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Bearing capacity of non-cohesive soil by variational method

1. Introduction

In the paper the method of evaluation of the bearing capacity of homogeneous non-cohesive soil under a strip foundation subjected to axial and inclined loads has been presented. The proposed method is based on the variational approach formulated originally by Garber and Baker (1977, 1979) for the subsoil loaded axially and vertically. The analytical solution to the problem consists in deriving the minimum value of external load which may lead the soil-foundation system to the limit equilibrium state. The variational method does not employ any constitutive law, except for the Coulomb yield condition. No constraint is impose for the character of the critical functions (except for the analyticallity). Therefore, critical load min Q, obtained as a result of the solution, is the smallest load, that may cause failure. This means that for the soil parameters (c, ϕ, γ) and the foundation parameters (B, H), if $Q \leq \min Q\ (c, \phi, \gamma, B, H)$ the soil-foundation system is stable regardless of the consitutive law of the soil, while for $Q > \min Q\ (c, \phi, \gamma, B, H)$ – its stability depends on the consitutive character of the medium.

2. Assumptions

In the course of the analysis the following assumptions were taken:

(i) soil is homogeneous, isotropic and non-cohesive,

(ii) plane strain conditions,

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(iii) free rotation and one–side collapse of foundation are enabled,

(iv) continuous slip surface connects one edge of the foundation to the ground,

(v) the soil above the foundation level is replaced by uniformly distributed surcharge $(\gamma * H)$,

(vi) solution will be sought in the class of perfectly smooth functions.

3. Formulation of problem

A shallow strip foundation of width $2B$ is located in soil at depth H below the surface. The soil parameters are: the effective unit weight γ and the angle of internal friction ϕ (cohesion $c = 0$). Load is applied in the center of the foundation at an angle β. The calculation scheme is shown in fig.1. The state of limit equilibrium in the subsoil

