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*ANDRZEJ WITA, AGNIESZKA CZACHOWSKA-AUSSENBERG**

Optimal Ground Water Pumping Model

Abstract

The paper presents possibilities of application of linear programming to optimize water pumping from an aquifer using vertical wells. An optimization model is described, in which so-called "response coefficients" characterizing lowering of the ground water table in some points of the aquifer, resulting from unit water discharge from individual wells, are used. The methods of calculating response coefficients are given for both confined and unconfined ground water flow. The objective function and constraints for the problem of minimization of total discharge from a group of wells dewatering an excavation have been formulated. The results of application of the optimization model to the problem of dry dock dewatering are presented. To verify the optimal solution resulting from the model, a forecast of ground water level changes has been calculated.

Key words: ground water flow, mathematical model, optimization, linear programming, response matrix method

***A. WITA, A. CZACHOWSKA-AUSSENBERG, Institute of Meteorology and Water Management, Warsaw, Poland.**

1. Introduction

An important development in the field of application of numerical simulation models to hydrology has been achieved in recent years. Simulation models are used with increasing frequency to study possibilities to attain a desired water management goal. Direct application of simulation models to such cases requires repeated simulation under different manners of operation of existing or designed objects impacting the conditions of ground water flow. A natural consequence of ground water systems modelling development is the tendency to use standard optimization methods such as linear and nonlinear programming. For the problem of ground water flow an objective function and constraints must be formulated; the constraints can be created using mathematical models, analytical solutions etc. Thus we deal with a substantially new situation – a model of optimal management of ground water flow. Two basic methods of development of such models, using linear programming, may be distinguished.

The first one, called "embedding method" (Jones et al. 1987), consist in incorporation of the system of hydraulic equations describing the ground water flow into the optimization model. Linear algebraic equations resulting from the application of the method of finite difference or finite element are in most cases incorporated as constraints into the optimization model. The quantities of discharge or recharge at grid nodes of the hydrological model usually become decision variables. Such a model is composed of an objective function and a set of hundreds or even thousands of constraints according to the number of space nodes and time steps in the time interval considered.

The second method of modelling, called "response matrix method", consists in determination of so-called response coefficients characterizing changes of water level in some points of the aquifer, resulting from water pumping from or recharging in individual wells with a unit pumping rate. The response coefficients can be defined using analytical solutions (if existing) or obtained from numerical simulation or determined by *in situ* experiments (Cicioni et al. 1982). The coefficients are only calculated once and used to determine the objective function and basic constraints. The quantities of water pumping or recharge for each well are decision variables in the model of optimal management of ground water flow. Special points, called also critical points, are arbitrarily chosen. One should be aware that the optimal solution of the problem depends on the number and localisation of the critical points. The number of constraints depends on the number of critical points, but when comparing with the first method the number is usually considerably smaller. Instead of "response matrix" the notions of "sensitivity matrix" (Jones et al. 1987, Schwartz 1977) and "algebraic technological function" (Maddock 1972) are used.

In this paper some exemplary problems of linear programming for ground water flow are formulated. Next, are presented methods of response coefficients calculation using analytical solutions for steady and unsteady, confined and unconfined ground water flow. The results of calculation using the response matrix method for the case of excavation dewatering optimization are further presented.

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2. Linear Programming in Problems of Ground Water Flow

Let us consider the situations presented in Figures 1 and 2. There is a group of NW wells and NP critical points. The optimization problem consists in finding the discharge (or recharge) for each well provided the constraints concerning a drawdown (or ground water level) are satisfied in all critical points and the total discharge (or recharge) of all wells attains minimum (or maximum) value. The linear programming problem for this case is as follows:

$$F = \sum_{j=1}^{NW} Q_j = \text{max or min}; \qquad (1)$$

with constraints:

$$s_k \leq s_{k\,\max}, \qquad k = 1, 2, \ldots, NP \qquad (2$$

or/and

$$s_k \geq s_{k\,\min}, \qquad k = 1, 2, \ldots, NP \qquad (3$$

and

$$Q_{j\,\min} \leq Q_j \leq Q_{j\,\max}, \qquad j = 1, 2, \ldots, NW \qquad (4$$

where:

F	–	objective function (total discharge of all wells),
Q_j	–	discharge (or recharge) value for well number j,
s_k	–	drawdown in critical point number k,
$s_{k\,\max}, s_{k\,\min}$	–	maximum and minimum admissible value of drawdown in point number k,
$Q_{j\,\max}, Q_{j\,\min}$	–	maximum and minimum admissible value of discharge for well number j.

