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Transient Vibrations of a Simple Structure and Initial Generation of Water Waves in a Layer of Fluid

1. Introduction

Vibrations of structures immersed in fluid and generation of water waves are of practical importance. Many engineering problems of this kind are associated with the construction and exploitation of offshore structures. From the theoretical point of view, such problems are complicated and difficult to solve, since in a general case we have to deal with coupled problems of the structure-fluid interaction in the presence of a free surface of the fluid. There are no mathematical methods which provide effective solutions when finite displacements of a structure and non-linear free surface conditions are taken into account. Thus, many simplifying assumptions are introduced to obtain a reasonable description of a physical situation. Usually such simplifications assume some kind of linearization of boundary conditions. Moreover, it is frequently assumed that displacements of a structure are infinitesimal and the fluid flow is potential. Such a linear theory of vibrations and generation of water waves gives results which in some cases fit experimental data while in other cases may be used as a starting point for consideration of non-linear effects. But, even in the case of the linearized theory, constructing a general solution is a difficult task, especially when unsteady problems are considered. The transient vibrations of a structure submerged in fluid, the box drop problem and the problem of forming a surface gravitational wave starting from rest are examples of the latter cases.

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The aim of the present work is to investigate transient vibrations of a rigid block with a single degree of freedom submerged in fluid of constant depth. Special attention will be paid to the examination of the added mass of fluid which is not constant during vibrations. The problem is related to initial generation of water waves by a piston type wave maker and, therefore, in the second part of the paper, an initial generation of water waves in a channel of constant depth is considered.

Several unsteady problems of this kind were discussed by Lamb (1975). Within the framework of the linear theory Stocker (1957) analyzed a variety of problems involving the initial motion of fluid. He obtained the closed form solutions for imposed initial conditions at the starting point. The important cases of waves due to disturbances at a point on the free surface were studied in detail. Some general theorems associated with the initial value problems in hydrodynamics may be found in Wehausen and Laitone (1960). These authors studied many time-dependent Green functions and quoted extensive literature on the subject. Recently, a number of papers have appeared where more particular problems of steady and unsteady motion are investigated. Biesel and Suquet (1951) formulated the theory of water waves generated in a flume of constant depth by a moving rigid plate. Fontanet (1961) derived the second order solution to the wave maker problem based on a Lagrangian formulation. His solution, however, is difficult to evaluate as compared to the more familiar Eulerian description. Waves generated by a single pressure impulse were investigated by Massel (1968). The function of velocity potential was obtained by adapting the time dependent Green function to the boundary and initial conditions. The same author formulated and solved the problem of generation of surface waves due to longitudinal (Massel 1970a) and, side – (Massel 1970b) ship launching. The problem was solved under the assumption that the distribution of pressure on the hull body may be approximated by a linear function. In his next paper (Massel 1972), the problem of generation of surface waves by the motion of a rigid body was considered. The problem of water waves generated by landslides in reservoirs was investigated by Noda (1969). He considered two types of landslides: a vertical one modelled by a box falling vertically and a horizontal one, modelled by a wall moving horizontally. A theoretical solution to the time dependent boundary displacement was examined against experimental data. The generation of water waves by a piston-type wave-maker starting from rest was investigated by Madsen (1970). The theoretical results obtained were compared with experimental data. The experiments showed second order effects in wave amplitudes and thus an approximate second order wave maker theory was discussed.

Concerning vibrations in fluid, the very important and earliest contribution is that of Westergaard (1933). He described the hydrodynamical forces acting on dams due to seismic vibrations. The solution was obtained within the linear theory of compressible fluid neglecting surface waves. A similar problem of vibrations of dams during earthquakes was considered by Nath (1973). He also presented an extensive survey of research in structure-fluid interaction. Norwood and Warren (1975) analyzed the transient response of a plate–fluid system to stationary and moving loads. They investigated possible wave motion in a fluid half-space which supports an infinite elastic plate. Wilde (1988) presents a review of papers on related steady state problems.

2. Transient Vibrations of a Rigid Block in Fluid

We will confine our attention to the plane problem of vibrations of a rigid block in a semi-infinite layer of fluid as shown in Fig. 1. The block with mass M is supported

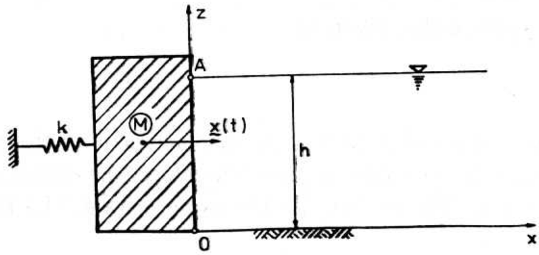

Fig. 1. The block–fluid system

by a linear spring characterized by the elastic coefficient k. It is assumed that there is no friction and no fluid flow between the block and the bottom of the layer. The compressibility of the fluid and its viscosity are neglected and the potential theory is adopted. The block and the fluid are at rest for time $t \leq 0$. The fluid flow is induced by horizontal vibrations of the block due to an assumed initial displacement from its equilibrium position and then, its abrupt release. In other words, the vibrations are due to a constant external force which is suddenly applied to the block and which remains unchanged during the time of vibrations. Within a small range of time from the beginning of the motion, this problem may also be considered as an impulsive generation of motion.

