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Random Water Wave Kinematics
Part 1. Theory

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Abstract

This paper presents the theoretical development of stochastic properties for orbital velocities of random water waves in intermediate water depth. Both emergence effect and weak nonlinear effects are studied. An analytic formula for probability distribution for velocities modified by the emergence effect, as well as by non-linearities of the wave motion in intermediate water depth is developed. This probability function gives us the first statistical moment, the second statistical moment for modified velocities in an *analytical form*, and by numerical integration the third statistical moment for modified velocities.

The theoretical formulae for the statistical moments for surface elevation and for velocities of up to the third order, with non-linearities of motion taken into account, in case the emergence effect can be neglected, i.e., below the surface layer, have been developed. This includes a generalised formula for free surface elevation setdown and calculation of the asymmetry of the horizontal velocity.

From the first statistical moment of the modified horizontal velocity, the mean flux between any two levels is obtained. When the integration is carried out from the bottom up to $+\infty$, the formula for total mean flux is obtained.

In Part 2 of this paper (Cieślikiewicz, Gudmestad 1994b) the theoretical predictions are compared with measured kinematics. Moreover, in the vicinity of the mean water level, currents in two different directions are noted. Firstly, the emergence effect gives rise to a current at the mean water level in the direction of the wave advance. Secondly, a flow in the opposite direction, interpreted as a return current in the wave flume, is noticed just below that level.

Theoretical prediction of the measured kinematics has allowed for a better estimation of the return flow in the wave flume.

1. Introduction

For many applications in coastal and offshore engineering, it is necessary to know the water wave kinematics under the waves (Tørum and Gudmestad 1990, Sarpkaya and Isaacson 1981). Normally a random offshore wave field is characterized by a sum of sinusoidal waves with individual energies given by the wave spectrum, however, the principal shortcoming of linear wave theory for irregular water waves is its inappropriateness for determining kinematic and dynamic parameters near the still water level.

Although some improved proposed methods exist for predicting kinematics in the vicinity of the water surface such as Wheeler's method (Wheeler 1970) and Gudmestad's method (Gudmestad 1990), it should be noted that these represent extrapolations of linear theory and do not satisfy the Laplace equation for fluid flow.

Another reason for deviation from linear theory is caused by the fact that the free surface fluctuates with time, such that a fixed point in space in the vicinity of the mean water level is not submerged at all times but emerges from the water in some phases of the wave motion. This makes the probability distributions and the corresponding spectral densities different from those of continuously submerged points. This *emergence effect* (Cieślikiewicz 1985, Cieślikiewicz and Massel 1988, Cieślikiewicz and Gudmestad 1993), as it shall be referred to in this study, is also known as *free surface fluctuations phenomenon* (Pajouhi and Tung 1975) and has been taken into account in this study following the approach of Tung (1975). This wave theory with the intermittency of submergence taken into account is, furthermore, referred to in the papers of Anastasiou *et al.* (1982a, b) and Isaacson and Baldwin (1990) as *the intermittent random wave theory*.

The influence of the emergence effect is essential when determining the stochastic characteristics of particle kinematics near the mean water level. The difference between properties when taking into account the emergence effect and when ignoring it decreases considerably for points far from the free surface. On the other hand, it is well known that wave energy is concentrated in the vicinity of the still water level. Thus, the hydrodynamics in this layer is of considerable importance and should therefore be determined with a high degree of accuracy.

Laboratory and field measurements of wave kinematics with emphasis given to the mean water level zone have been reported by Anastasiou *et al.* (1982a, b) and by Skjelbreia *et al.* (1989, 1991). These latter measurements will be used in Part 2 to demonstrate that the emergence effect modifies the statistics of linear wave theory near the mean water level according to theoretical predictions.

In addition to the emergence effect on statistics of wave particle kinematics, the influence of non-linearity of the motion itself will be discussed in a form similar to that of Longuet-Higgins (1963). The background for this study is presented in Figure 1.

2. Probability Distribution of Modified Random Process $\bar{Y}(z)$

In this section the formulae for the probability density and first two statistical moments of particle velocities will be derived.

Let us consider the orbital velocities $\mathbf{u}(\mathbf{x}, z, t)$, where $\mathbf{x} = [x_1, x_2]$ is the location vector on the horizontal plane, the z-axis is directed vertically upwards

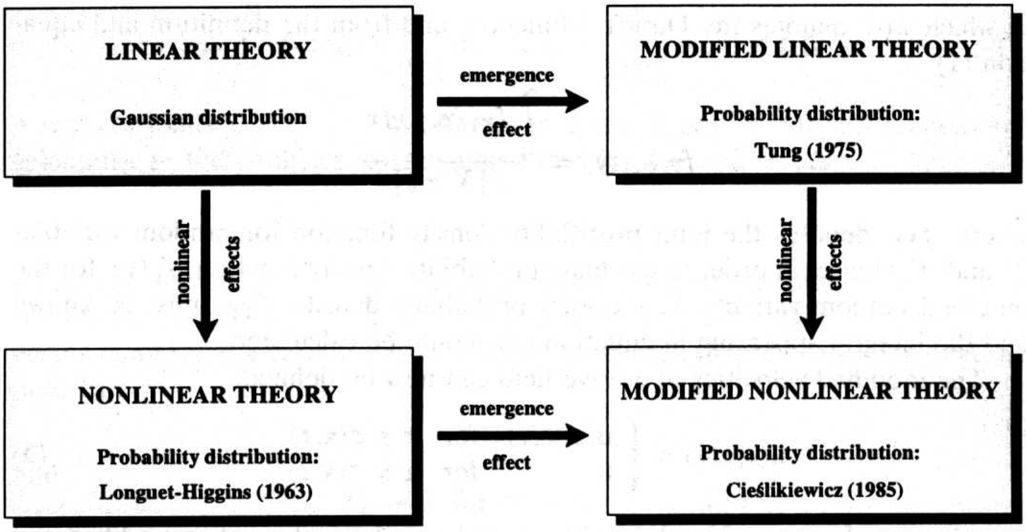

Fig. 1. Background for the study

and t is the time. It should be pointed out that the non-random parameter z of this random field is chosen from the interval $[-h, \zeta]$, where the upper limit ζ (the free surface of the wave) for a given $\mathbf{x}$ and t is a random variable (h is the water depth). This means in fact that $\mathbf{u}(\mathbf{x}, z, t)$ is not in agreement with the definition of a random field. In order to treat velocity as a random field it can be said that for points above the free surface, the velocity is equal to zero with the probability equal to one. This is not obvious from a "philosophical" point of view because it is difficult to speak about zero velocity if there is no object (i.e. no water particles in this case) for which the velocity is measured. But this is consistent in the experimental sense since when the point under consideration is above the free surface, the velocimeter will show zero value.

For the random process $Y(z)$, where $z \in (-\infty, X]$, let us then introduce the random process $\overline{Y}_X(z)$ for $z \in R^1$ modified due to the random variable X, as suggested by Tung (1975) such that:

$$\overline{Y}_X(z) = \begin{cases} Y(z) & \text{for} \quad z \leq X, \\ 0 & \text{for} \quad z > X. \end{cases} \tag{1}$$

For the sake of simplicity, the subscript X will be omitted henceforth.

For any $z \in (-\infty, X]$ the probability density function for random variable $\overline{Y}(z)$ may be estimated, by the theorem of total probability, according to the following equation

$$f_{\overline{Y}}(y) = f_{\overline{Y}|X<z}(y)\, P[X < z] + f_{\overline{Y}|X \geq z}(y)\, P[X \geq z], \tag{2}$$

where the conditional probability densities may be written as

$$f_{\overline{Y}|X<z}(y) = \delta(y) \tag{3}$$

in which $\delta(\cdot)$ denotes the Dirac's δ-function, and from the definition and equation (1)

$$f_{\overline{Y}|X\geq z}(y) = \frac{\int_z^{\infty} f_{XY}(x,y)dx}{P[X \geq z]}, \tag{4}$$

where f_{XY} denotes the joint probability density function for random variables X and Y. Hence, in order to calculate probability density function $f_{\overline{Y}}(y)$ for the modified random variable $\overline{Y}$, the joint probability density f_{XY} must be known and the integral appearing in equation (4) should be calculated.

The modified velocities of a wave field can now be defined:

$$\overline{\mathbf{u}}(\mathbf{x},z,t) = \begin{cases} \mathbf{u}(\mathbf{x},x,t) & \text{for} \quad z \leq \zeta(\mathbf{x},t), \\ \mathbf{0} & \text{for} \quad z > \zeta(\mathbf{x},t), \end{cases} \tag{5}$$

in which $\mathbf{u} = [u_1, u_2, u_3]$ and $\overline{\mathbf{u}} = [\bar{u}_1, \bar{u}_2, \bar{u}_3]$ are the vectors of unmodified and modified water wave orbital velocities, respectively.

Note that the original notation from Tung (1975) $\overline{\mathbf{u}} = \mathbf{u}\mathcal{H}(\zeta - z)$ in which $\mathcal{H}(\cdot)$ is the Heaviside unit step function is not used since in the authors' opinion $\mathbf{u}(\mathbf{x},z,t)$ for $z > \zeta(\mathbf{x},t)$ is not well defined.

It is assumed that $\zeta(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},z,t)$ and $\overline{\mathbf{u}}(\mathbf{x},z,t)$ are stationary in time and homogeneous with respect to $\mathbf{x}$. Thus, the results obtained for the random variable X and processes Y and $\overline{Y}$ may be applied directly to the free surface elevations ζ and components of $\mathbf{u}$ and $\overline{\mathbf{u}}$, owing to the analogous form of equations (1) and (5).

If the *non-linearity of the wave motion* is taken into account, the random field of the free surface elevation may be presented, after Longuet-Higgins (1963), by the following formula:

$$\zeta(\mathbf{x},t) = \sum_{i=1}^{N} \alpha_i(\mathbf{x},t)\xi_i + \sum_{i,j=1}^{N} \alpha_{ij}(\mathbf{x},t)\xi_i\xi_j + \sum_{i,j,k=1}^{N} \alpha_{ijk}(\mathbf{x},t)\xi_i\xi_j\xi_k + \cdots, \tag{6}$$

where $\alpha_i(\mathbf{x},t)$, $\alpha_{ij}(\mathbf{x},t), \ldots$ are non-random functions, while ξ_i are random variables, assumed to be independent and symmetrically distributed around zero. It is suggested that a similar representation may be used for the non-linear random field of particle velocity $\mathbf{u}(\mathbf{x},z,t)$. Thus, in order to determine the stochastic characteristics of random water wave field, let us examine the random variable X and the process Y (being a function of, say, z) assuming that they can be written in a form analogous to (6):

$$X = \sum_{i=1}^{N} \alpha_i\xi_i + \sum_{i,j=1}^{N} \alpha_{ij}\xi_i\xi_j + \sum_{i,j,k=1}^{N} \alpha_{ijk}\xi_i\xi_j\xi_k + \cdots, \tag{7}$$

$$Y(z) = \sum_{i=1}^{N} \beta_i(z)\xi_i + \sum_{i,j=1}^{N} \beta_{ij}(z)\xi_i\xi_j + \sum_{i,j,k=1}^{N} \beta_{ijk}(z)\xi_i\xi_j\xi_k + \cdots, \tag{8}$$

where constants α_i, $\alpha_{ij}, \dots$ and functions $\beta_i(z)$, $\beta_{ij}(z), \dots$ are not random and symmetric in their suffices. It is assumed that for $i = 1, 2, \dots, N$

$$< \xi_i > = 0, \tag{9}$$

where the symbol $< \cdot >$ denotes the expected value of the quantity enclosed in the bracket. Moreover, let us assume that with each value of i is associated a vector $\boldsymbol{\kappa}_i$ and that a certain function $F(\boldsymbol{\kappa})$ exists such that the following relations hold true

$$< \xi_i^2 > = V_i \tag{10}$$

and

$$\left. \begin{aligned} & V_i \to 0 \ \text{ for } N \to \infty \\ & \sum_{\boldsymbol{\kappa} \in d\boldsymbol{\kappa}} V_i = F(\boldsymbol{\kappa}) d\boldsymbol{\kappa} + O(d\boldsymbol{\kappa}^2) \end{aligned} \right\} \tag{11}$$

for any small but fixed region $d\boldsymbol{\kappa}$. V_i is the *variance* of the random variable ξ_i.