The objective function (1) should be maximized, if a problem of intensive exploı tation of a water intake and protection of some area (e.g. valuable from the environmental point of view) against dewatering is put forward. If a problem of some area (e.g. civil engineering excavation) dewatering is considered, the objective function should be minimized. The constraints (2) in the first and (3) in the latter case must be taken into consideration. The constraints (4) express in both cases that discharges

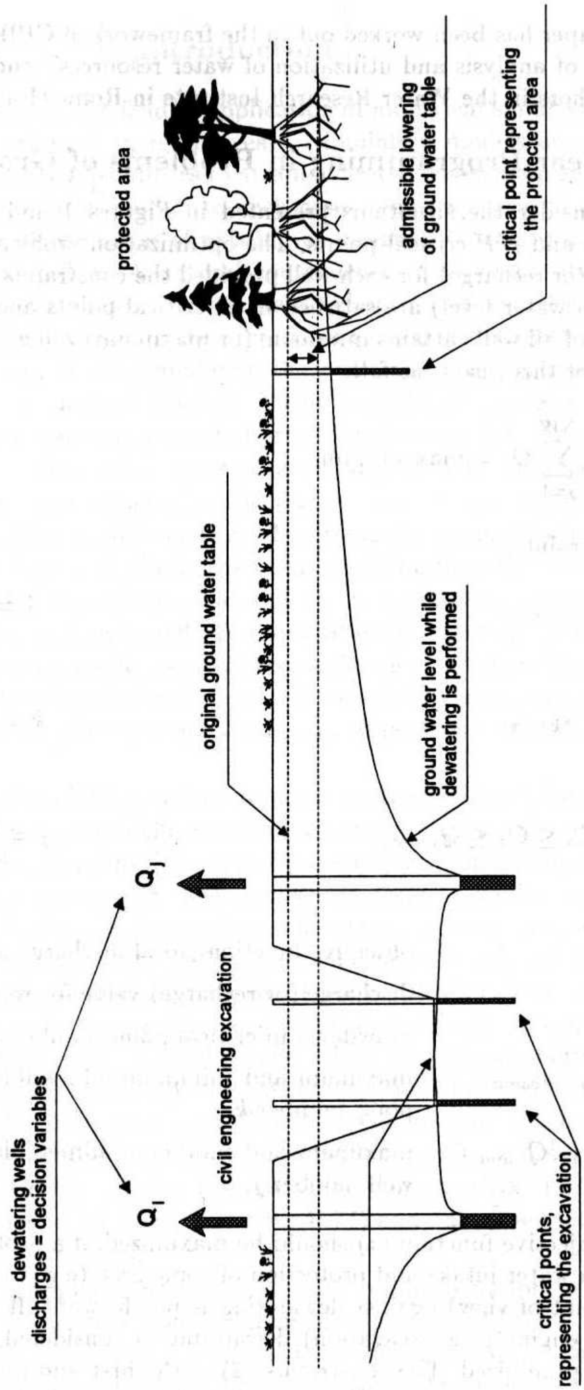

Fig. 1. Optimization problem of excavation dewatering

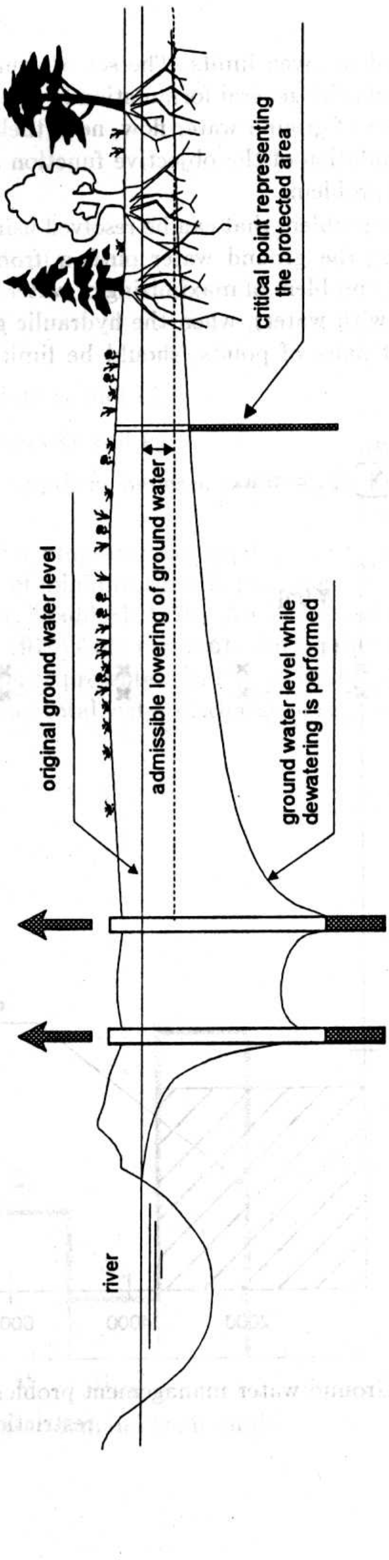

Fig. 2. Optimization problem in case of ground water intake

are included in given limits. The set of equations and constraints (1), (2), (3), and (4) represents the general formulation of linear programming for problems of optimal management of ground water flow, nevertheless another types of constraints and different formulation of the objective function are possible according to the needs of a considered problem.

