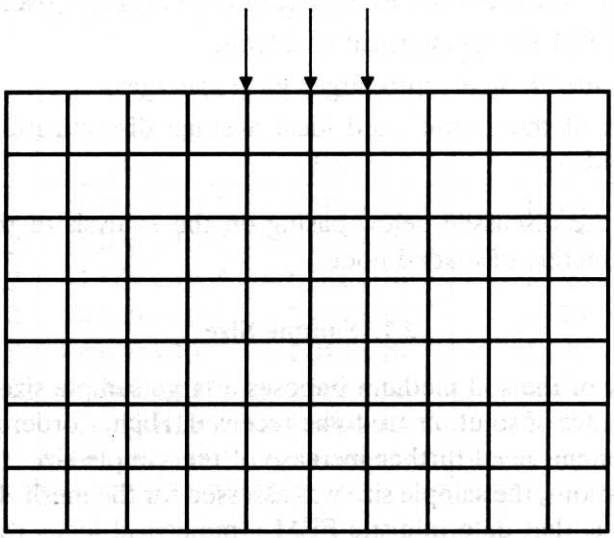

Fig. 1b. Mesh A, three force load

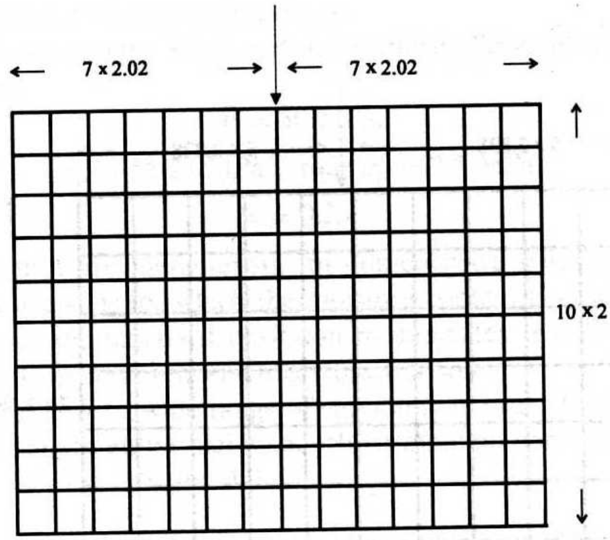

Fig. 1c. Mesh B, one force load

2. Stochastic Calculation

Stochastic calculation embraced a wide range of problems such as (Różyński 1992):

(I) sample size,
(II) "tail cutting" of the covariance matrix of local averages over finite elements,
(III) stochastic FEM for symmetrical problems,
(IV) grouping finite elements into larger local averages,
(V) comparison of the "point" and local average discretization results of the random field.

All these issues are discussed below basing on the analysis of the displacement distribution parameters of loaded nodes.

2.1. Sample Size

Strong variability of the soil medium imposes a large sample size, the larger, the more exact estimates of solution are to be received. Higher order parameters, like third central moment, need further increase of the sample size.

In our calculations, the sample size was assessed for the mesh A and three force load. It is obvious, that deterministic FEM symmetrical loads result in identical displacements in relation to the line of symmetry. Thus, the ratio:

$$R(\lambda) = \frac{\lambda_{\text{left}}}{\lambda_{\text{right}}} \tag{2}$$

of the distribution parameters of these displacements should be a good measure of the accuracy of the sample size. The results of the investigation are presented in Table 1.

Table 1. Sample size for mesh A

β	$E[v]$		σ_V		$R(E[v])$	$R(\sigma_v)$	sample size
	left	right	left	right			
0.1	−0.1186	−0.1188	0.02484	0.02470	0.998	1.006	3000
0.2	−0.1182	−0.1179	0.01755	0.01685	1.003	1.036	3000
0.3	−0.1164	−0.1165	0.01283	0.01266	0.999	1.013	3000
0.4	−0.1159	−0.1161	0.01051	0.01067	0.998	0.985	3000
0.5	−0.1152	−0.1152	0.00885	0.00865	1.000	1.023	2500
1.0	−0.1141	−0.1142	0.00488	0.00502	0.999	0.972	2000
2.0	−0.1134	−0.1135	0.00257	0.00273	0.999	0.941	1500
4.0	−0.1131	−0.1131	0.00140	0.00137	1.000	1.022	1000

The results from the above table indicate that the sample size should be considerable, to ensure accuracy of standard deviations. However, the sample size is far too small to guarantee accuracy of estimates of higher order parameters. This is shown in Table 2 where estimates of the third central moment μ_3, obtained for the sample size given in Table 1, are shown.