Since the second order approximation is assumed, we examine the following random variable

$$X = \sum_{i=1}^{N} \alpha_i\xi_i + \sum_{i,j=1}^{N} \alpha_{ij}\xi_i\xi_j \tag{12}$$

and the process

$$Y(z) = \sum_{i=1}^{N} \beta_i(z)\xi_i + \sum_{i,j=1}^{N} \beta_{ij}(z)\xi_i\xi_j. \tag{13}$$

The mean values, and the central statistical moments, as well as the joint central statistical moments of the second and third order for X and Y can be easily expressed in terms of the variances V_i by use of (12) and (13). Since we are interested in the limit $N \to \infty$, the terms $< \xi_i^4 >$ and $< \xi_i^6 >$ can be neglected and only terms of the lowest order of V_i will be left. Using the notation

$$\mu_{mn} =< (X- < X >)^m (Y- < Y >)^n > \tag{14}$$

the statistical moments for X and Y are found to be given by

$$\left. \begin{aligned} & m_{10} =< X >= \sum_{i=1}^{N} \alpha_{ii} V_i \\ & \mu_{20} = \sigma_X^2 = \sum_{i=1}^{N} \alpha_i\alpha_i V_i + 2 \sum_{i,j=1}^{N} \alpha_{ij}\alpha_{ij} V_i V_j \simeq \sum_{i=1}^{N} \alpha_i\alpha_i V_i \\ & \mu_{30} = 6 \sum_{i,j=1}^{N} \alpha_i\alpha_j\alpha_{ij} V_i V_j \end{aligned} \right\}, \tag{15}$$

$$
\left.\begin{aligned}
m_{01}(z) &= \langle Y(z) \rangle = \sum_{i=1}^{N} \beta_{ii}(z) V_i \\
\mu_{02} &= \sigma_Y^2 = \sum_{i=1}^{N} \beta_i(z)\beta_i(z) V_i \\
&\quad + 2 \sum_{i,j=1}^{N} \beta_{ij}(z)\beta_{ij}(z) V_i V_j \simeq \sum_{i=1}^{N} \beta_i(z)\beta_i(z) V_i \\
\mu_{03} &= 6 \sum_{i,j=1}^{N} \beta_i(z)\beta_j(z)\beta_{ij}(z) V_i V_j
\end{aligned}\right\}, \tag{16}
$$

$$
\left.\begin{aligned}
\mu_{11}(z) &= \sum_{i=1}^{N} \alpha_i \beta_i(z) V_i + 2 \sum_{i,j=1}^{N} \alpha_{ij} \beta_{ij}(z) V_i V_j \simeq \sum_{i=1}^{N} \alpha_i \beta_i(z) V_i \\
\mu_{21}(z) &= 2 \sum_{i,j=1}^{N} \alpha_i \alpha_j \beta_{ij}(z) V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_i(z) \alpha_{ij} V_i V_j \\
\mu_{12}(z) &= 2 \sum_{i,j=1}^{N} \alpha_{ij} \beta_i(z)\beta_j(z) V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_j(z)\beta_{ij}(z) V_i V_j
\end{aligned}\right\}. \tag{17}
$$

The probability densities for X and Y, $f_X(x)$ and $f_Y(y;z)$, respectively, as well as the joint probability density $f_{XY}(x,y;z)$ may be found by following Longuet-Higgins (1963). Introducing:

$$
Z(\gamma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\gamma^2}{2}} \tag{18}
$$

the second-order approximation is:

$$
f_X(x) = \frac{1}{\sigma_X} Z(x') \left[1 + \frac{1}{6} \lambda_{30} H_3(x') \right], \tag{19}
$$

$$
f_Y(y;z) = \frac{1}{\sigma_Y(z)} Z(y') \left[1 + \frac{1}{6} \lambda_{03}(z) H_3(y') \right] \tag{20}
$$

and

$$
\begin{aligned}
f_{XY}(x,y;z) = {} & \frac{1}{\sigma_X \sigma_Y(z) \sqrt{\Delta(z)}} Z(y')\, Z(\eta) \Bigg[1 + \frac{1}{6} \Bigg(\lambda_{30} H_{30} + 3\lambda_{21}(z) H_{21} \\
& + 3\lambda_{12}(z) H_{12} + \lambda_{03} H_{03} \Bigg) \Bigg],
\end{aligned} \tag{21}
$$

where

$$
\left.\begin{aligned}
&\lambda_{mn} = \frac{\mu_{mn}}{(\mu_{20}^m \mu_{02}^n)^{\frac{1}{2}}} \qquad \Delta = 1 - \lambda_{11}^2 \\
&x' = \frac{x - m_{10}}{\sigma_X} \quad y' = \frac{y - m_{01}}{\sigma_Y} \quad \eta(x', y'; \lambda_{11}) = \frac{x' - \lambda_{11} y'}{\sqrt{\Delta}}
\end{aligned}\right\}. \tag{22}
$$

Note that m_{10}, σ_X, λ_{30} are constants while m_{01}, σ_Y, λ_{03} are functions of z.

In (19) and (20) $H_3(\cdot)$ denotes a Hermite polynomial of the 3rd degree. Note that for the calculation of Hermite polynomials of the nth degree, the following relation is used:

$$(-1)^m \frac{\partial^m}{\partial x^m} e^{-\frac{x^2}{2}} = H_m(x) e^{-\frac{x^2}{2}} \tag{23}$$

in which it is assumed that $\partial^0/\partial x^0$ is a neutral operator. Two-dimensional equivalents of the Hermite polynomials appearing in (21) may be calculated using the relation:

$$(-1)^{m+n} \frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} e^{-\frac{1}{2}[y^2+\eta^2(x,y;r)]} = H_{mn}(x,y;r) e^{-\frac{1}{2}[y^2+\eta^2(x,y;r)]}, \tag{24}$$

where $r = \lambda_{11}(z)$ is the coefficient of cross-correlation between X and Y.

From (19) it is found that the probability that the random variable X exceeds the value z is equal to

$$P[X \geq z] = \int_z^\infty f_X(x)\, dz = Q^*(z'). \tag{25}$$

Here $z' = (z - m_{10})/\sigma_X$ and

$$Q^*(\gamma) = Q(\gamma) + \frac{1}{6}\lambda_{30} H_2(\gamma) Z(\gamma), \tag{26}$$

in which

$$Q(\gamma) = \int_\gamma^\infty Z(z)\, dz. \tag{27}$$

Calculation of the conditional density in equation (4) with f_{XY} in the form given by (21) reduces to evaluation of the following integrals:

$$\int_{x=z}^{+\infty} H_{mn}(x',y';r)\, Z(y')\, Z[\eta(x',y';r)]\, dx. \tag{28}$$

There are two kinds of such integrals—when $m \geq 1$ and when $m = 0$. In the first case we obtain

$$\begin{aligned} &\int_{x=z}^{+\infty} H_{mn}(x',y';r)\, Z(y')\, Z[\eta(x',y';r)]\, dx \\ &= -\frac{\sigma_X}{2\pi} \int_{x'=z'}^{+\infty} \frac{\partial}{\partial x'} \left[H_{m-1n}(x',y';r)\, e^{-\frac{1}{2}[y'^2+\eta^2(x',y';r)]} \right] dx' \\ &= \sigma_X H_{m-1n}(z',y';r)\, Z(y')\, Z[\eta(z',y';r)]. \end{aligned} \tag{29}$$

In the case of $m = 0$ we can write

$$\int_{x=z}^{+\infty} H_{0n}(x', y'; r) Z(y') Z[\eta(x', y'; r)]\, dx$$

$$= \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(y')\, Z(y') \int_{x=z}^{+\infty} (-1)^k \frac{\partial^k}{\partial y'^k} Z[\eta(x', y'; r)]\, dx$$

$$= \sigma_X \sqrt{\Delta} \left\{ H_n(y')\, Z(y')\, Q[\eta(z', y'; r)] \right.$$

$$- \frac{1}{\sqrt{2\pi}} Z(y') \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(\frac{r}{\sqrt{\Delta}}\right)^k H_{n-k}(y')$$

$$\left. \times \int_{\gamma=\eta(z',y';r)}^{+\infty} \frac{\partial}{\partial \gamma} \left[(-1)^{k-1} \frac{\partial^{k-1}}{\partial \gamma^{k-1}} e^{-\gamma^2/2} \right] d\gamma \right\} \tag{30}$$

and then, by equation (23), one can obtain the following expression for the integral:

$$\int_{x=z}^{+\infty} H_{0n}(x', y'; r)\, Z(y')\, Z[\eta(x', y'; r)] dx$$

$$= \sigma_X \sqrt{\Delta}\, Z(y') \left\{ H_n(y')\, Q[\eta(z', y'; r)] + Z[\eta(z', y'; r)] \right.$$

$$\left. \times \sum_{k=1}^{n} (-1)^k \binom{n}{k} \left(\frac{r}{\sqrt{\Delta}}\right)^k H_{n-k}(y')\, H_{k-1}[\eta(z', y'; r)] \right\}, \tag{31}$$

in which $H_0(x) = 1$.

Equation (21), combined with the use of (29) and (31) (where by equation (25) $P[X < z] = 1 - Q^*(z')$), leads to the probability density for $\overline{Y}(z)$ in the form obtained by Cieślikiewicz (1985)

$$f_{\overline{Y}}(y; z) = [1 - Q^*(z')]\delta(y) + \frac{1}{\sigma_Y(z)} Z(y') \left\{ \left[1 + \frac{1}{6}\lambda_{03}(z) H_3(y') \right] \times Q[\eta(z', y'; r)] \right.$$

$$\left. + \frac{1}{6\sqrt{\Delta}} G(y', z') Z[\eta(z', y'; r)] \right\}. \tag{32}$$

Function G in the above formula is:

$$G(y', z') = \lambda_{30} H_{20}(z', y'; r) + 3\lambda_{21} H_{11}(z', y'; r) + 3\lambda_{12} H_{02}(z', y'; r)$$

$$-\lambda_{03}\frac{r}{\Delta}\left\{r^2 H_2[\eta(z',y';r)] - 3\sqrt{\Delta} r H_1[\eta(z',y';r)]H_1(y')\right.$$
$$\left. + 3\Delta H_2(y')\right\}. \tag{33}$$

Note that (32) represents an extension of (20) by including emergence effects (Figure 1).