Another problem that can be resolved using the linear programming, is controlling and limiting the ground water outflow from some chosen area (Figure 3). Let us consider the problem of maximizing of water discharge from an aquifer (or recharging an aquifer with water), when the hydraulic gradient in the aquifer area, represented by a set of pairs of points, should be limited. The problem can be formulated as follows:

$$F = \sum_{j=1}^{NW} Q_j = \max; \tag{5}$$

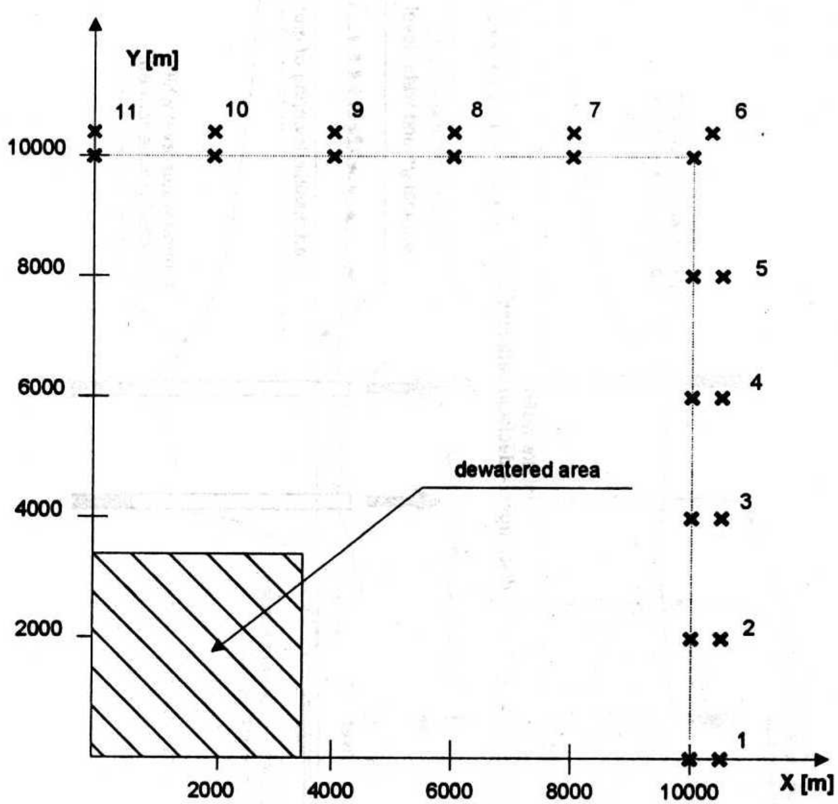

Fig. 3. Ground water management problem accounting for hydraulic gradient restriction

with constraints:

$$J_k \leq J_{k\,\max}, \tag{6}$$

where:

J_k – gradient between k^{th} pair of points,

$$J_k = (h_{k1} - h_{k2})/L_k, \tag{7}$$

L_k – distance between points k_1 and k_2,

h_{k1}, h_{k2} – hydraulic head in points k_1 and k_2,

$J_{k\,\max}$ – maximum admissible gradient between points k_1 and k_2.