2.1. Impulsive Generation of Motion of the Block–Fluid System

Let us consider the case in which a finite impulse of an external force is applied to the block in a horizontal direction. Since an impulse is regarded as an infinitely great force acting for an infinitely short time, effects of finite forces during the interval are to be neglected. The external impulse creates the resultant impulsive pressure within the fluid. The latter problem of impulsive generation of the fluid motion starting from rest was considered by Lamb (1975) and we will follow his ideas. According to Lamb (1975), the impulsive pressure in the fluid should satisfy the Laplace equation. Moreover there is a simple relation between the pressure and the velocity potential. Denoting the impulsive pressure in fluid by I, that relation takes the form:

$$I = -\rho \cdot \phi + C \tag{1}$$

where ϕ is the velocity potential, ρ the fluid density and C a constant.

Let F denote the impulse of the external force P acting on the block during the time increment Δt. The corresponding velocity of the block may be formally written as:

$$v = a \cdot \Delta t \tag{2}$$

where a is the acceleration of the block.
Accordingly, the displacement of the block is

$$\Delta s = \frac{1}{2} \cdot a \cdot (\Delta t)^2 = \frac{1}{2} \cdot v \cdot \delta t \tag{3}$$

and the external work performed is given by:

$$L = P \cdot \Delta s = \frac{1}{2} \cdot F \cdot v. \tag{4}$$

It is seen, that for finite values of F and v the displacement of the block is a small quantity which in the limit $\Delta t \to 0$ falls to zero. Therefore, the elastic force due to the elongation of the spring also falls to zero. At the same instant, the kinetic energy of the block is:

$$T_1 = \frac{1}{2} \cdot M \cdot v^2. \tag{5}$$

In order to find the relevant kinetic energy of the fluid we shall adopt the potential formulation. Thus, the energy may be expressed in the following form (Stocker 1957):

$$T_2 = \frac{1}{2} \cdot \rho \oint_C \Phi \cdot \frac{\partial \Phi}{\partial n} dS, \tag{6}$$

where C denotes the contour of the fluid domain.

To calculate the latter energy, it is necessary to find a solution of the Laplace equation for the velocity potential:

$$\nabla^2 \Phi = 0 \tag{7}$$

satisfying given boundary conditions. On the bottom of the layer there is no flow through the boundary and thus we have:

$$\left. \frac{\partial \Phi}{\partial z} \right|_{z=0} = 0. \tag{8}$$

On the line $z = h$ (Fig. 1) the pressure is constant and without loss of generality we may assume:

$$\Phi|_{z=h} = 0. \tag{9}$$

The boundary condition on the common surface of the block and the fluid reads:

$$\left. \frac{\partial \Phi}{\partial x} \right| = v. \tag{10}$$

To the above boundary conditions we have to add the condition that for $x \to \infty$ the potential and its derivatives die out. Keeping in mind the boundary conditions, the solution of the Laplace equation (7) may be constructed by means of the separation of variables. This simple procedure yields the solution:

$$\Phi = -\sum_{j=1}^{\infty} A_j \cdot \frac{1}{k_j} \cdot e^{-k_j x} \cdot \cos k_j z, \tag{11}$$

where

$$k_j = \frac{2j-1}{2h} \cdot \pi, \quad j = 1,\ 2,\ 3\ldots \tag{12}$$

are eigenvalues of the problems and A_j ($j = 1, 2, 3\ldots$) are constants to be determined from condition (10). Substituting (11) into (10) and making simple manipulations provides the relation:

$$A_j = \frac{4v}{\pi} \cdot \frac{(-1)^{j+1}}{(2j-1)}, \quad j = 1,\ 2,\ 3\ldots \tag{13}$$

Finally, from substitution of (13) into (11) it follows that:

$$\Phi = -\frac{8vh}{\pi^2} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^2} e^{-k_j x} \cdot \cos k_j z. \tag{14}$$

Having the solution, we can perform integration (6) to obtain:

$$T_2 = \frac{1}{2} \cdot \rho \cdot 16 \cdot \frac{h^2}{\pi^3} \cdot \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3} \cdot v^2. \tag{15}$$

The total kinetic energy of the block–fluid system can be written as:

$$T = T_1 + T_2 = \frac{1}{2} \left[M + \rho \cdot 16 \cdot \frac{h^2}{\pi^3} \cdot \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3} \right] v^2. \tag{16}$$

Since the system considered is conservative, the energy is equal to the external work (4). From comparison of equations (4) and (16) the following relation results:

$$F = M^* \cdot v, \tag{17}$$

where

$$M^* = M + \rho \cdot 16 \cdot \frac{h^2}{\pi^3} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3}. \tag{18}$$

It is seen, that motion of the rigid block in contact with fluid is similar to motion of the block without fluid but with its mass greater by the additional term:

$$M_a = \rho \cdot 16 \cdot \frac{h^2}{\pi^3} \cdot \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3}. \tag{19}$$

Eq. (19) describes the added mass of fluid. More precisely, the added mass obtained is the impulsive added mass of fluid. The corresponding angular frequency of vibrations of the block in contact with fluid is:

$$\omega = \sqrt{\frac{k}{M^*}}. \tag{20}$$

This frequency is smaller than the frequency of free vibration of the block without the fluid. It should be stressed, however, that the solution obtained may be used only for small lapse of time from the beginning of motion.