The probability density function (32) can be used in evaluation of the first two statistical moments of the modified process $\bar{Y}(z)$, i.e.

$$< \bar{Y} > = \int_{-\infty}^{+\infty} y f_{\bar{Y}}(y;z)\, dy \tag{34}$$

and

$$< \bar{Y}^2 > = \int_{-\infty}^{+\infty} y^2 f_{\bar{Y}}(y;z)\, dy. \tag{35}$$

In order to calculate these moments, the values of the following integrals

$$I_{mn}^k = \int_{-\infty}^{+\infty} y^k \int_{x=z'}^{+\infty} H_{mn}(x,y;r) Z(y) Z[\eta(x,y;r)]\, dx\, dy \tag{36}$$

for $k = 1,2$ and $m+n \leq 3$ should be known. Integration by parts results in

$$I_{mn}^1 = 0, \quad \text{for } n \geq 2 \quad \text{and} \quad I_{mn}^2 = 0, \quad \text{for } n \geq 3. \tag{37}$$

Successive integrals are given as

$$I_{30}^1 = \int_{-\infty}^{+\infty} y\, H_{20}(z',y;r)\, Z(y)\, Z[\eta(z',y;r)]\, dy$$
$$= \frac{\partial^2}{\partial z'^2}\left\{ Z(z') \int_{-\infty}^{+\infty} y\, Z[\eta(y,z';r)]\, dy \right\} = r\sqrt{\Delta}\, H_3(z')\, Z(z'), \tag{38}$$

where the following relation was used

$$Z(y)\, Z[\eta(x,y;r)] = Z(x)\, Z[\eta(y,x;r)]. \tag{39}$$

Furthermore

$$I_{21}^1 = -\int_{-\infty}^{+\infty} y \frac{\partial}{\partial y}\left\{ H_{10}(z',y;r)\, Z(y)\, Z[\eta(z',y;r)]\right\} dy. \tag{40}$$

Integrating by parts, using equation (39) leads to

$$I_{21}^{1} = -\frac{\partial}{\partial z'}\left\{ Z(z') \int_{-\infty}^{+\infty} Z[\eta(y, z'; r)]\, dy \right\} = \sqrt{\Delta}\, \dot{H}_1(z')\, Z(z'). \tag{41}$$

Since

$$\frac{\partial}{\partial y} Q[\eta(z', y; r)] = \frac{r}{\sqrt{\Delta}} Z[\eta(z', y; r)] \tag{42}$$

we have

$$I_{00}^{1} = -\sqrt{\Delta} \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial y} Z(y)\right) Q[\eta(z', y; r)]\, dy = r\sqrt{\Delta}\, Z(z'). \tag{43}$$

Successive integrals for $k = 2$ can be obtained in the same manner. Their values are as follows

$$\left.\begin{array}{ll} I_{00}^{2} = \sqrt{\Delta}\,[Q^{*}(z') + r^{2} H_1(z')\, Z(z')], & I_{30}^{2} = \sqrt{\Delta}\, H_4(z')\, Z(z') \\ I_{21}^{2} = 2\sqrt{\Delta} r\, H_2(z')\, Z(z'), & I_{12}^{2} = 2\sqrt{\Delta}\, Z(z') \end{array}\right\}. \tag{44}$$

By use of equations (37) and (44), we obtain from definitions (34) and (35), the mean value and the second moment of the modified process $\overline{Y}(z)$ in the form

$$< \overline{Y} >= m_{01} Q^{*}(z') + \sigma_Y Z(z') \left[r + \frac{1}{6}\left(r\lambda_{30} H_3(z') + 3\lambda_{21} H_1(z') \right) \right], \tag{45}$$

$$\begin{aligned} < \overline{Y}^{2} > = {} & (\sigma_Y^{2} + m_{01}^{2}) Q^{*}(z') + \sigma_Y^{2} Z(z') \left(r^{2} H_1(z') + \frac{1}{6}\lambda_{30} H_4(z') \right. \\ & \left. + r\lambda_{21} H_2(z') + \lambda_{12} \right) + 2 m_{01}\sigma_Y Z(z') \left[r + \frac{1}{6}\left(r\lambda_{30} H_3(z') \right.\right. \\ & \left.\left. + 3\lambda_{21} H_1(z') \right) \right]. \end{aligned} \tag{46}$$

The variance $\sigma_{\overline{Y}}^{2}(z)$ can be calculated using the following formula

$$\sigma_{\overline{Y}}^{2}(z) = < \overline{Y}^{2} > - < \overline{Y} >^{2}. \tag{47}$$

In (32) terms including λ_{mn} for $m + n = 3$ represent non-linear effects. If one omits these effects, the probability density given by equation (32) will assume a simpler form, as given by Tung (1975):

$$f_{\overline{Y}}(y; z) = [1 - Q(z')]\delta(y) + \frac{1}{\sigma_Y(z)} Z(y') Q[\eta(z', y'; r)], \tag{48}$$

where $y' = y/\sigma_Y$. Equation (48) represents the emergence effect in equation (32) (Figure 1). Similarly for moments $< \bar{Y} >$ and $< \bar{Y}^2 >$ we obtain:

$$< \bar{Y} >= r\sigma_Y Z(z'), \tag{49}$$

$$< \bar{Y}^2 >= \sigma_Y^2 Q(z') + \sigma_Y^2 r^2 z' Z(z'). \tag{50}$$

It can easily be shown that with $z \to -\infty$ the probability density for process $\bar{Y}$ given by (32) becomes equal to the density for the process Y as given by equation (20). From (45), (46) and (47) it follows that:

$$\lim_{z\to-\infty} < \bar{Y} >= m_{01}, \quad \lim_{z\to-\infty} < \bar{Y}^2 >= \sigma_Y^2 + m_{01}^2, \quad \text{and} \quad \lim_{z\to-\infty} \sigma_{\bar{Y}}^2 = \sigma_Y^2. \tag{51}$$

Thus the emergence effect ceases to be of significant importance for points located deeply below the free surface.

3. Application to Water Gravity Waves

In order to calculate quantities m_{10}, σ_X and λ_{30} appearing in (20) and (22) (remember that X and Y play the role of surface elevation ζ and velocity component in a wave field, respectively), the central statistical moments μ_{20} and μ_{30} for surface elevation should be known. Moreover, in order to determine the parameters of the probability function as given by equation (32) for modified velocities, the statistical moments for unmodified velocity quantities should be calculated together with the joint moments for them and ζ, up to the third order inclusive. Starting from the basic hydrodynamic equation for fluid flow, i. e. the Laplace equation, and the non-linear boundary conditions at the free surface, these statistical parameters can be calculated based on the free surface spectral density. This is carried out in this section after Cieślikiewicz (1989).

Consider a random field of surface waves propagating over a horizontal bottom in which the surface displacement $\zeta(\mathbf{x}, t)$ as well as particle velocity $\mathbf{u}(\mathbf{x}, z, t)$ are stationary in time and homogeneous with respect to $\mathbf{x}$. Let the free surface be represented by $z = \zeta(\mathbf{x}, t)$. With the assumption of irrotational motion and an incompressible fluid the continuity equation assumes the form

$$\Delta\, \phi(\mathbf{x}, z, t) = 0, \tag{52}$$

where $\phi(\mathbf{x}, z, t)$ denotes the velocity potential, i.e. $\mathbf{u} = \nabla\phi$.

The kinematic and dynamic boundary conditions at the free surface, that must be satisfied by ϕ and ζ are

$$\frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla(\zeta - z) = 0, \quad \text{for } z = \zeta, \tag{53}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\zeta = 0, \quad \text{for } z = \zeta, \tag{54}$$

while at the bottom we have

$$\frac{\partial \phi}{\partial z} = 0, \quad \text{for } z = -h. \tag{55}$$

In the second-order approximation, functions ϕ and ζ may be represented as follows

$$\zeta = \zeta^{(1)} + \zeta^{(2)}, \tag{56}$$

$$\phi = \phi^{(1)} + \phi^{(2)}. \tag{57}$$

The linear parts of these functions are

$$\zeta^{(1)} = \sum_{p=1}^{N'} a_p \cos \chi_p, \tag{58}$$

$$\phi^{(1)} = \sum_{p=1}^{N'} c_p \frac{\operatorname{ch} k_p(z+h)}{\operatorname{ch} k_p h} \sin \chi_p, \tag{59}$$

where

$$c_p = \frac{a_p \omega_p}{\widetilde{k}_p} \tag{60}$$

and the phase angle

$$\chi_p = \mathbf{k}_p \cdot \mathbf{x} - \omega_p t + \theta_p. \tag{61}$$

In the above formulae $\mathbf{k}_p = [k_{p1}, k_{p2}]$ denotes the wavenumber vector ($k_p = |\mathbf{k}_p|$), ω_p is the angular frequency, while θ_p is the phase shift. The following notation was also introduced

$$\widetilde{k}_p = k_p \operatorname{th} k_p h. \tag{62}$$

For any $p = 1, \ldots, N'$ the angular frequency ω_p and wavenumber k_p satisfy the dispersion relation

$$\omega_p^2 = g\widetilde{k}_p \tag{63}$$

in which g denotes acceleration of gravity.

It is also assumed that the constant amplitudes a_p and the phase shifts θ_p are chosen randomly so that $a_p \cos\theta_p$ and $a_p \sin\theta_p$ are zero-mean, statistically independent and jointly normal, with θ_n uniformly distributed (Longuet-Higgins 1963). Assuming furthermore that

$$V_p \to 0, \text{ for } N' \to \infty, \tag{64}$$

where $V_p = \frac{1}{2} < a_p^2 >$, the following relation holds true

$$\sum_{\mathbf{k} \in d\mathbf{k}} V_p = F^{(1)}(\mathbf{k})\, d\mathbf{k}, \tag{65}$$

in which $F^{(1)}(\mathbf{k})$ is a first-order part of the free surface elevation spectrum $F(\mathbf{k})$ (see Appendix A).

From the system of equations (52) to (55), it follows that $\zeta^{(2)}$ and $\phi^{(2)}$ must satisfy

$$\Delta\phi^{(2)} = 0, \tag{66}$$

$$\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(2)} = -\frac{\partial}{\partial t}\left(\nabla\phi^{(1)}\right)^2 - \zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(1)}, \text{ for } z = 0, \tag{67}$$

$$\zeta^{(2)} = -\frac{1}{g}\left[\frac{\partial\phi^{(2)}}{\partial t} + \frac{1}{2}\left(\nabla\phi^{(1)}\right)^2 + \zeta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial z\,\partial t}\right]_{z=0}, \tag{68}$$

and

$$\frac{\partial\phi^{(2)}}{\partial z} = 0, \quad \text{for } z = -h. \tag{69}$$

Substituting (58) and (59) in (67) gives

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(2)} = -\frac{1}{2}\sum_{p,q} c_p c_q \Big\{\Big[&(\omega_p - \omega_q)(\mathbf{k}_p\cdot\mathbf{k}_q + \widetilde{k}_p\widetilde{k}_q) \\ &+ (\omega_q R_p - \omega_p R_q)\Big]\sin(\chi_p - \chi_q) + \Big[(\omega_p + \omega_q)(\mathbf{k}_p\cdot\mathbf{k}_q - \widetilde{k}_p\widetilde{k}_q) \\ &+ (\omega_q R_p + \omega_p R_q)\Big]\sin(\chi_p + \chi_q)\Big\}, \quad \text{for } z = 0,\end{aligned} \tag{70}$$

where

$$R_p = \frac{1}{2}\left(k_p^2 - \widetilde{k}_p^2\right), \quad \text{for } p = 1, \ldots, N'. \tag{71}$$