At the end of this brief review of linear programming application to problems of ground water flow we present an example of minimization of operating cost of a group of wells (Deninger 1970, Jones et al. 1987, Maddock 1972). An additional constraint regarding the total water discharge (Bear 1979) from the intake during the operational period has been introduced. The situation is presented in Fig. 2, while the diagram in Fig. 4 illustrates requirements concerning total water discharge in consecutive time intervals. The objective function is then:

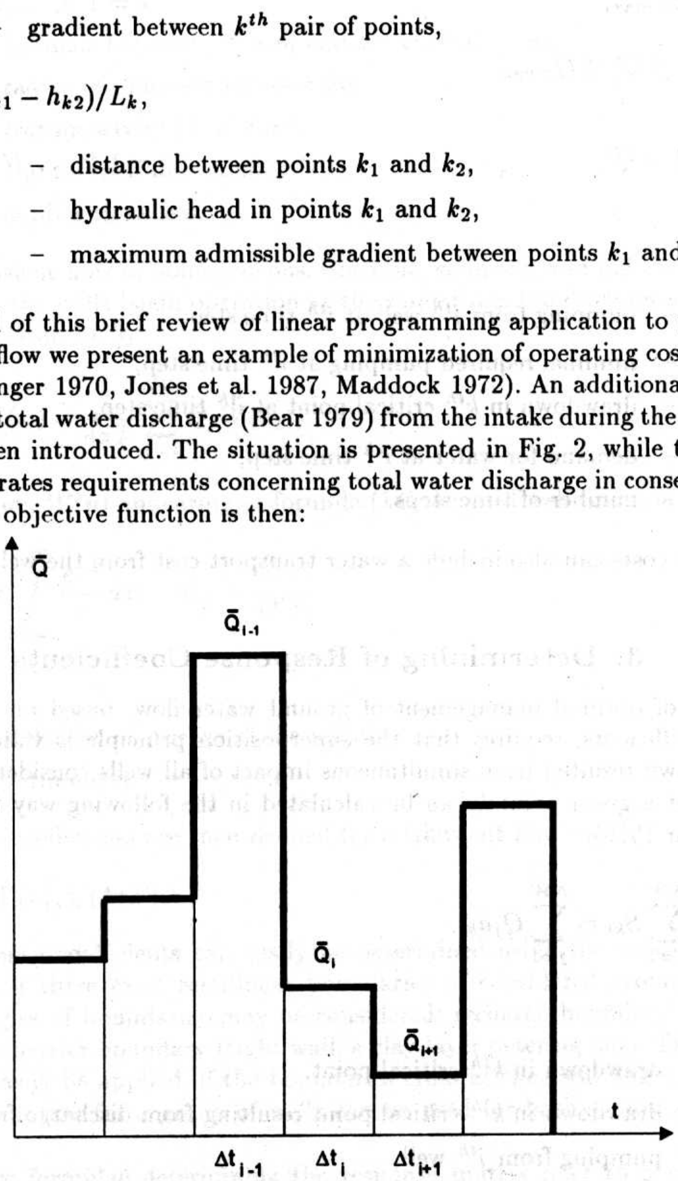

Fig. 4. Diagram of water demand from intake

$$F = \sum_{j=1}^{NW} c_j Q_j = \min; \tag{8}$$

where: c_j is unit cost of water acquisition from well number j, with constraints:

$$S_k^i \leq S_{k\,\max}, \qquad k = 1, 2, \ldots, NP \tag{9}$$

$$Q_{j\,\min} \leq Q_j^i \leq Q_{j\,\max} \tag{10}$$

$$\sum_{j=1}^{NW} Q_j^i = \overline{Q}^j; \qquad i = 1, 2, \ldots, NT \tag{11}$$

where:

- Q_j^i – pumping from j^{th} well at i^{th} time step,
- $Q_{\min}^i$ – minimal required pumping at i^{th} time step,
- S_k^i – drawdown in k^{th} critical point at i^{th} time step,
- $\overline{Q}^j$ – demand for water at i^{th} time step,
- NT – number of time steps.

The unit costs can also include a water transport cost from the well to a destination.

3. Determining of Response Coefficients

The model of optimal management of ground water flow, based on the notion of response coefficients, requires that the superposition principle is valid with regard to a drawdown resulted from simultaneous impact of all wells considered. Thus, the drawdown at a given point k can be calculated in the following way for a confined aquifer (Bear 1979):

$$S_k = \sum_{j=1}^{NW} S_{ki} = \sum_{j=1}^{NW} Q_j a_{ki} \tag{12}$$

where:

- S_k – drawdown in k^{th} critical point,
- S_{ki} – drawdown in k^{th} critical point resulting from discharge from j^{th} well,
- Q_j – pumping from j^{th} well,
- a_{ki} – response coefficient of j^{th} well impacting k^{th} critical point.