2.2. Undamped Vibrations of the Block–Fluid System

As was mentioned earlier, the problem of transient vibrations of the block–fluid system due to the initial displacement of the block may be solved directly by integration of the differential equations of motion of the system with respect to time. Assuming the potential flow and omitting surface gravitational waves, the differential equation describing the block motion is:

$$M \cdot \ddot{\underset{\sim}{x}} + k \cdot \underset{\sim}{x} = -W, \tag{21}$$

where $\underset{\sim}{x}$ is the displacement of the block referring to its neutral position and W is the resultant force of the fluid pressure acting on the block. As in the previous case, the function of velocity potential may be written in the form:

$$\Phi = -\sum_{j=1}^{\infty} A_j(t) \cdot \frac{1}{k_j} \cdot e^{-k_j x} \cdot \cos k_j z, \tag{22}$$

where k is defined by (12) and $A_j(t)$ $(j = 1, 2, 3 \ldots)$ are functions dependent on time. The boundary condition on the common surface of the block and fluid yields:

$$\dot{\underset{\sim}{x}} = \left.\frac{\partial \Phi}{\partial x}\right|_{x=0} = \sum_{j=1}^{\infty} A_j(t) \cdot \cos k_j z, \tag{23}$$

where $\dot{\underset{\sim}{x}}$ is the velocity of the block. From Eq. (23) it follows that:

$$A_j(t) = 2 \cdot \frac{(-1)^{j+1}}{k_j h} \cdot \dot{\underset{\sim}{x}}, \quad j = 1,\ 2,\ 3 \ldots \tag{24}$$

Substitution of (24) into (22) gives:

$$\Phi = -\frac{8 \cdot h \cdot \dot{\underset{\sim}{x}}}{\pi^2} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j=1}}{(2j-1)^2} \cdot e^{-k_j x} \cdot \cos k_j z. \tag{25}$$

The pressure of the fluid is given by Stocker (1957):

$$p = -\rho \cdot \phi = \frac{8 \cdot \rho \cdot h \cdot \ddot{\underset{\sim}{x}}}{\pi^2} \cdot \sum_{j=1}^{\infty} \cdot \frac{(-1)^{j+1}}{(2j-1)^2} \cdot e^{-k_j x} \cdot \cos k_j z. \tag{26}$$

Now, the resultant of the pressure acting on the block may be calculated as:

$$W = \int_0^h p \bigg|_{z=0} dz = 16 \cdot \rho \cdot \frac{h^2}{\pi^3} \cdot \ddot{\underset{\sim}{x}} \sum_{j=1}^{\infty} \frac{1}{(2j-1)^3}. \tag{27}$$

Substitution of (27) into (21) gives:

$$M^* \cdot \ddot{\underset{\sim}{x}} + k \cdot \underset{\sim}{x} = 0. \tag{28}$$

where the mass M^* is defined by (18).

The general solution of the homogeneous differential equation (28) is:

$$\underset{\sim}{x}\ (t) = A \cdot \cos\omega t + B \cdot \sin\omega t, \tag{29}$$

where ω is defined by (20) and A and B are constants. Assuming that at the starting point ($t = 0^+$) we have the initial conditions:

$$\underset{\sim}{x}\ (t = 0) = \underset{\sim}{x_0}\ , \quad \underset{\sim}{\ddot{x}_0}\ (t = 0) = 0, \tag{30}$$

the solution (29) assumes the form:

$$\underset{\sim}{x}\ (t) = \underset{\sim}{x_0}\ \cdot \cos\omega t. \tag{31}$$

As seen, the solution obtained describes the free undamped vibrations of the block with a mass which equals the sum of its own mass M and the attached mass of fluid M_a. For the considered case of zero pressure on the upper boundary of the layer, the added mass of fluid is exactly equal to the impulsive mass of fluid obtained previously. The solution was constructed under the assumption that there were no surface gravitational waves. In reality, such vibrations of the block may generate surface waves and thus a damping of vibrations would occur. The damping results from transmission of energy from the block to infinity by means of generated waves. Thus, the next case to be considered when the surface gravitational waves are taken into account.

2.3. Damped Vibrations of the Block–Fluid System

Consider now the same problem of initial vibrations of the block–fluid system but with wave condition on the upper surface of the layer. The equation of motion (21) is still in force, but the resultant pressure force W should be calculated in a different way. Instead of the boundary condition (9) we have the following (Stocker 1957).

$$\left(\ddot{\Phi} + g \cdot \frac{\partial \Phi}{\partial z} = 0 \right) \bigg|_{z=h} \tag{32}$$

where g is the gravitational acceleration.

In order to find a solution of the problem for arbitrary time it is convenient to divide the velocity potential into two parts:

$$\Phi = \psi + \phi \tag{33}$$

each of them satisfying the Laplace equation:

$$\nabla^2\phi = 0, \quad \nabla^2\psi = 0 \tag{34}$$

and, the relevant boundary conditions:

$$\frac{\partial \phi}{\partial x}\bigg|_{x=0} = 0, \quad \psi\bigg|_{z=h} = 0. \tag{35}$$

The boundary condition on the upper surface of the layer assumes the form:

$$\left(\ddot{\phi} + g \cdot \frac{\partial \phi}{\partial z} + g \cdot \frac{\partial \psi}{\partial z}\right)\bigg|_{z=h} = 0. \tag{36}$$

The remaining boundary conditions are:

$$\frac{\partial \phi}{\partial z}\bigg|_{z=h} = 0, \qquad \frac{\partial \psi}{\partial z}\bigg|_{z=0} = 0 \tag{37}$$

and:

$$\frac{\partial \psi}{\partial x}\bigg|_{x=0} = \dot{\underset{\sim}{x}} \,. \tag{38}$$

Both the potentials should disappear in infinity. The potential ψ and corresponding fluid pressure are given by (25) and (26) respectively. In order to find the second potential ϕ the Fourier cosine transform is applied, namely (Sneddow 1951):

$$\begin{aligned} \phi^* &= \int_0^\infty \phi(x,\ z,\ t) \cdot \cos(sx)\ dx \\ \phi &= \tfrac{2}{\pi} \cdot \int_0^\infty \phi^*(s,\ z,\ t) \cdot \cos(sx) ds \end{aligned} \tag{39}$$