The solution to (70) that satisfies boundary condition (69) may be written in the following form:

$$\phi^{(2)}(\mathbf{x}, z, t) = \frac{1}{2}\sum_{p,q}\frac{c_p c_q}{\sqrt{g}}\left[C_{pq}^{-}(z)\sin(\chi_p - \chi_q) + C_{pq}^{+}(z)\sin(\chi_p + \chi_q)\right], \tag{72}$$

where for $p \neq q$

$$C_{pq}^{\pm}(z) = \frac{B_{pq}^{\pm}}{\sqrt{\widetilde{k}_p} \pm \sqrt{\widetilde{k}_q}}\,\frac{\operatorname{ch} k_{pq}^{\pm}(z+h)}{\operatorname{ch} k_{pq}^{\pm}h}, \tag{73}$$

$$\begin{aligned}B_{pq}^{\pm} = \Bigg[&\left(\sqrt{\widetilde{k}_p} \pm \sqrt{\widetilde{k}_q}\right)^2 (\mathbf{k}_p\cdot\mathbf{k}_q \mp \widetilde{k}_p\widetilde{k}_q) \\ &+ \left(\sqrt{\widetilde{k}_p} \pm \sqrt{\widetilde{k}_q}\right)\left(\sqrt{\widetilde{k}_q}R_p \pm \sqrt{\widetilde{k}_p}R_q\right)\Bigg]\left[\left(\sqrt{\widetilde{k}_p} \pm \sqrt{\widetilde{k}_q}\right)^2 - \widetilde{k}_{pq}^{\pm}\right]^{-1}\end{aligned} \tag{74}$$

and for $p = q$

$$\left.\begin{aligned} C^{+}_{qq} &= \frac{B^{+}_{qq}}{2\sqrt{\widetilde{k}_q}} \frac{\operatorname{ch} 2k_q(z+h)}{\operatorname{ch} 2k_q h} & C^{-}_{qq} &= 0 \\ B^{+}_{qq} &= \frac{3}{2} k_q^2 \frac{1 - \operatorname{th}^4 k_q h}{\operatorname{th}^2 k_q h} & B^{-}_{qq} &= 0 \end{aligned}\right\}, \tag{75}$$

in which

$$\left.\begin{aligned} \mathbf{k}^{-}_{pq} &= \mathbf{k}_p - \mathbf{k}_q & \mathbf{k}^{+}_{pq} &= \mathbf{k}_p + \mathbf{k}_q \\ k^{-}_{pq} &= |\mathbf{k}^{-}_{pq}| & k^{+}_{pq} &= |\mathbf{k}^{+}_{pq}| \\ \widetilde{k}^{-}_{pq} &= k^{-}_{pq} \operatorname{th} k^{-}_{pq} h & \widetilde{k}^{+}_{pq} &= k^{+}_{pq} \operatorname{th} k^{+}_{pq} h \end{aligned}\right\}. \tag{76}$$

Substituting (58), (59) and (72) into equation (68) we find the second-order correction term for the free surface elevation

$$\begin{aligned} \zeta^{(2)}(\mathbf{x}, t) = \frac{1}{2} \sum_{p,q} \frac{a_p a_q}{\sqrt{\widetilde{k}_p \widetilde{k}_q}} \Big\{ \Big[& B^{-}_{pq} + B^{+}_{pq} - \mathbf{k}_p \cdot \mathbf{k}_q + (\widetilde{k}_p + \widetilde{k}_q) \sqrt{\widetilde{k}_p \widetilde{k}_q} \Big] \\ & \times \cos \chi_p \cos \chi_q + \Big[B^{-}_{pq} - B^{+}_{pq} - \widetilde{k}_p \widetilde{k}_q \Big] \sin \chi_p \sin \chi_q \Big\}. \end{aligned} \tag{77}$$

It can easily be seen that expressions (77) for $\zeta^{(2)}$ and (72) for $\phi^{(2)}$, in the limit $h \to \infty$, become equal to those found by Cieślikiewicz (1985) for deep water.

In a second-order approximation, the free surface elevation ζ can be expressed in a form similar to (12). For the special point $\mathbf{x} = \mathbf{0}$ and time $t = 0$ we can write

$$\zeta = \sum_{i=1}^{N} \alpha_i \xi_i + \sum_{i,j=1}^{N} \alpha_{ij} \xi_i \xi_j, \tag{78}$$

where

$$N = 2N' \quad \text{and} \quad \xi_i = \begin{cases} a_i \cos \theta_i & i = 1, \ldots, N', \\ -a_{i-N'} \sin \theta_{i-N'} & i = N'+1, \ldots, 2N', \end{cases} \tag{79}$$

while

$$\alpha_i = \begin{cases} 1 & i = 1, \ldots, N', \\ 0 & i = N'+1, \ldots, 2N', \end{cases} \tag{80}$$

$$\alpha_{ij} = \begin{cases} \dfrac{1}{2\sqrt{\widetilde{k}_i\widetilde{k}_j}}\left[B_{ij}^- + B_{ij}^+ - \mathbf{k}_i \cdot \mathbf{k}_j + (\widetilde{k}_i + \widetilde{k}_j)\sqrt{\widetilde{k}_i\widetilde{k}_j}\right] & i, j = 1, \ldots, N', \\ \dfrac{1}{2\sqrt{\widetilde{k}_i\widetilde{k}_j}}\left[B_{ij}^- - B_{ij}^+ - \widetilde{k}_i\widetilde{k}_j\right] & i, j = N'+1, \ldots, 2N', \\ 0 & \text{otherwise.} \end{cases} \tag{81}$$

It can be shown by using equation (75) that the constants α_{ii} have the following form

$$\alpha_{ii} = \begin{cases} \dfrac{\widetilde{k}_i}{4}\,\dfrac{\mathrm{th}^4 k_i h - 2\mathrm{th}^2 k_i h + 3}{\mathrm{th}^4 k_i h} & i = 1, \ldots, N', \\ \dfrac{\widetilde{k}_i}{4}\,\dfrac{\mathrm{th}^4 k_i h - 3}{\mathrm{th}^4 k_i h} & i = N'+1, \ldots, 2N'. \end{cases} \tag{82}$$

Since the representation (78) of the free surface elevation is identical with formula (12), we can use equations (15) when determining the statistical moments for that random variable. We have thus

$$m_{10} = <\zeta> = \sum_{i=1}^{N} \alpha_{ii}\, V_i = \sum_{p=1}^{N'} \frac{\widetilde{k}_p}{2}\,\frac{\mathrm{th}^2 k_p h - 1}{\mathrm{th}^2 k_p h}\, V_p, \tag{83}$$

$$\mu_{20} = \sum_{i=1}^{N} \alpha_i^2\, V_i = \sum_{p=1}^{N'} V_p, \tag{84}$$

$$\mu_{30} = 6 \sum_{i,j=1}^{N} \alpha_i \alpha_j \alpha_{ij}\, V_i V_j = 6 \sum_{p,q=1}^{N'} \alpha_{pq}\, V_p V_q. \tag{85}$$

Due to relation (65), the above formulae can be rewritten in integral form as

$$m_{10} = \frac{1}{2}\int_{\mathbf{k}} \widetilde{k}(1 - \mathrm{th}^{-2} kh) F^{(1)}(\mathbf{k})\, d\mathbf{k}, \tag{86}$$

$$\mu_{20} = \int_{\mathbf{k}} F^{(1)}(\mathbf{k})\, d\mathbf{k} \tag{87}$$

and

$$\mu_{30} = \int_{\mathbf{k}}\int_{\mathbf{k}'} K_{30}(\mathbf{k}, \mathbf{k}')\, F^{(1)}(\mathbf{k})\;\; F^{(1)}(\mathbf{k}')\, d\mathbf{k}\, d\mathbf{k}', \tag{88}$$

where

$$K_{30}(\mathbf{k},\mathbf{k}') = \frac{3}{\sqrt{\widetilde{k}\widetilde{k}'}} B(\mathbf{k},\mathbf{k}'), \tag{89}$$

$$B(\mathbf{k},\mathbf{k}') = B^-(\mathbf{k},\mathbf{k}') + B^+(\mathbf{k},\mathbf{k}') - \mathbf{k}\cdot\mathbf{k}' + (\widetilde{k}+\widetilde{k}')\sqrt{\widetilde{k}\widetilde{k}'}, \tag{90}$$

$$B^{\pm}(\mathbf{k},\mathbf{k}') = \left[\left(\sqrt{\widetilde{k}} \pm \sqrt{\widetilde{k}'}\right)^2 (\mathbf{k}\cdot\mathbf{k}' \mp \widetilde{k}\widetilde{k}') + \left(\sqrt{\widetilde{k}} \pm \sqrt{\widetilde{k}'}\right)\left(\sqrt{\widetilde{k}'}R \pm \sqrt{\widetilde{k}}R'\right)\right] \times \left[\left(\sqrt{\widetilde{k}} \pm \sqrt{\widetilde{k}'}\right)^2 - |\mathbf{k}\pm\mathbf{k}'|\,\mathrm{th}\,|\mathbf{k}\pm\mathbf{k}'|h\right]^{-1}. \tag{91}$$

In these formulae

$$R = \frac{1}{2}(k^2 - \widetilde{k}^2) \qquad R' = \frac{1}{2}(k'^2 - \widetilde{k}'^2) \tag{92}$$

and

$$\widetilde{k} = |\mathbf{k}|\,\mathrm{th}\,|\mathbf{k}|h \qquad \widetilde{k}' = |\mathbf{k}'|\,\mathrm{th}\,|\mathbf{k}'|h. \tag{93}$$

Expression (86) for the mean water level in the case of idealised narrow spectrum $F^{(1)}(\mathbf{k}) = \sigma_0^2\delta(\mathbf{k}-\mathbf{k}_0)$ gives $<\zeta> = -\sigma_0^2 k_0/\,\mathrm{sh}\,2k_0h$ in which $k_0 = |\mathbf{k}_0|$. If a is a slowly changing random wave amplitude, we obtain the well known formula for mean water level setdown $<\zeta> = -\frac{<a^2>}{2}\frac{k_0}{\mathrm{sh}\,2k_0h}$. It can be noted, that for deep water, with $h\to\infty$ the mean value of free surface elevation tends to zero

$$\lim_{h\to\infty} m_{10} = 0. \tag{94}$$

It can be shown, after some algebra, that representation (13) exists for the velocity potential. For special point $\mathbf{x}=\mathbf{0}$ and time $t=0$ we then have

$$\phi(z) = \sum_{i=1}^{N}\beta_i(z)\xi_i + \sum_{i,j=1}^{N}\beta_{ij}(z)\xi_i\xi_j, \tag{95}$$

where

$$\beta_i = \begin{cases} 0 & i = 1,\ldots,N', \\ -\sqrt{\dfrac{g}{\widetilde{k}_i}}\,\dfrac{\mathrm{ch}\,k_i(z+h)}{\mathrm{ch}\,k_ih} & i = N'+1,\ldots,2N', \end{cases} \tag{96}$$

$$\beta_{ij} = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[C_{ij}^-(z) - C_{ij}^+(z)\right] & i = 1,\ldots,N', \quad j = N'+1,\ldots,2N', \\ -\dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[C_{ij}^-(z) + C_{ij}^+(z)\right] & i = N'+1,\ldots,2N', \quad j = 1,\ldots,N', \\ 0 & \text{otherwise}. \end{cases} \tag{97}$$

3.1. Particle Velocity

If we write the orbital velocity as the sum of the linear part and second-order correction term, we have

$$\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)}, \quad \text{where} \quad \mathbf{u}^{(1)} = \nabla\phi^{(1)}, \quad \mathbf{u}^{(2)} = \nabla\phi^{(2)}. \tag{98}$$