For a steady state flow in homogeneous and isotropic porous medium the response coefficients are calculated as follows (Bear 1979):

$$a_{kj} = \ln(R/r_{kj})/(2\pi T) \qquad \text{for } r_{kj} \leq R \tag{13}$$

where:

- r_{kj} – distance between j^{th} well and k^{th} critical point,
- R – radius of influence for all wells,
- T – transmissivity ($T = Km$),
- K – aquifer hydraulic conductivity,
- m – aquifer thickness.

For a transient flow in homogeneous, isotropic, confined, and infinite in the plane aquifer, when the wells begin operation at the same time, the drawdown in k^{th} point is defined as (Bear 1979):

$$S_k^i = S_k(t_i) = \frac{t}{4\pi T}\sum_{j=1}^{NW} Q_j W(u_{kj}^i) \tag{14}$$

The function $W(u)$ appearing in formula (14) is called well function and is:

$$W(u) = \int_{x=u}^{\infty} \frac{e^{-x}}{x} dx; \quad u_{kj}^i = \frac{r_{kj}^2}{4T}\frac{S}{t_i} \tag{15}$$

where:

- S – coefficient of storage of the aquifer,
- t_i – i^{th} time step.

The response coefficients are then defined for a transient flow as:

$$a_{kj}^i = W(u_{kj}^i)/(4\pi T) \tag{16}$$

The response coefficients can easily be determined using the image-well system (Bear 1979), if there exist rectilinear boundaries of considered ground water flow area. Two types of boundaries may be considered: recharge boundary (e.g. river or lake bank) or barrier boundary (tight wall, a clay layer petering out). The image-well system can easily be applied, if the boundaries cross at the right angle. In such case an analytic solution for a group of wells can be used for both steady and transient flow.

The above formulae determining the response matrix refer to a confined flow. A solution for a group of wells co-operating in an unconfined flow is given e.g. by

Bear (1979). If Dupuit's approximation is used for a steady state flow, the solution is as follows:

$$H_0^2 - h_k^2 = \sum_{j=1}^{WN} Q_j \ln(R/r_{kj})/(\pi K) \tag{17}$$

while for transient flow:

$$H_0^2 - h_k^2(t_i) = \sum_{j=1}^{WN} Q_j W(u_{kj}^i)/(2\pi k) \tag{18}$$

where:

$H_0 = \text{const}$ – original water table over an impervious bottom,

h_k – water table in k^{th} critical point.

In order to solve the linear programming problem a new variable, namely $\nu_k = H^2 - h_k^2$ should be introduced as a new constraint. Introducing it into the expression (18) one obtains:

$$\nu_k = \sum_{j=1}^{NW} Q_j \ln(R/r_{kj})/(\pi K) = \sum_{j=1}^{NW} Q_j a_{kj} \tag{19}$$

where

$$a_{kj} = \ln(R/r_{kj})/(\pi K) \tag{20}$$

and for transient flow:

$$\nu_k^i = \sum_{j=1}^{NW} Q_j W(u_{kj}^i)/(2\pi K) = \sum_{j=1}^{NW} Q_j a_{kj}^i \tag{21}$$

where

$$a_{kj}^i = W(u_{kj}^i)/(2\pi K). \tag{22}$$

The relation between a real drawdown s_k and the new variable ν_k is as follows:

$$s_k = H_0 - h_k = \nu_k/(H_0 + h_k) \tag{23}$$

or

$$\nu_k = s_k(2H_0 - s_k). \tag{24}$$

Solving the equations (24) with regard to s_k one obtains expressions for s_k:

$$s_k = H_0 - (H_0^2 - \nu_k)^{1/2} \tag{25}$$

Thus, when considering the linear programming problem for unconfined ground water flow, a constraint with regard to the new variable ν must be formulated.

4. Numerical Example

To present the practical possibility of application of optimization methods an example concerning dewatering of an excavation with vertical wells has been calculated. The data for the calculation is taken from the paper by Aquado et al. (1974). The authors of that publication used the first method (embedding method), described above in the introduction, to design a dewatering system for a dry dock excavation 1000 m long and 90 m wide. The aquifer parameters were the following: aquifer thickness 36 m, hydraulic conductivity 10.18 m/day, and coefficient of storage $S = 0.2$. Taking into account the symmetry of the area considered, only one quarter of it has been taken for further considerations. The situation is presented in Fig. 5. The boundary conditions are assumed at a small distance from the excavation bank (about 100 m). The boundary coordinates are $X = 0$ m and $Y = 0$ m for recharge boundary and $X = 600$ m and $Y = 150$ m for barrier boundary (axis of symmetry). Aquado et al. (1974) found the optimal solution for steady state flow. The wells were localized in nodes of a rectangular grid of 40 m by 10 m. They were situated as close to the excavation border as possible from the point of view of the assumed grid. The layout of the wells is shown in Fig. 5, the wells coordinates are specified in Table 1. In order to establish the time necessary to lower the water table inside the dewatered area to a desired level a simulation model for transient flow using the finite difference method was used.