Table 2. Estimates of μ_3 for sample size from Table 1

β	$\mu_{3\text{left}}$	$\mu_{3\text{right}}$	$R(\mu_3)$
0.1	$-6.086 * 10^{-5}$	$-4.815 * 10^{-5}$	1.264
0.2	$-1.076 * 10^{-5}$	$-5.762 * 10^{-6}$	1.867
0.3	$-1.661 * 10^{-6}$	$-1.384 * 10^{-6}$	1.200
0.4	$-8.490 * 10^{-7}$	$-7.900 * 10^{-7}$	1.075
0.5	$-4.630 * 10^{-7}$	$-4.040 * 10^{-7}$	1.145
1.0	$-4.300 * 10^{-8}$	$-5.500 * 10^{-8}$	0.782
2.0	$-2.100 * 10^{-9}$	$-4.000 * 10^{-9}$	0.512
4.0	not computed	not computed	not computed

An increase of accuracy in the estimates of μ_3 could be attained for much greater sample sizes. This, however, is not very likely at least for standard PC equipment, because very large samples require too much computer time. Nevertheless, it is worth computing even inaccurate estimates of μ_3, as they are accurate enough to state whether the distribution received can be treated as gaussian or not. In our case all values of μ_3 are negative, therefore the distribution of the displacements is evidently non-gaussian.

2.2. "Tail Cutting" in the Covariance Matrix of Local Averages

In stochastic FEM, the continuous random field should be transformed into discrete, finite random variable by the theory of local averages. In this approach, finite elements are associated with the random field local averages and their variances and covariances are obtained by either analytical or numerical integration, described in detail in Part I. Intuitively we know that the covariances of remote local averages are close to zero and their influence on the final results of the Monte Carlo simulation should be insignificant. It may thus seem at first glance, that we can simply reduce these covariances to zero. However, this is not always permissible.

Numerical experiments were performed for the covariance matrices of local averages in which the covariances less than 10% of the local average variance σ_u^2, were reduced to zero. An attempt was then made to diagonalize these matrices. Unfortunately they all lost their positive definiteness: some elements of the diagonalized matrices turned out to be negative and an ordinary simulation procedure ceased to work. It had therefore to be modified by extorting the semi-positive definiteness of the covariance matrices. The details of such a modification can be found in (Wilde 1981).

Table 3 contains the results of calculations for a one force load, mesh A and both types of covariance matrices: with and without "tail cutting".

Table 3. Influence of "tail cutting" on stochastic FEM results mesh A, one force load

	without "tail cutting"		with "tail cutting"		errors	
β	$E[v]$	σ_v	$E[v]$	σ_v	$E[v]$	σ_v
0.1	−0.07298	0.01562	−0.07311	0.01578	+0.20%	+1.0%
0.2	−0.07223	0.01204	−0.07192	0.01188	−0.40%	−1.3%
0.3	−0.07156	0.00988	−0.07155	0.00967	−0.01%	−2.1%
0.4	−0.07152	0.00837	−0.07121	0.00813	−0.40%	−2.9%
0.5	−0.07118	0.00746	−0.07085	0.00695	−0.50%	−6.8%
1.0	−0.07024	0.00435	−0.06975	0.00413	−0.70%	−5.1%
2.0	−0.06957	0.00235	−0.06959	0.00216	+0.03%	−8.0%
4.0	−0.06939	0.00125	−0.06935	0.00121	−0.05%	−3.2%

We can see from Table 3 that the "tail cutting" influence is rather small. In most cases, the distribution parameters for simulations including "tail cuting", are slightly underestimated (negative error signs). In a few, they are slightly overestimated. The assumption that the elimination of covariances smaller than 10% of σ_u^2 is permissible therefore seems to be reasonable.