Differentation of equations (59) and (72) (for $\mathbf{x} = \mathbf{0}$ and $t = 0$) gives

$$u_\nu(z) = \sum_{i=1}^{N} \beta_i^{u_\nu}(z)\xi_i + \sum_{i,j=1}^{N} \beta_{ij}^{u_\nu}(z)\xi_i\xi_j, \tag{99}$$

where

$$\beta_i^{u_\nu} = \begin{cases} \sqrt{\dfrac{g}{\widetilde{k}_i}}\, k_{i\nu}\, \dfrac{\operatorname{ch} k_i(z+h)}{\operatorname{ch} k_i h} & i = 1,\ldots,N', \\ 0 & i = N'+1,\ldots,2N', \end{cases} \tag{100}$$

for $\nu = 1, 2$

$$\beta_i^{u_3} = \begin{cases} 0 & i = 1,\ldots,N', \\ -\sqrt{g\widetilde{k}_i}\, \dfrac{\operatorname{sh} k_i(z+h)}{\operatorname{sh} k_i h} & i = N'+1,\ldots,2N', \end{cases} \tag{101}$$

$$\beta_{ij}^{u_\nu} = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[(k_{i\nu} - k_{j\nu})C_{ij}^{-}(z) + (k_{i\nu} + k_{j\nu})C_{ij}^{+}(z)\right] \\ \qquad\qquad i,j = 1,\ldots,N', \\ \dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[(k_{i\nu} - k_{j\nu})C_{ij}^{-}(z) - (k_{i\nu} + k_{j\nu})C_{ij}^{+}(z)\right] \\ \qquad\qquad i,j = N'+1,\ldots,2N', \\ 0 \qquad\qquad \text{otherwise}, \end{cases} \tag{102}$$

for $\nu = 1, 2$ and

$$\beta_{ij}^{u_3} = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[\widetilde{k}_{ij}^{-} D_{ij}^{-}(z) - \widetilde{k}_{ij}^{+} D_{ij}^{+}(z)\right] \\ \qquad\qquad i = 1,\ldots N', \quad j = N'+1,\ldots,2N', \\ -\dfrac{1}{2}\sqrt{\dfrac{g}{\widetilde{k}_i\widetilde{k}_j}}\left[\widetilde{k}_{ij}^{-} D_{ij}^{-}(z) + \widetilde{k}_{ij}^{+} D_{ij}^{+}(z)\right] \\ \qquad\qquad i = N'+1,\ldots,2N', \quad j = 1,\ldots,N', \\ 0 \qquad\qquad \text{otherwise}, \end{cases} \tag{103}$$

in which

$$D_{ij}^{\pm}(z) = C_{ij}^{\pm}(z)\, \frac{\operatorname{th} k_{ij}^{\pm}(z+h)}{\operatorname{th} k_{ij}^{\pm} h}. \tag{104}$$

In the above formulae, the subscript ν indicates the vector component, hence, $k_{i\nu}$ $(\nu = 1, 2)$ is equivalent to the notation $k_i = [k_{i1}, k_{i2}]$.

Making use of definitions (73) to (75) and (103), the values of the functions $\beta_{ij}^{u_\nu}(z)$ in the case $i = j$ can be calculated. For $\nu = 1, 2$ we obtain after some algebra

$$\beta_{ii}^{u_\nu} = \begin{cases} \frac{3}{4}\sqrt{g} k_{i\nu} \sqrt{\widetilde{k}_i}\, \Omega \frac{\operatorname{ch} 2k_i(z+h)}{\operatorname{ch} 2k_i h} & i = 1, \ldots, N', \\ -\frac{3}{4}\sqrt{g} k_{i\nu} \sqrt{\widetilde{k}_i}\, \Omega \frac{\operatorname{ch} 2k_i(z+h)}{\operatorname{ch} 2k_i h} & i = N'+1, \ldots, 2N', \end{cases} \tag{105}$$

where

$$\Omega = \frac{1 - \operatorname{th}^4 k_i h}{\operatorname{th}^4 k_i h} \tag{106}$$

and for vertical velocities

$$\beta_{ii}^{u_3} = 0. \tag{107}$$

Now, let us calculate the statistical moments of orbital velocities. From equations (16), taking into account expressions for coefficients α and functions β^{u_ν}, we have

$$m_{01}^{u_\nu} = < u_\nu > = \sum_{i=1}^{N} \beta_{ii}^{u_\nu} V_i = 0, \quad \text{for } \nu = 1, 2, 3, \tag{108}$$

$$\mu_{02}^{u_\nu} = \sum_{i=1}^{N} \beta_i^{u_\nu} \beta_i^{u_\nu} V_i = \sum_{p=1}^{N'} \frac{g}{\widetilde{k}_p} k_{p\nu}^2 \frac{\operatorname{ch} 2k_p(z+h) + 1}{\operatorname{ch} 2k_p h + 1} V_p, \quad \text{for } \nu = 1, 2, \tag{109}$$

$$\mu_{02}^{u_3} = \sum_{i=1}^{N} \beta_i^{u_3} \beta_i^{u_3} V_i = \sum_{p=N'+1}^{2N'} g \widetilde{k}_p \frac{\operatorname{ch} 2k_p(z+h) - 1}{\operatorname{ch} 2k_p h - 1} V_p, \tag{110}$$

$$\mu_{11}^{u_\nu} = \sum_{i=1}^{N} \alpha_i \beta_i^{u_\nu} V_i = \sum_{p=1}^{N'} \sqrt{\frac{g}{\widetilde{k}_p}} k_{p\nu} \frac{\operatorname{ch} k_p(z+h)}{\operatorname{ch} k_p h} V_p, \quad \text{for } \nu = 1, 2, \tag{111}$$

$$\mu_{11}^{u_3} = 0, \tag{112}$$

$$\mu_{03}^{u_\nu} = 6 \sum_{i,j=1}^{2N'} \beta_i^{u_\nu} \beta_j^{u_\nu} \beta_{ij}^{u_\nu} V_i V_j = 3g\sqrt{g} \sum_{p,q=1}^{N'} \frac{k_{p\nu} k_{q\nu}}{\widetilde{k}_p \widetilde{k}_q} \Big[(k_{p\nu} - k_{q\nu})\, C_{pq}^{-}$$

$$+ (k_{p\nu} + k_{q\nu})\, C_{pq}^{+} \Big] \frac{\operatorname{ch} k_p(z+h) \operatorname{ch} k_q(z+h)}{\operatorname{ch} k_p h \, \operatorname{ch} k_q h} V_p V_q, \quad \text{for } \nu = 1, 2, \tag{113}$$

$$\mu_{03}^{u_3} = 6 \sum_{i,j=1}^{N} \beta_i^{u_3} \beta_j^{u_3} \beta_{ij}^{u_3} V_i V_j = 0, \tag{114}$$

$$\begin{aligned}
\mu_{21}^{u_\nu} &= 2 \sum_{i,j=1}^{N} \alpha_i \alpha_j \beta_{ij}^{u_\nu} V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_j^{u_\nu} \alpha_{ij} V_i V_j \\
&= \sum_{p,q=1}^{N'} \sqrt{\frac{g}{\widetilde{k}_p \widetilde{k}_q}} \left[(k_{p\nu} - k_{q\nu}) C_{pq}^- + (k_{p\nu} + k_{q\nu}) C_{pq}^+ \right] V_p V_q \\
&\quad + 2 \sum_{p,q=1}^{N'} \sqrt{\frac{g}{\widetilde{k}_p}} \frac{k_{q\nu}}{\widetilde{k}_q} \left[B_{pq}^- + B_{pq}^+ - \mathbf{k}_p \cdot \mathbf{k}_q + (\widetilde{k}_p + \widetilde{k}_q) \sqrt{\widetilde{k}_p \widetilde{k}_q} \right] \\
&\quad \times \frac{\operatorname{ch} k_q (z+h)}{\operatorname{ch} k_q h} V_p V_q, \quad \nu = 1, 2,
\end{aligned} \tag{115}$$

$$\mu_{21}^{u_3} = 2 \sum_{i,j=1}^{N} \alpha_i \alpha_j \beta_{ij}^{u_3} V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_j^{u_3} \alpha_{ij} V_i V_j = 0, \tag{116}$$

$$\begin{aligned}
\mu_{12}^{u_\nu} &= 2 \sum_{i,j=1}^{N} \alpha_{ij} \beta_i^{u_\nu} \beta_j^{u_\nu} V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_j^{u_\nu} \beta_{ij}^{u_\nu} V_i V_j \\
&= \sum_{p,q=1}^{N'} \sqrt{\frac{g}{\widetilde{k}_p \widetilde{k}_q}} k_{p\nu} k_{q\nu} \left[B_{pq}^- + B_{pq}^+ - \mathbf{k}_p \cdot \mathbf{k}_q \right. \\
&\quad \left. + (\widetilde{k}_p + \widetilde{k}_q) \sqrt{\widetilde{k}_p \widetilde{k}_q} \right] \frac{\operatorname{ch} k_p (z+h) \operatorname{ch} k_q (z+h)}{\operatorname{ch} k_p h \operatorname{ch} k_q h} V_p V_q \\
&\quad + 2 \sum_{p,q=1}^{N'} \frac{g}{\widetilde{k}_q \sqrt{\widetilde{k}_q}} k_{q\nu} \left[(k_{p\nu} - k_{q\nu}) C_{pq}^- + (k_{p\nu} + k_{q\nu}) C_{pq}^+ \right] \\
&\quad \times \frac{\operatorname{ch} k_q (z+h)}{\operatorname{ch} k_q h} V_p V_q
\end{aligned} \tag{117}$$

and

$$\begin{aligned}
\mu_{12}^{u_3} &= 2 \sum_{i,j=1}^{N} \alpha_{ij} \beta_i^{u_3} \beta_j^{u_3} V_i V_j + 4 \sum_{i,j=1}^{N} \alpha_i \beta_j^{u_3} \beta_{ij}^{u_3} V_i V_j \\
&= \sum_{p,q=N'+1}^{2N'} g \left[B_{pq}^- - B_{pq}^+ - \widetilde{k}_p \widetilde{k}_q \right] \frac{\operatorname{sh} k_p (z+h) \operatorname{sh} k_q (z+h)}{\operatorname{sh} k_p h \operatorname{sh} k_q h} V_p V_q
\end{aligned}$$

$$-2\sum_{p=1}^{N'}\sum_{q=N'+1}^{2N'}\frac{g}{\sqrt{\widetilde{k}_p}}\left[\widetilde{k}^-_{pq}\,D^-_{pq}(z)-\widetilde{k}^+_{pq}\,D^+_{pq}(z)\right]\frac{\operatorname{sh}k_q(z+h)}{\operatorname{sh}k_q h}\,V_pV_q. \quad (118)$$