Table 1

Optimal solution for excavation dewatering problem (variants 1 and 2)

Well number	Coordinates		Well discharge	
	X [m]	Y [m]	Variant 1	Variant 2
1	80	90	0.0	4,758.5
2	120	90	12,606.9	3,838.6
3	200	90	0.0	3,286.8
4	240	90	3,483.5	2,893.4
5	280	90	350.4	2,952.1
6	320	90	3,554.7	3,086.0
7	360	90	367.5	0.0
8	400	90	3,559.2	2,996.7
9	440	90	368.7	807.4
10	480	90	3,559.6	3,092.9
11	520	90	369.3	829.6
12	560	90	3,559.6	3,098.0
13	600	90	184.6	415.4
14	40	150	0.0	3,086.3
Total			31,964.0	35,141.7

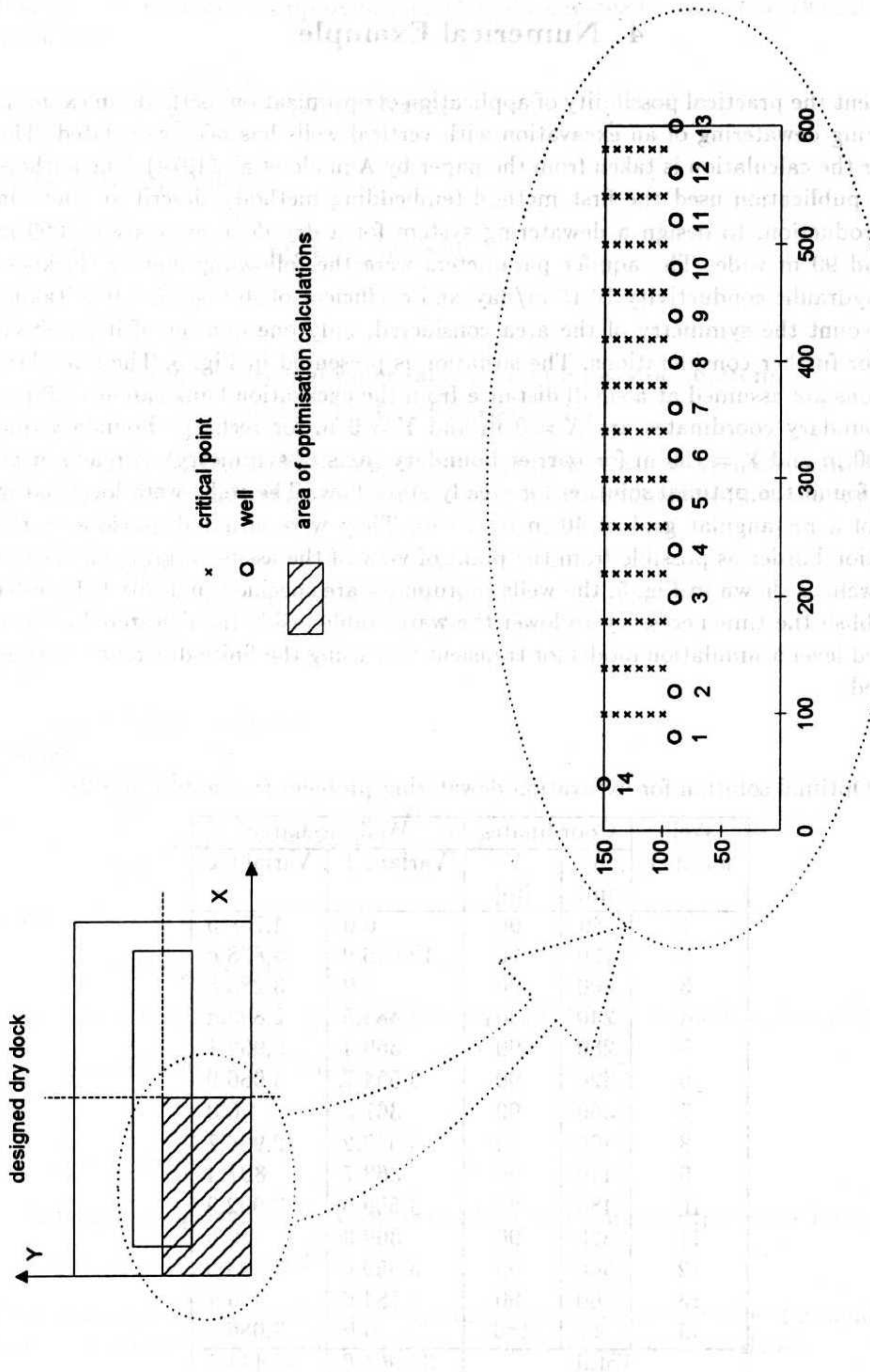

Fig. 5. Sketch diagram for optimization task of dry dock dewatering with layout of critical points

The authors of this paper have applied a method different from that by Aquado et al. (1974) to solve the same problem, namely the "response matrix" method. This approach is supported by the fact that it becomes possible to employ an analytical solution for determining the drawdown at any point of the considered area, and by possibility of taking into account the duration of excavation dewatering. The response coefficients have been calculated using formula (22) in conjunction with the image-well system described by Bear (1979). The relation (24) has been used to formulate the constraints. Special procedures calculating the response coefficient for the case of rectilinear boundaries and two programmes generating initial data acceptable by one of the linear programming packs (LP87) for IBM PC have been developed. The first programme enables the creating of data for the case when the well radius is unknown. The second one includes additional restrictions resulting from the well radius and imposed on the drawdown value in all wells. The drawdown in a well has been calculated using response coefficients between the point of location of this well and points of location of all other wells. The self-response coefficient for a well has been defined as the coefficient between the centre of the well and a point located at a distance equal to the well radius.

Using the programmes mentioned, the data for optimizing calculations has been prepared for two variants of the problem described by Aquado et al. (1974). The critical points representing the dewatered area have been localized inside the same area in nodes of the grid as assumed by Aquado et al. (1974), hence in the mesh 40 × 10 m. The total number of critical points is 78, their layout being presented in general outline on Fig. 5.