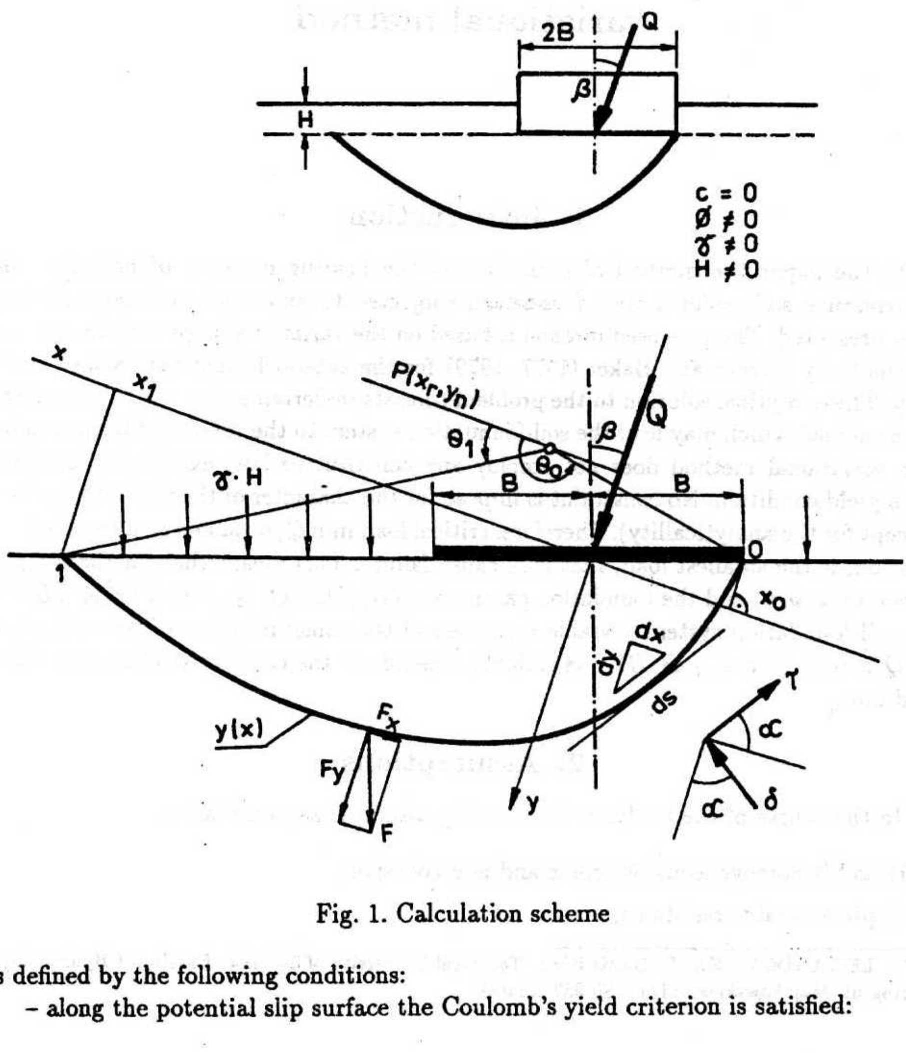

Fig. 1. Calculation scheme

is defined by the following conditions:

– along the potential slip surface the Coulomb's yield criterion is satisfied:

$$\tau = \sigma \tan \phi \tag{1}$$

where:

τ – the shear stress,
σ – the normal stress,
ϕ – the angle of internal friction.

– for the part of soil separated by the slip surface three equilibrium equations are valid:

$$Q - \int_s (\tau \sin\alpha + \sigma \cos\alpha)\, ds + \int_{x_o}^{x_1} F_y\, dx = 0 \tag{2}$$

$$\int_s (\sigma \sin\alpha - \tau \cos\alpha)\, ds - 1\int_{x_o}^{x_1} F_x\, dx = 0 \tag{3}$$

$$\int_s [(\tau \cos\alpha - \sigma \sin\alpha)\, y - (\tau \sin\alpha + \sigma \cos\alpha) x]\, ds +$$

$$+ \int_{x_o}^{x_1} F_x\, y dx + \int_{x_o}^{x_1} F_y x dx = 0 \tag{4}$$

where:

$$F_x = F\ sin\beta = (y \cos\beta - x\ \sin\beta + H)\ \gamma\ \sin\beta$$

$$F_y = F\ \cos\beta = (y\ \cos\beta - x\ \sin\beta + H)\ \gamma\ \cos\beta$$

$$\alpha = \arctan(dy/dx)$$

s– length of arc along the slip line

Eqs (2) and (3) represent the conditions of force equilibrium with respect to axis OY and OX, respectively. Eq. (4) is the moment equilibrium condition. Introduction of the eq. (1) to the eqs (2)–(4) yields the following equations:

$$Q - \int_s [\sigma(\cos\alpha + \psi \sin\alpha)\, ds + \int_{x_o}^{x_1} [(y \cos\beta - x\ \sin\beta + H)\gamma \cos\beta]\, dx = 0 \tag{5}$$

$$\int_s [\sigma(\sin\alpha - \psi \cos\alpha) ds - \int_{x_o}^{x_1} [(y \cos\beta - x \sin\beta + H)\gamma \sin\beta]\, dx = 0 \tag{6}$$

$$\int_s [\sigma(\psi \cos\alpha - \sin\alpha)]y - \sigma(\cos\alpha + \psi\sin\alpha)x]ds +$$

$$+ \int_{x_o}^{x_1} [(y\cos\beta - x\ \sin\beta + H)\ \gamma\ \sin\beta\ y]\ dx +$$

$$+ \int_{x_o}^{x_1} [(y\cos\beta - x\ \sin\beta + H)\ \gamma\ \cos\beta\ x]\ dx = 0 \qquad (7)$$

where: $\psi = \tan\phi$

The mathematical problem has been formulated in the following way: determine two functions $y(x)$ and $\sigma(x)$, which minimize the functional Q, eq. (5), and satisfy two integral equations (6) and (7). Then, using eq. (5) calculate min Q, which represents the bearing capacity. This is a variational problem of the isoperimetric type with one variable point which is point 1.