Rewriting the above formulae in integral form, and using (65), we obtain

$$m_{01}^{u_\nu}=0, \quad \text{for } \nu=1,2,3, \quad (119)$$

$$\mu_{02}^{u_\nu}=g\int_{\mathbf{k}}\frac{k_\nu^2}{\widetilde{\widetilde{k}}}\,\frac{\operatorname{ch}2k(z+h)+1}{\operatorname{ch}2kh+1}\,F^{(1)}(\mathbf{k})\,d\mathbf{k}, \quad \nu=1,2, \quad (120)$$

$$\mu_{02}^{u_3}=g\int_{\mathbf{k}}\widetilde{k}\,\frac{\operatorname{ch}2k(z+h)-1}{\operatorname{ch}2kh-1}\,F^{(1)}(\mathbf{k})\,d\mathbf{k}, \quad (121)$$

$$\mu_{11}^{u_\nu}=\int_{\mathbf{k}}\sqrt{\frac{g}{\widetilde{\widetilde{k}}}}k_\nu\,\frac{\operatorname{ch}k(z+h)}{\operatorname{ch}kh}\,F^{(1)}(\mathbf{k})\,d\mathbf{k}, \quad \nu=1,2, \quad (122)$$

$$\mu_{11}^{u_3}=0. \quad (123)$$

For statistical moments of the 3rd order we can write

$$\mu_{mn}^{u_\nu}=\int_{\mathbf{k}}\int_{\mathbf{k}'}K_{mn}^{u_\nu}(\mathbf{k},\mathbf{k}';z)F^{(1)}(\mathbf{k})\,F^{(1)}(\mathbf{k}')\,d\mathbf{k}\,d\mathbf{k}', \quad (124)$$

where $m,n=0,\ldots,3$ and $m+n=3$, and where functions $K_{mn}^{u_\nu}$ can be written in the following form:

$$K_{03}^{u_\nu}(\mathbf{k},\mathbf{k}';z)=3g\sqrt{g}\,\frac{k_\nu k'_\nu}{\widetilde{\widetilde{kk'}}}\left[(k_\nu-k'_\nu)\,C^-(\mathbf{k},\mathbf{k}';z)+(k_\nu+k'_\nu)\,C^+(\mathbf{k},\mathbf{k}';z)\right]$$
$$\times\frac{\operatorname{ch}k(z+h)\operatorname{ch}k'(z+h)}{\operatorname{ch}kh\operatorname{ch}k'h}, \quad \text{for } \nu=1,2, \quad (125)$$

in which $\mathbf{k}=[k_1,k_2]$ and

$$C^\pm(\mathbf{k},\mathbf{k}';z)=\frac{B^\pm(\mathbf{k},\mathbf{k}')}{\sqrt{\widetilde{k}}\pm\sqrt{\widetilde{k}'}}\,\frac{\operatorname{ch}k^\pm(z+h)}{\operatorname{ch}k^\pm h} \quad (126)$$

and

$$k^\pm=|\mathbf{k}^\pm|=|\mathbf{k}\pm\mathbf{k}'|. \quad (127)$$

Next

$$K_{03}^{u_3}=0, \quad (128)$$

$$K_{21}^{u_\nu}(\mathbf{k},\mathbf{k}';z) = \sqrt{\frac{g}{\widetilde{kk'}}}\left\{(k_\nu - k'_\nu)\,C^-(\mathbf{k},\mathbf{k}';z) + (k_\nu + k'_\nu)\,C^+(\mathbf{k},\mathbf{k}';z)\right.$$

$$\left. + \frac{2k'_\nu}{\sqrt{\widetilde{k}'}}\,B(\mathbf{k},\mathbf{k}')\,\frac{\operatorname{ch} k'(z+h)}{\operatorname{ch} k'h}\right\}, \quad \text{for } \nu = 1, 2, \tag{129}$$

$$K_{21}^{u_3} = 0, \tag{130}$$

$$K_{12}^{u_\nu}(\mathbf{k},\mathbf{k}';z) = \frac{gk'_\nu}{\widetilde{k}'\sqrt{\widetilde{k}}}\,\frac{\operatorname{ch} k'(z+h)}{\operatorname{ch} k'h}\left\{\frac{k_\nu}{\sqrt{\widetilde{k}}}\,B(\mathbf{k},\mathbf{k}')\,\frac{\operatorname{ch} k(z+h)}{\operatorname{ch} kh}\right.$$

$$\left. + 2\left[(k_\nu - k'_\nu)\,C^-(\mathbf{k},\mathbf{k}';z) + (k_\nu + k'_\nu)\,C^+(\mathbf{k},\mathbf{k}';z)\right]\right\},$$

$$\text{for } \nu = 1, 2, \tag{131}$$

$$K_{12}^{u_3}(\mathbf{k},\mathbf{k}';z) = g\,\frac{\operatorname{sh} k'(z+h)}{\operatorname{sh} k'h}\left\{\left[B^-(\mathbf{k},\mathbf{k}') - B^+(\mathbf{k},\mathbf{k}') - \widetilde{kk'}\right]\right.$$

$$\times\frac{\operatorname{sh} k(z+h)}{\operatorname{sh} kh} - \frac{2}{\sqrt{\widetilde{k}}}\left[|\mathbf{k}-\mathbf{k}'|\operatorname{th}|\mathbf{k}-\mathbf{k}'|h\,D^-(\mathbf{k},\mathbf{k}';z)\right.$$

$$\left.\left. - |\mathbf{k}+\mathbf{k}'|\operatorname{th}|\mathbf{k}+\mathbf{k}'|h\,D^+(\mathbf{k},\mathbf{k}';z)\right]\right\}, \tag{132}$$

where functions D^+ and D^- are defined as follows:

$$D^\pm(\mathbf{k},\mathbf{k}';z) = C^\pm(\mathbf{k},\mathbf{k}';z)\,\frac{\operatorname{ch} k^\pm(z+h)}{\operatorname{ch} k^\pm h}. \tag{133}$$

It can be seen from the above formulae that the *weak non-linearities* (within the frame of the adopted approximation) do not affect the mean values $m_{01}^{u_\nu}$ of particle velocities, which remain equal to zero (equation (119)). Variances $\mu_{02}^{u_\nu}$ given by (120) and (121) of the orbital velocities are left unchanged (see also Appendix A). However, the non-linearity of the motion leads to non-zero skewness for horizontal velocity (see $\mu_{03}^{u_\nu}$ for $\nu = 1, 2$ given by (124) and (125)). Skewness of the vertical component still remains equal to zero (see $\mu_{03}^{u_3}$ given by (124) and (130)).

The influence of the *emergence effect* on the orbital velocities, as given in (5), however, leads to a modification of the probability density for the velocity from that expressed by the equation (20) to the form given by (32). It results also in a non-zero mean value given by equation (45) for the horizontal velocities and in modified variances (47) for both horizontal and vertical velocity components. It should be noted from (45) that the mean value of the vertical component remains equal to zero in view of (123) and (130).

The above analysis yields the conclusion that the effectiveness and accuracy of evaluation of statistical properties of velocities depends very strongly on the spectral density function of surface elevation. Therefore, in Part 2 where measured data are examined, the spectral analysis will be described in some detail.

4. Mass Transport

In this section the total mean flux in the first order approximation will be considered. We will operate in the Eulerian frame. Although Tung's (1975) results clearly show a positive mean value of the horizontal orbital velocity in the near surface zone, to the authors' knowledge, this has not been discussed and interpreted as a current induced by waves until the recent study of Cieślikiewicz and Gudmestad (1994a). In Part 2 a comparison of velocity profile measured in a wave flume and a theoretically predicted mean velocity profile will be presented. Agreements are good for points situated *above* mean water level. Mean values of the horizontal component measured *below* the mean water level were influenced by the return current in the wave flume that appeared to be non-uniformly distributed along the vertical axis. It will be suggested here that the return flow noticed in the wave flume is induced by the current corresponding to the mean velocity profile obtained by taking into account the emergence effect.

The results of this section are obtained from the linear wave theory but they are non-linear quantities in the sense that they involve the wave amplitude to the second power. Taking into account the emergence effect consists in considering a modified velocity $\overline{\mathbf{u}}(\mathbf{x}, z, t)$ defined by (5) rather than the velocity $\mathbf{u}(\mathbf{x}, z, t)$. Equation (5) makes relation between $\overline{\mathbf{u}}$ and ζ non-linear even though the relation between $\mathbf{u}$ and ζ is linearized. This kind of non-linearity is due to surface effects and determines the character of the quantities derived. This non-linearity is mainly important in the vicinity of the mean water level.

As stated above, in this section the formula for total mean flux for linear random wave theory modified by taking into account the emergence effect is derived, but in order to document that the statistical mean value of horizontal velocity can be interpreted as the mass transport velocity, in the next subsection the same approach as for random waves will be used in the case of deterministic small-amplitude wave. It will be shown that this leads to known and well-interpreted formulae for mass flux usually obtained in the Lagrangian frame.

4.1. Digression on a Simple Harmonic Wave in a Deterministic Case

In this section the mean flux for a linear deterministic wave will be calculated. Consider a unidirectional progressive small-amplitude wave of the form

$$\zeta(x,t) = a\cos(kx - \omega t), \tag{134}$$

where a is the wave amplitude, k is the wave number related to the angular frequency of the wave ω by the dispersion relation

$$\omega^2 = gk \operatorname{th} kh, \tag{135}$$

in which h is the water depth that is constant by assumption. The associated horizontal velocity under the wave is given by

$$u(x,z,t) = \frac{gak}{\omega}\frac{\operatorname{ch} k(z+h)}{\operatorname{ch} kh}\cos(kx - \omega t), \quad \text{for } z \in [-h, \zeta(x,t)]. \tag{136}$$

Introduce extension of u on the z-domain $[-h, \infty)$ by the definition

$$\bar{u}(x,z,t) = \begin{cases} u(x,z,t) & \text{for} \quad z \le \zeta(x,t), \\ 0 & \text{for} \quad z > \zeta(x,t). \end{cases} \tag{137}$$

The mean value of u over a wave period T of a deterministic wave is

$$m_{\bar{u}}^{(\mathrm{d})}(z) = \langle \bar{u}(x,z,t) \rangle^{(d)} = \frac{1}{T}\int_{-T/2}^{T/2} \bar{u}(x,z,t)\,dt, \tag{138}$$

where the symbol $\langle\cdot\rangle^{(\mathrm{d})}$ denotes the mean value over a wave period of the deterministic function enclosed in the bracket. In view of (136) and (137)

$$m_{\bar{u}}^{(\mathrm{d})}(z) = \begin{cases} \dfrac{1}{T}\displaystyle\int_{t_1(z)}^{t_2(z)} u(x,z,t)\,dt & \text{for} \quad |z| \le a, \\ 0 & \text{for} \quad |z| > a, \end{cases} \tag{139}$$

where t_1 and t_2 are such that

$$\begin{cases} z = \zeta(x,t_1) = \zeta(x,t_2), \\ z \le \zeta(x,t), \quad \text{for } t_1 \le t \le t_2. \end{cases} \tag{140}$$

Carrying out the integration in (139) yields

$$m_{\bar{u}}^{(\mathrm{d})}(z) = \begin{cases} \dfrac{gak}{\pi\omega}\dfrac{\operatorname{ch} k(z+h)}{\operatorname{ch} kh}\sin\left(\arccos\dfrac{z}{a}\right) & \text{for} \quad |z| \le a, \\ 0 & \text{for} \quad |z| > a. \end{cases} \tag{141}$$

Note that for $z = 0$ we obtain $m_{\overline{u}}^{(\mathrm{d})}(0) = |u(x, 0, t)|/\pi$ which is "almost" the maximum value for $m_{\overline{u}}^{(\mathrm{d})}(z)$ since for $|z| \leq a$ the approximate value of $\operatorname{ch} k(z + h)/\operatorname{ch} kh \simeq 1$. Figure 2 presents the mean velocity profile given by (141).[1]

To obtain the total mean flux $q^{(\mathrm{d})}$ at a fixed position x (Eulerian frame) we perform the following integration

$$q^{(d)} = \int_{-h}^{\zeta(x,t)} m_{\overline{u}}^{(\mathrm{d})}(z)\, dz. \tag{142}$$

In view of (141)

$$q^{(\mathrm{d})} = \int_{-a}^{a} \frac{gak}{\pi\omega} \frac{\operatorname{ch} k(z+h)}{\operatorname{ch} kh} \sin\left(\arccos\frac{z}{a}\right) dz. \tag{143}$$

By substituting $\theta = \arccos\frac{z}{a}$ we obtain

$$q^{(\mathrm{d})} = \frac{gak}{2\pi\omega \operatorname{ch} kh} \int_{0}^{\pi} \left(e^{k(h+a\cos\theta)} + e^{-k(h+a\cos\theta)}\right) \sin^2\theta\, d\theta. \tag{144}$$

In view of the following integral representation for the modified Bessel function $I_\nu(f)$ for any integer ν

$$I_\nu(f) = \frac{\left(\frac{f}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{0}^{\pi} e^{\pm f\cos\theta} \sin^{2\nu}\theta\, d\theta \tag{145}$$

the integral $q^{(\mathrm{d})}$ may be finally written as

$$q^{(\mathrm{d})} = \frac{ga}{\omega} I_1(ak). \tag{146}$$

Thus, the flow of mass $M^{(\mathrm{d})} = \rho q^{(\mathrm{d})}$ is equal to

$$M^{(\mathrm{d})} = \frac{\rho g a}{\omega} I_1(ak), \tag{147}$$

where ρ is the water density.