First the optimization calculations have been performed without limiting drawdown and discharge from dewatering wells and assuming a dewatering time equal to 30 days. The results are presented in Table 1. The total discharge (value of the objective function) is 31,964 m^3/day. For 3 wells (1, 3, 14) the total discharge is nil, while in one of them (2) the solution gives very high discharge, namely 12,606.9 m^3/day. This optimal solution appears to be hardly useful from the practical point of view, because an attempt to pump the water using calculated discharges may lead in some wells to such lowering of the water table that it attains the aquifer base. It depends on dimensions of dewatering wells (radius, screen length etc.). Aquado et al. (1974) were using the "embedding method" and in consequence their set of constraints included differential equations formulated for individual nodes of the aquifer area grid. The optimized discharge in a grid node corresponded to the whole region of node influence – in this case to 10 × 40 m. They did not deal with any parameters of dewatering wells thus no constraints were imposed on physical reality of drawdown in the dewatering wells. They should rather tell about water discharge from the region of node influence than from the well placed in the node. The variety of the response matrix method, presented above, enables easy incorporation of additional constraints imposed on drawdown in dewatering wells and simultaneous respect of duration of the excavation dewatering.

In the second variant of calculations it has been assumed that drawdown in any dewatering well cannot exceed 36 m and additional constraints have been incorporated assuming that any dewatering well is a fully penetrating well with radius of 0.5 m. The optimal solution calculated for this formulation is presented in Table 1. This time the total discharge is 35,141.7 m^3/day. There appears only one well (7) with null discharge, and the discharge of remaining wells is much more homogenous than before (from 415 to 4,758 m^3/day). It is worth noting that for wells with radius of 0.1 or 0.2 m no solution satisfying the assumed constraints can be achieved.