The transform of the Laplace equation for the potential ϕ leads to the ordinary differential equation:

$$\frac{\partial^2 \phi^*}{\partial z^2} - s^2 \cdot \phi^* = 0. \tag{40}$$

Taking into account the bottom boundary condition and its Fourier transform, the solution to the differential equation (40) may be written in the form:

$$\phi^*(s,z,t) = A(s,t) \cdot \cos h(sz), \tag{41}$$

where $A(s,t)$ is a constant of integration. The Fourier transform of the boundary condition (36) is:

$$\left(\dot{\phi}^* + g \cdot \frac{\partial \phi^*}{\partial z} + g \cdot \frac{\partial \psi^*}{\partial z}\right)\bigg|_{z=h} = 0. \tag{42}$$

Now, we have to evaluate the Fourier transform of the potential (25). Simple manipulations yield:

$$\psi^*(s,z,t) = -\frac{4 \cdot \dot{\underset{\sim}{x}}}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)} \cdot \frac{1}{k_j^2 + s^2} \cdot \cos k_j z. \tag{43}$$

The derivative of the transform with respect to z, taken at $z = h$, is:

$$\frac{\partial \psi^*}{\partial z}\bigg|_{z+h} = \frac{2 \cdot \dot{\underset{\sim}{x}}}{h} \cdot \sum_{j+1}^{\infty} \frac{1}{k_j^2 + s^2}. \tag{44}$$

The series in (44) may be brought to the closed analytical form (Gradsztein, Rižik 1962):

$$\sum_{j=1}^{\infty} \frac{1}{k_j^2 + s^2} = \frac{h}{2 \cdot s} \cdot \tanh(sh), \tag{45}$$

and thus, equation (44) may be rewritten as:

$$\left.\frac{\partial \psi^*}{\partial z}\right|_{z=h} = \dot{\underset{\sim}{x}} \cdot \frac{\tanh(sh)}{s}. \tag{46}$$

Substitution of (41) and (46) into (42) gives:

$$\ddot{A} \cdot \cosh(sh) + A \cdot g \cdot s \cdot \sinh(sh) + \frac{g \cdot \dot{\underset{\sim}{x}}}{s} \cdot \tanh(sh) = 0. \tag{47}$$

Finally, the non-homogeneous second-order differential equation is written in the form:

$$\ddot{A} + r^2 \cdot A + F(s) \cdot \dot{\underset{\sim}{x}} = 0 \tag{48}$$

where:

$$r^2 = g \cdot s \cdot \tanh(sh), \qquad F(s) = \frac{g}{s} \cdot \frac{\tanh(sh)}{\cosh(sh)}. \tag{49}$$

The general solution to equation (48) is:

$$A = C_1 \cdot \cos rt + c_2 \cdot \sin rt - \frac{F(s)}{r} \cdot \int_0^t \dot{\underset{\sim}{x}}(\xi) \cdot \sin r(t-\xi) d\xi. \tag{50}$$

In this way, the Fourier transform of the potential ϕ is expressed as follows:

$$\begin{aligned} \phi^*(s,z,t) &= [C_1 \cdot \cos rt + C_2 \cdot \sin rt + \\ &- \left. \frac{F(s)}{r} \cdot \int_0^t \dot{\underset{\sim}{x}}(\xi) \cdot \sin r(t-\xi) d\xi \right] \cdot \cosh(sz). \end{aligned} \tag{51}$$

The constants C_1 and C_2 are to be found from the initial conditions:

$$\begin{aligned} \phi(t=0)|_{z=h} = 0, &\rightarrow \phi^*(t=0)|_{z=h} = 0 \\ \dot{\phi}(t=0)|_{z=h} = 0, &\rightarrow \dot{\phi}^*(t=))|_{z=h} = 0. \end{aligned} \tag{52}$$

It follows from (52) that $C_1 = C_2 = 0$ and, finally:

$$\phi^*(s,z,h) = -\frac{F(s)}{r} \cdot \int_0^t \dot{\underset{\sim}{x}}(\xi) \cdot \sin r(t-\xi) d\xi \cdot \cosh(sz) \tag{53}$$

The inverse Fourier transform of (53) provides the solution:

$$\phi(x,z,t) = -\frac{2\cdot g}{\phi}\cdot\int\limits_0^\infty \frac{\tanh(sh)}{s}\cdot\frac{\cosh(sz)}{\cosh(sh)}\times$$
$$\times\ \frac{1}{r}\cdot\int\limits_0^t \dot{\underset{\sim}{x}}\ (\xi)\cdot\sin r(t-\xi)d\xi\cdot\cos(sx)ds. \tag{54}$$

Having the solutions (25) and (54) we can write the expression for the pressure:

$$p = -\rho\cdot\phi = \frac{8\cdot\rho\cdot\ \ddot{\underset{\sim}{x}}}{\pi^2}\cdot\sum_{j=1}^\infty\frac{(-1)^{j+1}}{(2j-1)^2}\cdot e^{-k_jx}\cdot\cos k_jz+\frac{2\cdot\rho\cdot g}{\pi}\times$$
$$\times\int\limits_0^\infty\frac{\tanh(sh)}{s}\cdot\frac{\cosh(sz)}{\cosh(sh)}\cdot\int\limits_0^t\dot{\underset{\sim}{x}}\ (\xi)\cdot\cos r(t-\xi)d\xi\cdot\cos(sx)ds. \tag{55}$$