For the modified Bessel function $I_\nu(f)$ the following series expansion holds true

$$I_\nu(f) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{f}{2}\right)^{\nu+2n}. \tag{148}$$

[1]The mean velocity profile presented in this figure corresponds to the experimental data of I18 random wave case described in Part 2 of this paper. Namely, for the significant height H_S and the peak period T_p taken from Table 1 of Part 2, $a = H_S/2$, $\omega = 2\pi/T_p$, $h = 1.3$ m and k is obtained from the dispersion relation (135).

Therefore, the flow of mass $M^{(d)}$ with an accuracy to the second order in a is equal to

$$M^{(d)} = \frac{\rho g a}{\omega} \frac{ak}{2} \left[1 + \frac{1}{2} \left(\frac{ak}{2} \right)^2 + \cdots \right] \simeq \frac{\rho g a^2}{2} \frac{k}{\omega} = \frac{E}{C}, \tag{149}$$

a result first presented by Starr (1947) where E is the average energy per unit surface area and C is the phase velocity. Note that the above approximation may also be easily obtained directly from (143) by assuming that $\operatorname{ch} k(z+h)/\operatorname{ch} kh \simeq 1$ for $|z| \le a$ (see Figures 2 and 3).

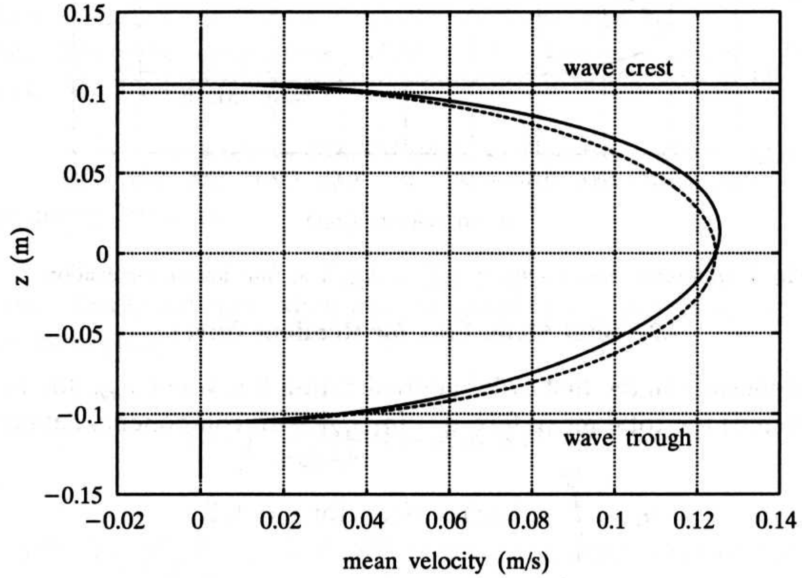

Fig. 2. Theoretical mean velocity profile for deterministic small-amplitude wave; – – – – with an approximation $\operatorname{ch} k(z+h)/\operatorname{ch} kh \simeq 1$

In equation (149) we have obtained, in the Eulerian frame, the classical second-order Stokes expression for the total mean flux of a small-amplitude wave train obtained in the Lagrangian frame. As mentioned above, this expression was first developed by Starr (1947) but in a very different way, starting from the integral $\int_0^L \int_{-h}^{\zeta} (u_x^2 + u_z^2)\, dz\, dx$ in the moving frame and showing, by the use of Green's theorem, its equality to $\int_{-h}^{\zeta} u\, dz$ at an arbitrary vertical. The present approach is direct in the sense that we have used only the definition of the total mean flux and the expression for horizontal velocity.

The objective of this analysis of the deterministic small-amplitude wave case has been to document that the non-zero mean value of the horizontal velocity introduced by taking into account the emergence effect should be treated as a *mass flux velocity*.

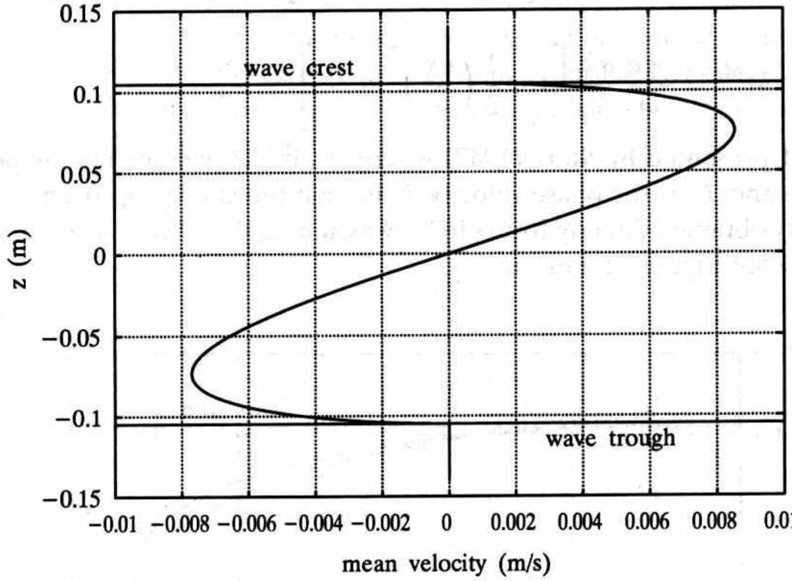

Fig. 3. Difference $m_{\overline{u}}(z) - m_{\overline{u}}(z)|_{\mathrm{ch}\,k(z+h)/\mathrm{ch}\,kh \simeq 1}$ as function of z-elevation

4.2. Total Mean Flux for Random Waves

Now, let us consider in the first order approximation (i.e. when λ_{mn} for $m+n = 3$ are neglected) the total mean flux $\mathbf{q} = [q_1, q_2]$ with components defined as

$$q_\nu = \int_{-h}^{\infty} < \bar{u}_\nu(z) > dz, \quad \text{for } \nu = 1, 2. \tag{150}$$

By using equations (49) and (122) the above integrals may be rewritten as follows

$$\begin{aligned} q_\nu &= \frac{1}{\sigma_\zeta} \int_{-h}^{\infty} Z(z') \left[\int_{\mathbf{k}} \sqrt{\frac{g}{k}} k_\nu \frac{\mathrm{ch}\, k(z+h)}{\mathrm{ch}\, kh} F^{(1)}(\mathbf{k})\, d\mathbf{k} \right] dz \\ &= \frac{1}{\sigma_\zeta} \int_{\mathbf{k}} \sqrt{\frac{g}{k}} \frac{k_\nu}{\mathrm{ch}\, kh} F^{(1)}(\mathbf{k}) \left[\int_{-h}^{\infty} \mathrm{ch}\, k(z+h) Z(z') dz \right] d\mathbf{k}, \quad \text{for } \nu = 1, 2, \end{aligned} \tag{151}$$

in which $\sigma_\zeta = \sqrt{\mu_{20}}$ is the standard deviation of the surface elevation ζ. The integration over z can be performed without difficulty resulting in the following form of the total mean flux q_ν

$$q_\nu = \int_{\mathbf{k}} \sqrt{\frac{g}{k}} \frac{k_\nu}{\mathrm{ch}\, kh} W(-h; \sigma_\zeta, k)\, F^{(1)}(\mathbf{k})\, d\mathbf{k}, \tag{152}$$

where the function W is defined as

$$W(z_a;\sigma,k) = \int_{z_a/\sigma}^{\infty} \operatorname{ch} k(z+h) Z(z)\, dz$$

$$= \frac{1}{2}\exp\left[\frac{\sigma^2 k^2}{2}\right]\left\{e^{kh}\, Q(z_a/\sigma + k\sigma) + e^{-kh}\, Q(z_a/\sigma - k\sigma)\right\}. \qquad (153)$$

The more general form of the function $W(z_a, z_b; \sigma, k) = \int_{z_a/\sigma}^{z_b/\sigma} \operatorname{ch} k(z+h) Z(z)\, dz$ can be found in the paper by Cieślikiewicz and Massel (1988) and may be used if need be to calculate the flux between elevations z_a and z_b.

If we take polar co-ordinates (k, θ) in the $\mathbf{k}$-plane, we can introduce the directional spectrum $\hat{F}(\omega, \theta)$ by

$$F(\mathbf{k})\, d\mathbf{k} = F(k,\theta) k\, dk\, d\theta = \hat{F}(\omega,\theta)\, d\omega\, d\theta \qquad (154)$$

and the dispersion relation

$$\omega^2(k) = gk \operatorname{th} kh. \qquad (155)$$

In the one-directional case, when $\hat{F}(\omega, \theta)$ vanishes everywhere except for $\theta = \theta_0$, we have from (152)

$$q = \int_0^{\infty} \frac{gk}{\omega \operatorname{ch} kh} W(-h; \sigma_\zeta, k) S^{(1)}(\omega)\, d\omega, \qquad (156)$$

where $S^{(1)}(\omega) = \int_{-\pi}^{\pi} \hat{F}^{(1)}(\omega,\theta)\, d\omega\, d\theta$ denotes the linear part of the frequency spectrum $S(\omega) = \int_{-\pi}^{\pi} \hat{F}(\omega,\theta)\, d\omega\, d\theta$. In the integral (156) the wave number k is related to the angular frequency ω by the dispersion relation (155).

In the assumed approximation, the difference between $S(\omega)$ and $S^{(1)}(\omega)$ can be neglected (see Appendix A). Assuming further that

1° $k_p \sigma_\zeta \ll 1$ (where k_p is the wave number corresponding to the peak frequency) and that the spectrum $S(\omega)$ decays quickly enough for $\omega \to \infty$;

2° the water depth h is large enough (in practice it is sufficient to have $h > 3\sigma_\zeta$)

one can set in (156) $Q(-h' \pm k\sigma_\zeta) \simeq 1$ $(h' = h/\sigma_\zeta)$. Thus, the total mean flux obtains the following approximate form

$$q \simeq \int_0^{\infty} \frac{gk}{\omega} \exp\left[\frac{\sigma_\zeta^2 k^2}{2}\right] S(\omega)\, d\omega. \qquad (157)$$

Note that the above expression is accurate for infinite water depth $h \to \infty$. Because it is assumed that $\sigma_\zeta k$ has a small value for k such that the corresponding ω (through the dispersion relation (155)) gives a value of $S(\omega)$ which is not infinitely small, we can find an even simpler form of (157). Expanding $\exp(\cdot)$ into Taylor series gives

$$\exp\left[\frac{\sigma_\zeta^2 k^2}{2}\right] = 1 + \frac{\sigma_\zeta^2 k^2}{4} + \cdots \simeq 1, \tag{158}$$

thus

$$q \simeq \int_0^\infty \frac{gk}{\omega} S(\omega)\, d\omega, \tag{159}$$

which is the result obtained by Phillips (1960).