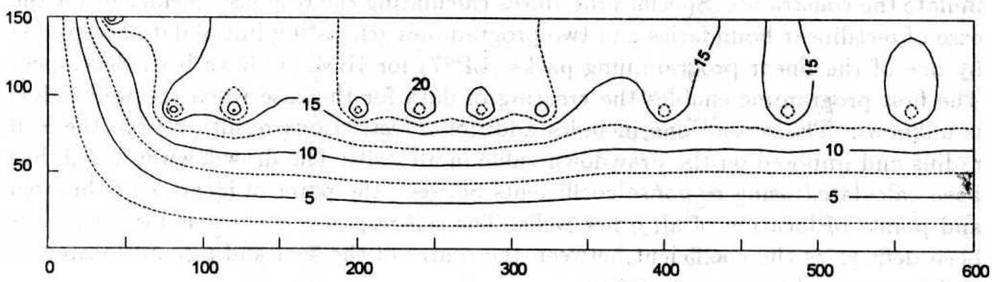

Fig. 6. Forecast of ground water table distribution after 30 days of pumping (variant 2)

Some verifying calculations have been performed with regard to the optimal solution obtained in the second variant. A forecast using the finite difference method has been calculated for 10 × 10 m mesh grid assuming in grid nodes well discharges taken from Table 1 (variant 2). The forecast results are presented in form of water table lines in Fig. 6. It can be noticed that the required drawdown (over 15 m) has been attained in the desired area. Some insignificant deviations of the drawdown can be explained by the fact that the method used for the forecast calculation is different from the one used for response coefficients and by imperfection of the interpolation procedure determining the water table lines.

5. Conclusions

Application of the response matrix method to a model of optimal management of water pumping from an aquifer is especially recommended, if it is possible to use analytical formulae to calculate the response coefficients. Otherwise, methods of numerical simulation can be used. The number of constraints is usually much smaller than in the case of considering of difference equations as constraints in the linear programming. The maximum number of constraints in the method used in this paper is equal to the total number of critical points (confinement imposed on drawdown) and tripled number of wells (upper and lower limit of well discharge and drawdown limit for every well). The optimal solution is influenced by the number and layout of the critical points, so an important problem is proper choice of these. A moderate number of constraints enables utilisation of IBM PC XT/AT class computers for data preparation and problem resolving that has been confirmed by the calculations presented here.

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Summary

The possibility of applying ground water flow optimizing methods using linear programming is presented in the article. Optimization of water pumping from an aquifer using vertical wells is considered. Two methods of applying the linear programming in ground water flow problems are described in the introduction. The first ("embedding method") consists in incorporation of hydraulic equations into the optimization model. Linear algebraic equations resulting from e.g. the finite difference method become the linear programming constraints. Pumping or recharge rates in the nodal points are usually decision variables. The second method is based on so-called response coefficients ("response matrix method"), which define changes of ground water level in arbitrary chosen points ("critical points") of the aquifer, resulting from unit rate pumping from each well. The coefficients are only once calculated and then used to determine the objective function and constraints. The number of constraints depends on the number of critical points, but anyway it is much smaller than the first method requires. Examples of the application of linear programming methods to problems of pumping from an aquifer are presented in Chapter 2. Among other things the objective function and constraints are defined for the problem of an excavation dewatering using vertical wells with an optimal total pumping rate. Methods of response coefficients determining for steady and transient flow in confined and unconfined aquifer are presented in Chapter 3.

A numerical example illustrating practical possibilities of application of optimization models is presented in Chapter 4. For calculations the response matrix method has been applied to solve the optimization problem of dry dock dewatering, known from other publication. Computer programmes for IBM PC/AT generating data applicable for standard linear programming packs and calculating the response coefficients have been developed. The results of application of the optimization model in two variants to the problem of dry dock dewatering are presented and a forecast of ground water level changes has been calculated in order to verify the optimal solution obtained.

Model optymalnego sterowania poborem wody z warstwy wodonośnej

Streszczenie

W artykule przedstawiono możliwości zastosowania programowania liniowego do optymalizacji poboru wody z warstwy wodonośnej za pomocą studni pionowych. Omówiono model optymalizacyjny wykorzystujący tzw. współczynniki wpływu, charakteryzujące zmiany poziomu zwierciadła wody w wybranych punktach warstwy wodonośnej, spowodowane jednostkowym poborem wody z poszczególnych studni. Podano sposób obliczania współczynników wpływu w warunkach przepływu w ograniczonej warstwie wodonośnej oraz dla przepływu za swobodnym zwierciadłem. Sformułowano funkcję celu i ograniczenia dla zadania minimalizacji sumarycznego wydatku zespołu studni odwadniających projektowany wykop. Przedstawiono wyniki obliczeń optymalizacyjnych dla zadania odwodnienia suchego doku. Wykonano prognozę zmian zwierciadła wód podziemnych w celu sprawdzenia uzyskanego rozwiązania optymalnego.