The main "tail cutting" advantage is that the computation time of the covariances of local averages is considerably reduced. This reduction is more meaningful for meshes consisting of a large number of elements.

2.3. Stochastic FEM in Symmetrical Problems

In FEM analyses, it is not rare that symmetrical problems are solved in which the system, medium-load, has a line of symmetry. In soil mechanics this line is usually vertical. In cases of symmetrical problems, ordinary, deterministic FEM uses only one symmetrical part of the half-space of the medium and loads.

In stochastic FEM the application of symmetrical analysis is equal to the assumption that the elements, whose centroids lie on the same horizontal line and are equally distant from the line of symmetry, are fully correlated. A direct consequence of such an assumption is the vanishing of horizontal displacement standard deviations, σ_h, for nodes lying on the line of symmetry. However, the negative consequences are much greater.

In Table 4 the results of full and symmetrical analyses for one force load and mesh A are compared. It is striking how much the results are distorted by the full correlation between pertinent elements.

Table 4. Comparison between half-space and full-space calculation results, mesh A, one force load

	full analysis		one part symmetrical analysis		errors	
β	$E[v]$	σ_v	$E[v]$	σ_v	$E[v]$	σ_v
0.1	−0.07298	0.01562	−0.07377	0.01749	+1.0%	+12%
0.2	−0.07223	0.01204	−0.07286	0.01368	+0.9%	+15%
0.3	−0.07156	0.00988	−0.07234	0.01132	+1.0%	+13%
0.4	−0.07152	0.00837	−0.07176	0.00975	+0.4%	+16%
0.5	−0.07118	0.00746	−0.07145	0.00861	+0.4%	+16%
1.0	−0.07024	0.00435	−0.07028	0.00552	+0.2%	+27%
2.0	−0.06957	0.00235	−0.06971	0.00330	+0.2%	+41%
4.0	−0.06939	0.00125	−0.06945	0.00177	+0.1%	+42%

All the results of the one part symmetrical analysis are highly and for greater β very highly overestimated. The overestimation of σ_v is so high that it disqualifies one part symmetrical analysis completely.

The errors for the expected values of displacements are not high but it does not pay to apply stochastic FEM to obtain first moments of the probability distributions only.

2.4. Grouping Finite Elements into Larger Local Averages

The mesh that is deterministically optimum usually consists of different types of elements, thus it can be difficult to compute the parameters of local averages. However, a simplified analysis can be carried out. It consists in grouping several finite elements into larger ones that form local averages, rectangular or square if possible. The random field is then discretized with respect to greater local averages, thus the elements that form greater local averages are fully correlated.

The influence of grouping was investigated utilizing the results for mesh B. It consists of 140 elements such that $l = 2.02$ m (horizontal) and $t = 2$ m (vertical). At first the analysis for all 140 elements, treated as local averages, was carried out. Then each four elements were grouped into 35 local averages such that $L = 2\,l$, $T = 2\,t$. The results are compared in Table 5.

Table 5. Comparison of results for local averages of grouped and nongrouped elements, mesh B, one force load

	nongrouped		grouped		errors	
β	$E[v]$	σ_v	$E[v]$	σ_v	$E[v]$	σ_v
0.1	−0.08050	0.01831	−0.08076	0.01785	+0.2%	−2.5%
0.2	−0.07998	0.01544	−0.07947	0.01320	−0.6%	−14.0%
0.3	−0.07934	0.01156	−0.07975	0.01094	−0.7%	−5.3%
0.4	−0.07882	0.01056	−0.07846	0.00935	−0.5%	−11.0%
0.5	−0.07840	0.00869	−0.07752	0.00820	−1.0%	−5.6%
1.0	−0.07746	0.00554	−0.07689	0.00500	−0.7%	−9.8%
2.0	−0.07657	0.00310	−0.07628	0.00282	−0.4%	−6.0%
4.0	−0.07637	0.00179	−0.07620	0.00156	−0.2%	−13.0%

We can see that values of $E[v]$ estimates are very little underestimated (for $\beta = 0.1$ overestimated). Standard deviations σ_v are, in turn, underestimated considerably. This does not, however, disqualify this approach completely. It only indicates that the grouping should be avoided in the vicinity of loaded nodes. Further investigations would seem necessary to find the areas for which the grouping will cause less serious errors.