The fluid pressure acting on the block is:

$$p(x=0) = \frac{8\cdot\rho\cdot h\cdot\ \ddot{\underset{\sim}{x}}}{\pi^2}\cdot\sum_{j=1}^\infty\frac{(-1)^{j+1}}{(2j-1)^2}\cdot\cos k_jz+\frac{2\cdot\rho\cdot g}{\pi}\times$$
$$\times\int\limits_0^\infty\frac{\tanh(sh)}{s}\cdot\frac{\cosh(sz)}{\cosh(sh)}\cdot\int\limits_0^t\dot{\underset{\sim}{x}}\ (\xi)\cdot\cos r(t-\xi)d\xi ds \tag{56}$$

and the resultant of the pressure is given by the formula:

$$W = \frac{16\cdot\rho\cdot\ \ddot{\underset{\sim}{x}}}{\pi}\cdot\left(\frac{h}{\pi}\right)^2\cdot\sum_{j=1}^\infty\frac{1}{(2j-1)^3}+$$
$$+\ \frac{2\cdot\rho\cdot g}{\pi}\cdot\int\limits_0^\infty\left(\frac{\tanh(sh)}{s}\right)^2\cdot\int\limits_0^t\dot{\underset{\sim}{x}}\ (\xi)\cdot\cos r(t-\xi)d\xi ds. \tag{57}$$

It can be seen, that the first term on the right–hand side of (57) is equal to expression (27) obtained previously for the case of zero pressure on the upper boundary of the layer. Substituting (57) into (21) and making simple rearrangements one can obtain:

$$M^*\cdot\ \ddot{\underset{\sim}{x}}\ +k\cdot\ \underset{\sim}{x}\ +\frac{2\cdot\rho\cdot g}{\pi}\ \cdot\ \int\limits_0^\infty\left(\frac{\tan h(sh)}{s}\right)^2\times$$
$$\times\ \int\limits_0^t\dot{\underset{\sim}{x}}\ (\xi)\cdot\cos r(t-\xi)d\xi ds=0. \tag{58}$$

This integro–differential equation describes the transient vibrations of the block when surface gravitational waves are taken into account. The equation will be solved numerically by means of the θ-Wilson method (for details see Bathe 1982). In the numerical procedure the solution is constructed step by step for the chosen sequence

$t = 1\Delta t,\ 2\Delta t,\ 3\Delta t\ldots$ of time steps. Having the solution it is also possible to find the free surface elevation:

$$\eta(x,t) = \frac{2}{\pi} \cdot \int_0^\infty \frac{\tanh(sh)}{s} \cdot \int_0^t \dot{\underset{\sim}{x}}\,(\xi) \cdot \cos r(t-\xi)d\xi \cdot \cos(sx)ds \tag{59}$$

which, in turn, needs calculations of the improper integral. To find a value η for chosen parameters t and x it is necessary to resort to approximate integration. The procedure of the numerical integration will be described further.

2.4. Numerical Example

In order to illustrate the above theory some numerical calculations have been performed. For chosen values of M, k, h, which serve as input data, the displacement $\underset{\sim}{x}\,(t)$ of the block was obtained by means of approximate integration of the differential equation (58). Then, following the solution obtained, the free surface elevation for a chosen value of x was calculated. The displacement of the block is plotted in Fig. 2. Having

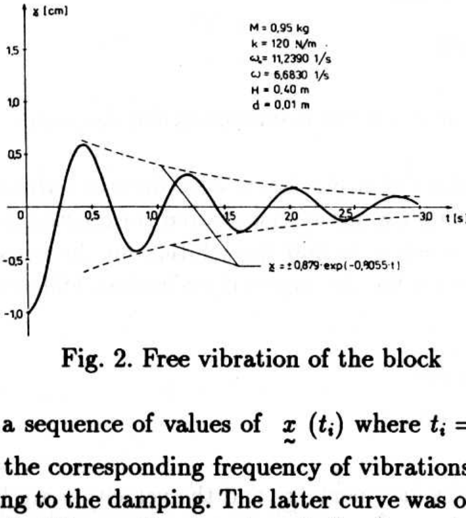

Fig. 2. Free vibration of the block

the displacement as a sequence of values of $\underset{\sim}{x}\,(t_i)$ where $t_i = i \cdot \Delta t$, $i = 1, 2\ldots$, it is possible to calculate the corresponding frequency of vibrations as well as the shape of envelope corresponding to the damping. The latter curve was obtained by means of the least square method. In this way the damping of vibrations of the block resulting from transmission of energy by the generated waves was described by means of a constant single damping coefficient. It can be seen, that the added mass of fluid for the case of surface gravitational waves is smaller than the impulsive added mass. Thus, the corresponding frequency of vibrations for the case is greater than that obtained for the case of zero pressure on the upper boundary. Fig. 3 shows the disturbance of the free surface elevation at $x =$ const. for the assumed time interval.

3. Initial Generation of Water Waves

Let us now consider the initial problem of forming surface waves in the semi-infinite layer of fluid starting from rest. The fluid flow is induced by horizontal displacement of

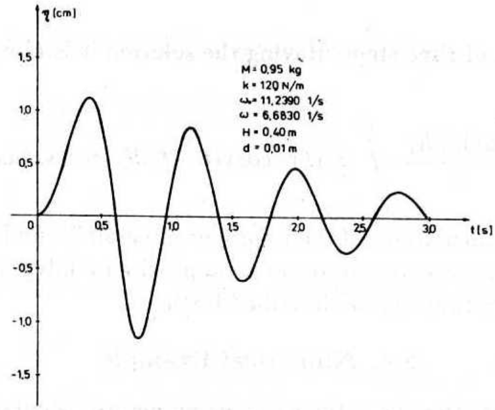

Fig. 3. Elevation of the free surface at $x = 20$ cm

a rigid vertical wall located at the beginning of the layer. The problem is similar to the previous one, but now the displacements of the rigid block are assumed to the known and described by the equation:

$$\underset{\sim}{x} = d \cdot (1 - \cos \omega t), \tag{60}$$

where d is the amplitude and ω is the assumed circular frequency of vibrations, respectively.