4. Variational analysis

In the analysis the system of non–dimensional parameters was introduced:

$$\bar{x} = \frac{x}{B};\ \bar{y} = \frac{y}{B};\ \bar{H} = \frac{H}{B};\ \bar{\sigma} = \frac{\sigma}{B\,\gamma};\ \bar{Q} = \frac{Q}{B^2\gamma} \qquad (8)$$

The basic equation *s* form the following set:

$$\bar{Q} = \int_{\bar{x}_o}^{\bar{x}_1} N\ (\bar{\sigma}, \bar{x}, \bar{y}, \bar{y}', \bar{H}, \psi, \beta)\ d\bar{x} =$$

$$= \int_{\bar{x}_o}^{\bar{x}_1} [\bar{\sigma}(\psi\bar{y}' + 1) - (\bar{y}\ \cos\beta - \bar{x}\ \sin\beta + \bar{H}]\ \cos\beta\ d\bar{x} \qquad (9)$$

$$\int_{\bar{x}_o}^{\bar{x}_1} 0(\bar{\sigma}, \bar{x}, \bar{y}, \bar{y}', \bar{H}, \psi, \beta)\ d\bar{x} =$$

$$= \int_{\bar{x}_o}^{\bar{x}_1} [\bar{\sigma}(\bar{y}' - \psi) - (\bar{y}\ \cos\beta - \bar{x}\ \sin\beta + \bar{H})\ \sin\beta]\ d\bar{x} = 0 \qquad (10)$$

$$\int_{\bar{x}_o}^{\bar{x}_1} P(\bar{\sigma}, \bar{x}, \bar{y}, \bar{y}', \bar{H}, \psi, \beta)\ d\bar{x} = \int_{\bar{x}_o}^{\bar{x}_1} \{\bar{\sigma}[\psi(\bar{y} - \bar{x}\bar{y}') - (\bar{x} + \bar{y}\bar{y}')] +$$

$$+ \ (\bar{y}\ \cos\beta - \bar{x}\ \sin\beta + \bar{H})\ \sin\beta\ \bar{y} + (\bar{y}\ \cos\beta - \bar{x}\ \sin\beta + \bar{H})\ \cos\beta\bar{x}\Big\} \times$$

$$\times\ d\bar{x} = 0 \qquad (11)$$

The solution to this variational problem is based on the application of the Lagrange's undetermined multipliers method, in which an intermediate function T is introduced:

$$\mathbf{T} = \mathbf{N} + \lambda_1 \cdot \mathbf{0} + \lambda_2 \cdot \mathbf{P} \tag{12}$$

where: λ_1, λ_2 – Lagrange's undetermined multipliers.
Thus, the solution to the problem is resolved into finding the extremum of the functional T.
The critical functions $\sigma(\bar{x})$ and $\bar{y}(\bar{x})$, for which the limit equilibrium occurs, while Q reaches its minimum, must satisfy:
(i) the set of Euler's differential equations for the functional T:

$$\frac{\partial T}{\partial \bar{\sigma}} - \frac{d}{d\bar{x}}\left(\frac{\partial T}{\partial \bar{\sigma}'}\right) = 0 \tag{13}$$

$$\frac{\partial T}{\partial \bar{y}} - \frac{d}{d\bar{x}}\left(\frac{\partial T}{\partial \bar{y}'}\right) = 0 \tag{14}$$

(ii) two integral equations, eqs (10) and (11)
(iii) the boundary conditions for the points $\bar{x}_1$ i $\bar{x}_o$:
– geometrical conditions:

$$\begin{aligned} \bar{x}_o &= -\cos\beta \\ \bar{y}_o &= \bar{y}(\bar{x} = \bar{x}_o) = -\sin\beta \\ \bar{y}_1 &= \bar{y}(\bar{x} = \bar{x}_1) = \bar{x}_1 \tan\beta \end{aligned} \tag{15}$$