2.5. Comparison of "Point" and Local Random Field Discretization Results

The simplest method of random field discretization is to assume that the local average parameters are equal to the point parameters of the random field. Thus, the variance of the local average is assumed equal to the field variance σ_p^2:

$$\sigma_u^2 = \sigma_p^2. \tag{3}$$

Simultaneously, correlations between local averages are equal to the field correlation between the centroids of local averages. The discretization performed following such assumptions is called point random field discretization. It neglects the correlation decay of the random field within the local averages themselves and therefore leads to overestimation of the displacement distribution parameters. Table 6 exemplifies this overestimation.

Table 6. Comparison of results for point and local average discretization, mesh A, one force load

	local average discretization		point discretization		errors	
β	$E[v]$	σ_v	$E[v]$	σ_v	$E[v]$	σ_v
0.1	−0.07298	0.01562	−0.07418	0.01800	+1.6%	+15%
0.2	−0.07223	0.01204	−0.07285	0.01370	+0.9%	+14%
0.3	−0.07156	0.00988	−0.07301	0.01190	+2.0%	+20%
0.4	−0.07152	0.00837	−0.07307	0.01100	+2.2%	+32%
0.5	−0.07118	0.00746	−0.07256	0.01030	+1.9%	+38%
1.0	−0.07024	0.00435	−0.07235	0.00906	+3.0%	+85%
2.0	−0.06957	0.00235	−0.07210	0.00713	+3.6%	+204%
4.0	−0.06939	0.00125	−0.07210	0.00696	+3.9%	+457%

We can see that the results are so highly overestimated that the point discretization is useless even when the random field is strongly correlated (small β).

3. Conclusions

I. Application of Monte Carlo techniques to FEM requires an ample sample size to obtain reasonably accurate estimates of standard displacement deviations. It stretches from $n = 1000$ simulations for weakly correlated random fields ($\beta = 4$) to at least $n = 3000$ simulations for highly correlated ones ($\beta = 0.1$).

II. Estimates of parameters of a higher order are inaccurate and need much greater sample size. An increase of accuracy, however, is rather difficult to achieve at least for standard PC equipment. Estimates of the third central moment μ_3, though inaccurate, are worth computing, since they supply information as to whether a given distribution may be treated as gaussian or not.

III. "Tail cutting" in the covariance matrix of local averages results only in very slight underestimation of parameters and can be useful for large meshes by reducing numerical calculations of the covariances of local averages. It needs, however, a modified diagonalization procedure which enforces the matrix to become semipositively definite.

IV. Symmetrical problems must be analyzed in the whole so as not to fully correlate relevant local averages on both sides of the line of symmetry. Othervise the parameter overestimation is very high and the calculation results are worthless.

V. Grouping the finite elements into greater local averages results in underestimation of parameters, especially standard deviations. The expected values of displacements are only slightly underestimated.

To avoid excessive underestimation, the grouping should be avoided in the vicinity of loaded nodes. Further analysis would be necessary to find the regions in which the local averages could be grouped not exposing the final results to serious errors.

VI. "Point" discretization of the random field is not recommended, as the errors of overestimation are the highest of all numerical experiments performed. The results for even very strongly correlated fields, such that $\beta = 0.1$ are overassessed too much.

References

Różyński G. (1992): *Ocena wpływu charakterystyk pola losowego parametrów podłoża gruntowego na otrzymane wyniki w stochastycznej metodzie elementów skończonych*, praca doktorska IBW PAN (PhD Thesis, Inst. of Hydro-Engineering of the Polish Acad. of Sci.).

Różyński G., Knabe W. (1993): Stochastic FEM in Soil Mechanics, Part, I, Discretization of random fields, *Archives of Hydro-Engineering and Environmental Mechanics*, Vol. 41, No. 1–2,

Wilde P. (1981): *Dyskretyzacja pól losowych w obliczeniach inżynierskich*, PWN, Warszawa, Poznań.