In fact, the assumed form of displacement describes the harmonic generation of waves which for a long time after a motion has started is expected to reach a steady-state harmonic generation. Our aim is to find the solution for the velocity potential as well as the free surface elevation for the imposed excitation. Differentiation of (60) with respect to time leads to:

$$\begin{aligned} v &= \dot{\underset{\sim}{x}} = d \cdot \omega \cdot \sin \omega t \\ a &= \dot{\underset{\sim}{x}} = -d \cdot \omega^2 \cdot \cos \omega t. \end{aligned} \tag{61}$$

It can be seen, that the displacement $\underset{\sim}{x}$ and the velocity v are both equal to zero at $t = 0$. Like in the previous case it is convenient to divide the potential into two parts (see 33). According to the first equation (61) and the boundary conditions (35–39) we have:

$$\psi(x, z, t) = -\frac{8 \cdot d \cdot h \cdot \omega}{\pi^2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^2} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \sin \omega t. \tag{62}$$

To find the second potential ϕ we apply the Fourier cosine – transform to its Laplace equation. The resulting equations (40–42) are still valid, but now:

$$\psi^*(s, z, t) = -\frac{4 \cdot d \cdot \omega}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j-1)} \cdot \frac{1}{k_j^2 + s^2} \cdot \cos k_j z \cdot \sin \omega t. \tag{63}$$

Following the transform the derivative with respect to z, taken at $z = h$ is:

$$\left.\frac{\partial \psi^*}{\partial z}\right|_{z=h} + \frac{2 \cdot d \cdot \omega}{h} \sum_{j=1}^{\infty} \frac{1}{k_j^2 + s^2} \cdot \sin \omega t. \tag{64}$$

And, in view of (45), the equation may be rewritten as:

$$\left.\frac{\partial \psi^*}{\partial z}\right|_{z=h} = \frac{d \cdot \omega}{s} \tanh(sh) \cdot \sin \omega t. \tag{65}$$

Accordingly, in place of (48) the following equation holds:

$$\ddot{A} + r^2 \cdot A + F(s) \cdot \sin \omega t = 0, \tag{66}$$

where:

$$r^2 = g \cdot s \cdot \tanh(sh), \quad F(s) = \frac{g \cdot d \cdot \omega}{s} \cdot \frac{\tanh(sh)}{\cosh(sh)}. \tag{67}$$

The general solution of the non-homogeneous equation (66) for the case $r \neq \omega$ is:

$$A = C_1 \cdot \cos rt + C_2 \cdot \sin rt + \frac{F(s)}{\omega^2 - r^2} \cdot \sin \omega t. \tag{68}$$

And for the case $r = \omega$, we have:

$$A = C_1 \cdot \cos rt + C_2 \cdot \sin rt + \frac{F(s)}{4 \cdot \omega^2} \cdot [2 \cdot \omega \cdot t \cdot \cos \omega t - \sin \omega t] \tag{69}$$

where C_1 and C_2 are constants of integration. Let us now consider case (68). The corresponding solution is:

$$\phi^* = \left[C_1 \cdot \cos rt + C_2 \cdot \sin rt + \frac{F(s)}{\omega^2 - r^2} \cdot \sin \omega t\right] \cdot \cosh(sz). \tag{70}$$

From the initial conditions at $t = 0$ one has:

$$C_1 = 0, \quad C_2 = -\frac{\omega}{r} \cdot \frac{f(s)}{\omega^2 - r^2}. \tag{71}$$

Thus, the inverse Fourier transform of (70) leads:

$$\begin{aligned} \phi \; &= \; \frac{2 \cdot g \cdot d \cdot \omega}{\pi} \cdot \int_0^{\infty} \frac{\tanh(sh)}{s} \cdot \frac{\cosh(sz)}{\cosh(sh)} \cdot \frac{1}{\omega^2 - r^2} \times \\ &\times \left[\sin \omega t - \frac{\omega}{r} \cdot \sin rt\right] \cdot \cos(sx) ds. \end{aligned} \tag{72}$$

The final solution for the velocity potential is the sum of (67) and (72). According to the solutions and the linearized boundary condition on the upper surface of the layer, the free surface elevation is given by the formula:

$$\eta(x,t) = \frac{2 \cdot d \cdot \omega^2}{\pi} \cdot \int_0^{\infty} \frac{\tanh(sh)}{s} \cdot \frac{1}{\omega^2 - r^2} \cdot [\cos rt - \cos \omega t] \cdot \cos(sx) ds. \tag{73}$$

The equation (73) is similar to that of Madsen (1970). The solutions (72) and (73) are valid for $r \neq 0$. The case $r \neq 0$ leads to another solution of the problem. Since the latter case corresponds to the isolated point $r = \omega$ of zero measure, it does not alter the result of integrations (72) and (73) and therefore, solution (69) may be excluded from our considerations. Confining attention to solution (68) it is necessary to investigate the behaviour of the integrands in (72) and (73) in the neighbourhood of the point $r = \omega$. It can be seen, that in the limit $r \to \omega$, both denominators and numerators of the integrands are going to zero. A more detailed investigation of the behaviour shows, that the left-hand limit and the right-hand limit of the terms are finite and equal to the same values. Taking the limit of the integrand (72) one obtains:

$$\lim_{\substack{r \to \omega \\ r \gtrless \omega}} \frac{\sin \omega t - \frac{\omega}{r} \cdot \sin rt}{\omega^2 - r^2} = \frac{1}{2 \cdot \omega^2} \cdot [\omega \cdot t \cdot \cos \omega t - \sin \omega t]. \tag{74}$$

Similarly the limit in (73) yields:

$$\lim_{\substack{r \to \omega \\ r \gtrless \omega}} \frac{\cos rt - \cos \omega t}{\omega^2 - r^2} = \frac{1}{2 \cdot \omega^2} \cdot [\omega \cdot t \cdot \sin \omega t]. \tag{75}$$

Hence in conclusion, the integrands in (72) and (73) are bounded and continuous for finite values $x \geq 0$ and $t \geq 0$. Moreover, it is possible to show that the improper integrals are convergent for the finite values of the parameters. On the other hand, one may decide to divide each of the integrals (72) and (73) into two integrals following the two terms in the square brackets. In the latter case it is possible to show that the resulting improper integrals are convergent in the Cauchy principal-value sense. The solutions obtained are expressed in the form of improper integrals. To find their values for chosen parameters x and t we use the approximate numerical integration.