– the variational boundary condition for the point $\bar{x}_1$ – transversality condition:

$$T - \bar{y}'\frac{\partial T}{\partial \bar{y}'} - \bar{\sigma}'\frac{\partial T}{\partial \bar{\sigma}'}\bigg|_{\bar{x}=\bar{x}_1} \delta\bar{x}_1 + \frac{\partial T}{\partial \bar{y}'}\bigg|_{\bar{x}=\bar{x}_1} \delta\bar{y}_1 + \frac{\partial T}{\partial \bar{\sigma}'}\bigg|_{\bar{x}=\bar{x}_1} \delta\bar{\sigma}_1 = 0 \tag{16}$$

where: δ – variational operator
This equation means that the point 1 can move only along the soil surface line.
The functional T is independent of $\bar{\sigma}'$ and therefore the eqs (13)–(16) take the following forms:

$$\frac{\partial T}{\partial \bar{\sigma}} = 0 \tag{17}$$

$$\frac{\partial T}{\partial \bar{y}} - \frac{d}{d\bar{x}}\left(\frac{\partial T}{\partial \bar{y}'}\right) = 0 \tag{18}$$

$$T - \bar{y}'\frac{\partial T}{\partial \bar{y}'} + \tan\beta\frac{\partial T}{\partial \bar{y}'}\bigg|_{\bar{x}=\bar{x}_1} \delta\bar{x}_1 = 0 \tag{19}$$

It was convenient to pass to the polar coordinate system:

$$\bar{x} = \bar{r}\cos\theta + \frac{1}{\lambda} \tag{20}$$

$$\bar{y} = \bar{r}\sin\theta + \frac{\lambda_1}{\lambda_2} \tag{21}$$

As a result of the solution of the first Euler's differential equation the equation of the log spiral was obtained:

$$\bar{r}(\theta) = \bar{r}_o \exp[(\theta_o - \theta)\ \psi] \tag{22}$$

where: $(\bar{r}_o, \theta_o)$ – integration constants; polar coordinates of the point '0'
The solution of the second Euler's equation gives the function $\bar{\sigma}(\theta)$ of the form:

$$\bar{\sigma}(\theta) = \bar{r}_o A(\theta) + C\ \exp(2\theta\psi) + \frac{\bar{r}_o}{2\psi}\sin\beta\ \sin(\theta_o - \beta) - \frac{\bar{H}}{2\psi}\sin\beta \tag{23}$$

where:

$$A(\theta) = \frac{1}{1 + 9\psi^2}[\sin(\theta - 2\beta) - 3\psi\cos(\theta - 2\beta)]\exp[\theta_o - \theta)\psi] \tag{24}$$

C – integration constant, determined from the transversality condition
Finally, the solution to the problem is reduced to the determination of three parameters $(\bar{r}_o, \theta_o, \theta_1)$, appearing in eqs (22) and (23).

They can be calculated in the way of integration and then numerical solution of the system of three non–linear algebraic equations. The numerical analysis of the solution proved that the set of the parameters $(\bar{r}_o, \theta_o, \theta_1)$ always exists and is unique.

5. Numerical examples

Several calculations for various data sets were carried out. Some of the examples are presented in fig. 2.

6. Results and analysis

The calculations of the bearing capacity of a strip foundation resting on non-cohesive soil ($H = 0$) enabled the determination of the bearing capacity factor $N_{\gamma\beta} = f(\phi, \beta)$ for the Terzaghi classical formula. The results are presented in the convenient form for the engineering practice, (fig. 3). Fig. 4 shows the comparison between the variational solution, the other solutions and the experiments for the case of $\beta = 0$. Fig. 5 presents the results of variational method and those obtained by Tran-Vo-Nhiem, Lebeque and Sokolowski for the case of $\beta \neq 0$.