Note that by (122), the approximate total mean flux (159) can be written as

$$q \simeq \mu_{11}^u \,|_{z=0} = < \zeta u(x,0,t) >, \tag{160}$$

which is of a form analogous to that for the case of deterministic small-amplitude waves $q \simeq \overline{\zeta u(x,0,t)}$ (see Phillips 1977), where the bar denotes the mean value over a wave period.

For deep water waves (159) can be rewritten as $q = \int_0^\infty \omega S(\omega)\, d\omega$ which is the value of the spectral moment of the first order m_1.

Narrow-band Spectrum

Assume the following spectral density as an idealised narrow spectrum:

$$S(\omega) = \sigma_0^2 \delta(\omega - \omega_0), \tag{161}$$

σ_0^2 is the free surface variance. In application, the spectrum has small, but finite width, hence, (161) is an idealisation. For the spectrum (161) the approximate value of the total mean flux (159) obtains the form:

$$q \simeq \frac{gk_0}{\omega_0} \sigma_0^2, \tag{162}$$

where ω_0 and k_0 satisfy the dispersion relation (155). If a is slowly changing random wave amplitude, the following relation holds true

$$\sigma_0^2 = \frac{< a^2 >}{2}. \tag{163}$$

Therefore, equation (162) may be rewritten as

$$q \simeq \frac{g < a^2 >}{2} \frac{k_0}{\omega_0}. \tag{164}$$

Thus, total mean mass flux $M = \rho q$ is

$$M \simeq \frac{\rho g < a^2 > }{2} \frac{k_0}{\omega_0} = \frac{E}{C_0}, \tag{165}$$

where C_0 is the wave celerity.

It can be noticed that the above expression corresponds to formula (149) for deterministic small-amplitude waves. The correspondence is even higher than to the lowest order since the higher order term in (158), $\sigma_\zeta^2 k^2/4$, by use of equations (161) and (163) can be recognized in equation (149).

5. Conclusions

It is proposed that the values of a stationary and homogeneous non-linear random process can be expressed as a series of random functions while the upper boundary for which the process has physical values is expressed as another series of random variables. Through this proposal the expressions for general random fields can be applied to the process. Defining the modified water wave velocities of a wave field as the kinematics below the surface of the waves and zero above the surface, the theory and expressions for random fields can then be applied to water wave kinematics.

Starting from the basic equations of hydrodynamics, the statistical properties of wave kinematics can be expressed in terms of the linear part of the surface elevation spectrum and there is no need to make assumptions regarding the analytical expressions for the wave kinematics. It should be noted that this allows the prediction of the statistics of wave kinematics knowing only the parameters or the data that are required to establish the wave surface elevation spectrum.

The probability density function for modified particle velocity determined in this study appeared to be non-zero mean and skewed in the case of horizontal component, whilst zero mean and unskewed for vertical velocity.

An attempt is made in this paper to document that the non-zero mean velocity (in the Eulerian frame) resulting from including the emergence effect should be treated as a mass transport velocity. This is done by showing that the formulae obtained for the total mean flux in approximation lead to the known and well-interpreted formulae for mass flux usually obtained in the Lagrangian frame.

An alternative method for derivation of the mean water flux in the region near the mean water level is presented. The relevant formulae are developed in the Eulerian frame for random water waves. The approximate value of the total mean flux was previously known (Phillips 1960) but the approach presented in this paper is in the authors' opinion more direct in the sense that only the definition of the total mean flux and the expression for the horizontal particle velocity is needed. Moreover, traditional approaches allow us to treat only the *total flux* as a physical quantity "existing on a subset of zero measure", namely, *exactly* on the free surface of the wave. With such an approach we are not able, in the Eulerian frame, to discuss the distribution of the mean velocity in the free surface zone. We would then have the situation that the mean velocity along the vertical is everywhere equal to zero except at the free surface. In the present approach the mean velocity is "stretched out" from the exact location on the surface onto the *free surface zone*. More precisely, for the random wave case, theoretically a non-zero mean horizontal velocity should exist from $z = -h$ to infinity (due to the Gaussian distribution for free surface elevation, this zone should, however, in practice, be treated as the region near the mean water level). Moreover, we are able to calculate not only the total mean water flux but also the flux between two given z-elevations.

The theoretical results of this study are verified with measurements taken in a wave tank. This is reported in Part 2, where an attempt is also made to estimate the return current in the wave flume in Eulerian frame by taking into account the emergence effect.

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Appendix A

As mentioned in section 2, the random field of the free surface elevation $\zeta(\mathbf{x}, t)$ may be presented by equation (6). For this stationary and homogeneous random field, on the other hand, there exists the following Stieltjes-Fourier representation

$$\zeta(\mathbf{x}, t) = \int_{\mathbf{k}} dA(\mathbf{k}) \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t)\right], \tag{A1}$$

in which $dA(\mathbf{k})$ is a complex random field. The representation given by equation (A1) enables the defining the continuous spectral density $F(\mathbf{k})$ by

$$< dA(\mathbf{k})\, dA^*(\mathbf{k}') >= F(\mathbf{k})\, \delta(\mathbf{k} - \mathbf{k}')\, d\mathbf{k}\, d\mathbf{k}', \tag{A2}$$

where the asterisk denotes complex conjugate.

The spectrum $F(\mathbf{k})$ contains energy associated with both the first and the higher order, forced components (see Tick 1959). Thus we can write

$$F(\mathbf{k}) = F^{(1)}(\mathbf{k}) + F^{(2)}(\mathbf{k}) + \cdots, \tag{A3}$$

where $F^{(1)}(\mathbf{k})$ and $F^{(2)}(\mathbf{k})$ are the first and second order parts of the spectrum $F(\mathbf{k})$, respectively.

As was presented in section 3, for the free surface displacement ζ, the random variables ξ_i in equation (6), for say, $i = 1, \ldots, N = 2N'$ are specified according to Longuet-Higgins (1963) such that

— ξ_i can be divided into two groups: $i = 1, \ldots, N'$ and $i = N'+1, \ldots, N$

— their variances $V_i = \langle \xi_i^2 \rangle$ are

$$V_i = \frac{1}{2} \begin{cases} \langle a_i^2 \rangle & \text{for } i = 1, \ldots, N', \\ \langle a_{i-N'}^2 \rangle & \text{for } i = N'+1, \ldots, N, \end{cases} \tag{A4}$$

where a_i, $i = 1, \ldots, N'$, associated with wavenumber $\mathbf{k}_i$, are the first order amplitudes of ζ.

According to Longuet-Higgins (1963), the relation between the variances V_i and the spectral density $F(\mathbf{k})$ is such that when $N \to \infty$ each $V_i \to 0$ so that

$$F^{(1)}(\mathbf{k})\, d\mathbf{k} = \sum_{\substack{\mathbf{k}_i \in d\mathbf{k} \\ i \in \{1,\ldots,N'\}}} V_i = \sum_{\substack{\mathbf{k}_i \in d\mathbf{k} \\ i \in \{N'+1,\ldots,N\}}} V_i \tag{A5}$$

over any small but fixed region in the $\mathbf{k}$-plane. This means that when $N \to \infty$ (and $N' \to \infty$)

$$\int_{\mathbf{k}} F^{(1)}(\mathbf{k})\, d\mathbf{k} = \sum_{i=1}^{N'} V_i = \sum_{i=N'+1}^{N} V_i. \tag{A6}$$

Where series $\sum_i \ldots V_i$ appear, one can then obtain integrals by

$$\sum_{i=1}^{N'} \ldots V_i, \ \sum_{i=N'+1}^{N} \ldots V_i \ \underset{\substack{N' \to \infty \\ \text{or } N \to \infty}}{\longrightarrow} \int_{\mathbf{k}} \ldots F^{(1)}(\mathbf{k})\, d\mathbf{k}. \tag{A7}$$

The first three statistical moments of the random variable X will thereafter be discussed. It can be shown that these moments can be expressed in terms of V_i as follows:

$$\left. \begin{aligned} m_1 &= \sum_{i=1}^{N} \alpha_{ii} V_i + \cdots \\ \mu_2 &= \sum_{i=1}^{N} \alpha_i \alpha_i V_i + 2 \sum_{i,j=1}^{N} \alpha_{ij} \alpha_{ij} V_i V_j + \cdots \\ \mu_3 &= 6 \sum_{i,j=1}^{N} \alpha_i \alpha_j \alpha_{ij} V_i V_j + \cdots \end{aligned} \right\}. \tag{A8}$$

In a first order approximation one obtains

$$m_1 = 0, \quad \mu_2 = \sum_{i=1}^{N} \alpha_i \alpha_i V_i, \quad \mu_3 = 0. \tag{A9}$$

The next approximation is given by equations (15) which means that, in this approximation, terms of the order V^2 can be neglected when compared with terms which are of the order V.

By the assumption of stationarity and homogeneity one can, in the case when X is specified as the free surface elevation, consider the special point $\mathbf{x} = \mathbf{0}$ and time $t = 0$. For this case Longuet-Higgins (1963) has shown that

$$\alpha_i = \begin{cases} 1 & \text{for } i = 1, \ldots, N', \\ 0 & \text{for } i = N' + 1, \ldots, N. \end{cases} \tag{A10}$$

Thus, in view of (15) and (A7) in the second order

$$\mu_2 = \sum_{i=1}^{N'} V_i = \int_{\mathbf{k}} F^{(1)}(\mathbf{k})\, d\mathbf{k}, \tag{A11}$$

when $N' \to \infty$.

On the other hand, it is obvious that $\mu_2 = \int_{\mathbf{k}} F(\mathbf{k})\, d\mathbf{k}$ which means that the difference between the true value of the variance μ_2 and the calculated value as given by equation (A11) in view of equation (A8) is of the order

$$\int_{\mathbf{k}} F^{(2)}(\mathbf{k})\, d\mathbf{k} = \sum_{i,j=1}^{N'} \Omega_{ij} V_i V_j, \tag{A12}$$

where Ω_{ij} is a certain constant associated with the pair of wavenumber vectors $(\mathbf{k}_i, \mathbf{k}_j)$. In order to obtain the parameters of the probability distribution given by equation (32) (which are primarily given in the form of series $\sum_i \ldots V_i$, $\sum_{i,j} \ldots V_i V_j$), equation (A7) demonstrates that the linear part of the spectrum $F^{(1)}(\mathbf{k})$ must be known. In the authors' opinion, however, there is no need to split the spectrum into its first and second order parts for practical calculations. Within the assumed accuracy, the linear part of the spectrum can be approximated by the full spectrum. This is consistent with Longuet-Higgins' (1963) second order approximation given by equation (15) but contrary to Anastasiou *et al.* (1982b). In fact, under such an assumption (i.e. using $F(\mathbf{k})$ instead of $F^{(1)}(\mathbf{k})$), the first two statistical moments, which are of the order V, are obtained with an error of the order of V^2 (while for the free surface elevation ζ the "ideal" variance is obtained), and third moments, being of the order V^2, are calculated with an error of the order of V^4.