3.1. Numerical Integration

The solutions obtained so far are expressed in the form of improper integrals. To find their values we resort to approximate integration by means of the trapezoidal rule. Because of the trigonometric functions entering the integrals, the first step in the numerical integration is to find zeros of the integrands. Then, in the second step, values of the integrals over the spans between subsequent zeros are calculated. Finally, we form finite series which approximate the improper integrals. Following the procedure, let us consider the free surface elevation (73). Except for a small vicinity of the point $r = \omega$, it is convenient to split the integral into the set of integrals:

$$\eta(x,t) = \frac{2 \cdot d \cdot \omega^2}{\pi} \cdot (J_1 + J_2 + J_3), \tag{76}$$

where:

$$J_{1/2} = \int_0^\infty \frac{\tanh(sh)}{s} \cdot \frac{1}{\omega^2 - r^2} \cdot \cos(sx \pm rt) ds$$
$$J_3 = -\cos\omega t \cdot \int_0^\infty \frac{\tanh(sh)}{s} \cdot \frac{1}{\omega^2 - r^2} \cdot \cos(sx) ds. \tag{77}$$

Let us consider the integrals $J_{1/2}$ in detail. For the assumed values of x and t, the zeros of the integrand are roots of the equation:

$$|sx - rt| = N_j, \quad N_j = \frac{2j-1}{2} \cdot \pi, \quad j = 1, 2, 3 \dots \tag{78}$$

For $sx - rt > 0$, the roots are defined by intersections of the functions:

$$y_1(s) = \pm\sqrt{g \cdot s \cdot \tanh(sh)} \cdot t,$$
$$y_{2j}(s) = sx - N_j, \quad s \geq 0, \quad x \geq 0, \quad N_j > 0. \tag{79}$$

For $sx - rt < 0$, in place of the second of (79), we have:

$$y_{2j}(s) = sx + N_j, \quad j = 1, 2, 3 \dots \tag{80}$$

The functions $y_{2j}(s)$ form a set of straight lines with constant slope equal to x. The first function (79) is a parabola. The graphical illustration of the solution (79) is shown in Fig. 4. For $sh \gg 1$, we may assume $\tanh(sh) \cong 1$ and then use the approximate formula:

$$y_1(s) = \pm\sqrt{g \cdot s}. \tag{81}$$

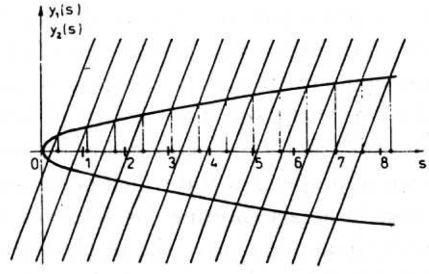

Fig. 4. Graphical solution of Eq. (78)

The latter case enables us to find closed analytical formulae describing the roots. Having the roots, say $s_1, s_2 \dots, s_m \dots$ we may find values of the integrals over each interval $(s_{i+1} - s_i)$, $i = 1, 2, 3 \dots$, and thus, to calculate the improper integral as to be the sum of the values. In this way, the improper integral is approximated by the series the terms of which have alternating signs and decrease in absolute value except for a finite number of the first terms. The similar considerations for the remaining integrals (76) lead to similar procedure and conclusions.

3.2. Numerical Example

To illustrate the considerations and procedures developed in the preceding sections, numerical calculations have been performed. The main aim of the calculations was to find the free surface elevation generated by the assumed excitation. The calculations were performed, according to the formula (73) for the following data:

$$\begin{aligned} &h = 0,40 \text{ m}, \quad d + 0,01 \text{ m}, \quad \lambda = 0,80 \text{ m}, \quad k_0 = \tfrac{2\cdot\pi}{\lambda} = 7,854\tfrac{1}{\text{m}} \\ &\omega = \sqrt{g \cdot k_0 \cdot \tanh(k_0 h)} = 8.761\tfrac{1}{s} \\ &T = \tfrac{2\cdot\pi}{\omega} = 40 \cdot \Delta t = 0.717 \text{ s}. \end{aligned} \tag{82}$$

Some of the results obtained in the computations are shown in Figures 5 and 6. The plots in Fig. 5 show the developing free surface elevation at chosen instants of time measured from the beginning of motion. Fig. 6 shows the spreading of the elevation calculated at partially equal time steps.

4. Conclusions

Solutions of the transient and initial problems considered in this paper wrere obtained by means of analytical formulation and Fourier transforms. The procedure applied to the transient vibrations of the block-fluid system has led to the integro-differential equation of the block motion. The equation was solved by means of numerical integration. From the computations it follows that the added mass of fluid, for the case of surface waves generated by the vibrations, is smaller than the impulsive added mass. The latter may be considered as the limiting case which is proper for higher frequencies of vibrations when the surface waves may be omitted. Following the computations performed it was possible to calculate an approximate envelope of the solution from which a substitute parameter of an apparent viscous damping may be obtained. It can be seen that for the considered case of rigid block with a single degree of freedom, the parameter substitutes the damping resulting from transmission of energy by the generated waves fairly well. For the case of initial generation of water waves with prescribed form excitation, the final solution for the free surface elevation is expressed in the form of an improper integral which needs approximate integration. Computation of the improper integrals entering the solutions obtained in this paper seems to be the main difficulty in application of the method proposed to more complicated cases for which a better way may be a discrete formulation. In the latter case, however, we badly need a rigorous analytical solution of a relevant simpler problem which may serve as the test of verification of the discrete methods. Thus, the method proposed in this paper may be useful in analysis of such phenomena.

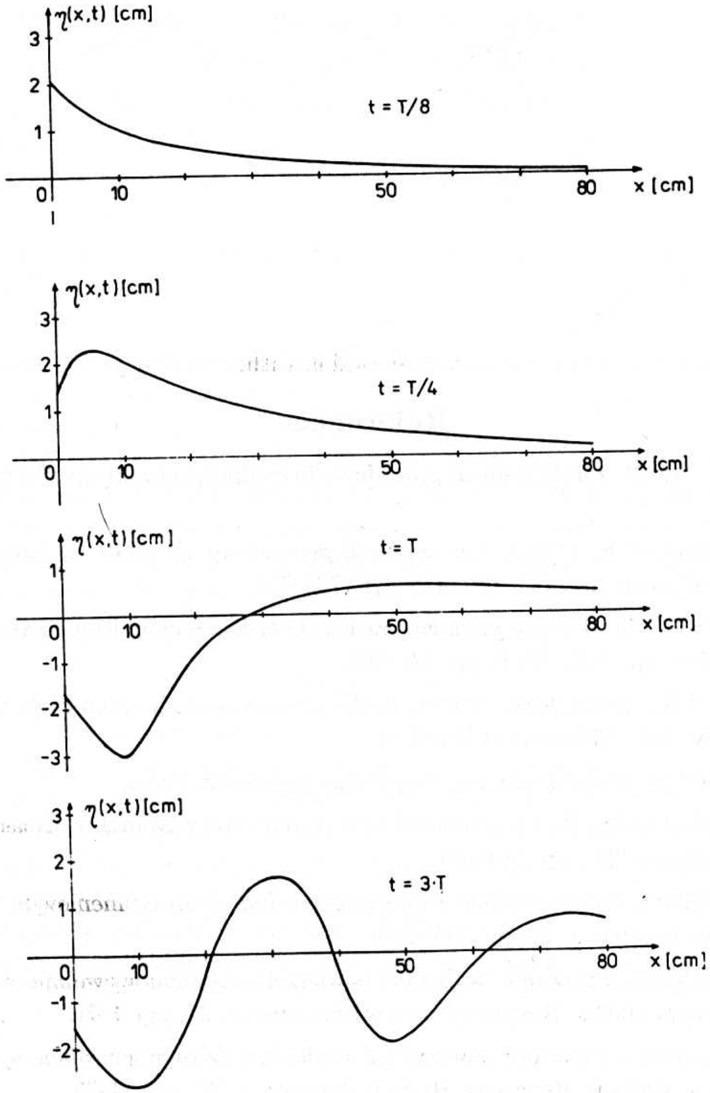

Fig. 5. The free surface elevation at chosen instants of time

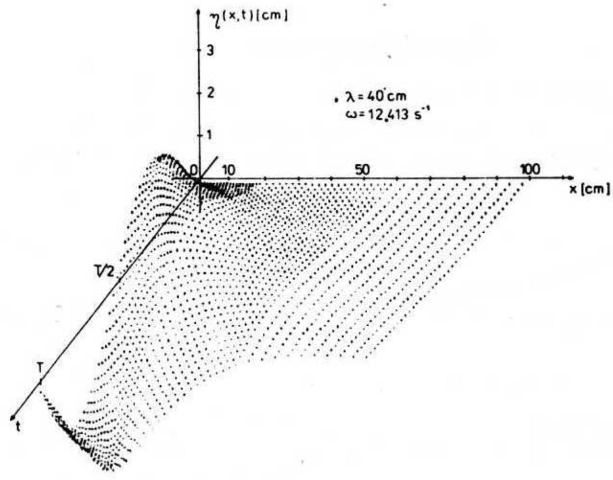

Fig. 6. Spreading of the free-surface elevation within the first period of oscilation

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Summary

The paper deals with transient vibrations of a rigid block and initial generation of waves in a layer of fluid of constant depth. The considerations are confined to the two-dimensional linearized problem of free vibrations of the block-fluid system starting from rest. Fourier transform technique is applied to solve the initial problems mentioned. Solution of the problems leads to improper integrals which need approximate numerical integrations. The considerations performed and formulae obtained are illustrated by numerical examples. The method proposed seems to be useful in applications especially in cases when the domain of solution is relatively regular.

Streszczenie

Praca zajmuje się zagadnieniem drgań tranzytywnych sztywnego bloku i generacją fal w paśmie cieczy o stałej głębokości. Rozważania ograniczono do dwuwymiarowego zlinearyzowanego problemu drgań swobodnych układu blok–ciecz rozpoczynającego ruch ze stanu spokoju. Do rozwiązania tych zagadnień zastosowano metodę transformacji Fouriera. Rozwiązanie problemu prowadzi do całek niewłaściwych, które wymagają przybliżonego numerycznego całkowania. Rozwiązania i otrzymane wyniki są zilustrowane przykładami numerycznymi. Zaproponowana metoda wydaje się być wygodna w zastosowaniach szczególnie wówczas, gdy obszar rozwiązania jest względnie regularny.