The comparative analysis shows that the variational method gives for $\beta = 0$ the values of the bearing capacity factor N_γ which are in general consistent with the experimental data and higher than those obtained by the other authors.

For $\beta \neq 0$ the $N_{\gamma\beta}$ – values are higher than the values given by Tran-Vo-Nhiem, Sokolowski and Lebeque (for the range $\phi > 25° - 35°$). The existing difference in the results can be explained by the assumption of the smoothness of the slip line, but this finds its confirmation in the experimental observations (Jumikis, Chummar, the authors).

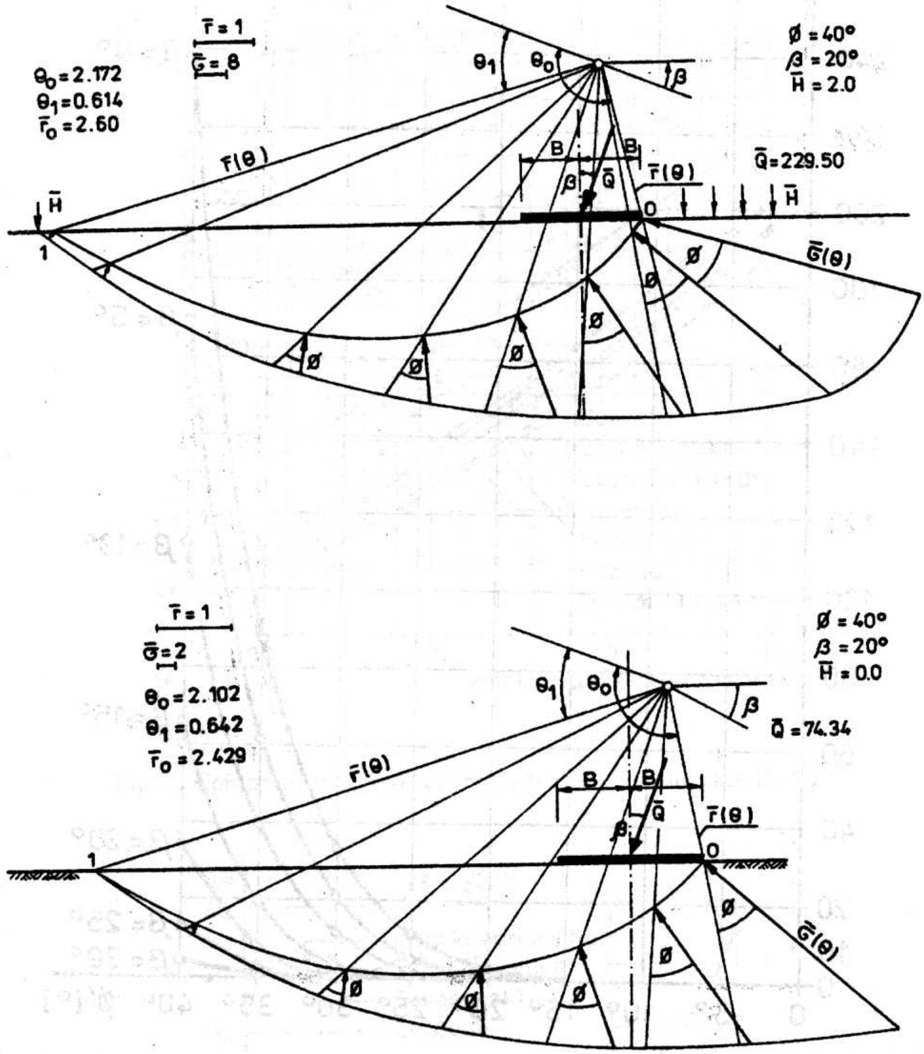

Fig. 2. Critical line $r(\theta)$ and distribution of normal component of critical stress $\sigma(\theta)$.

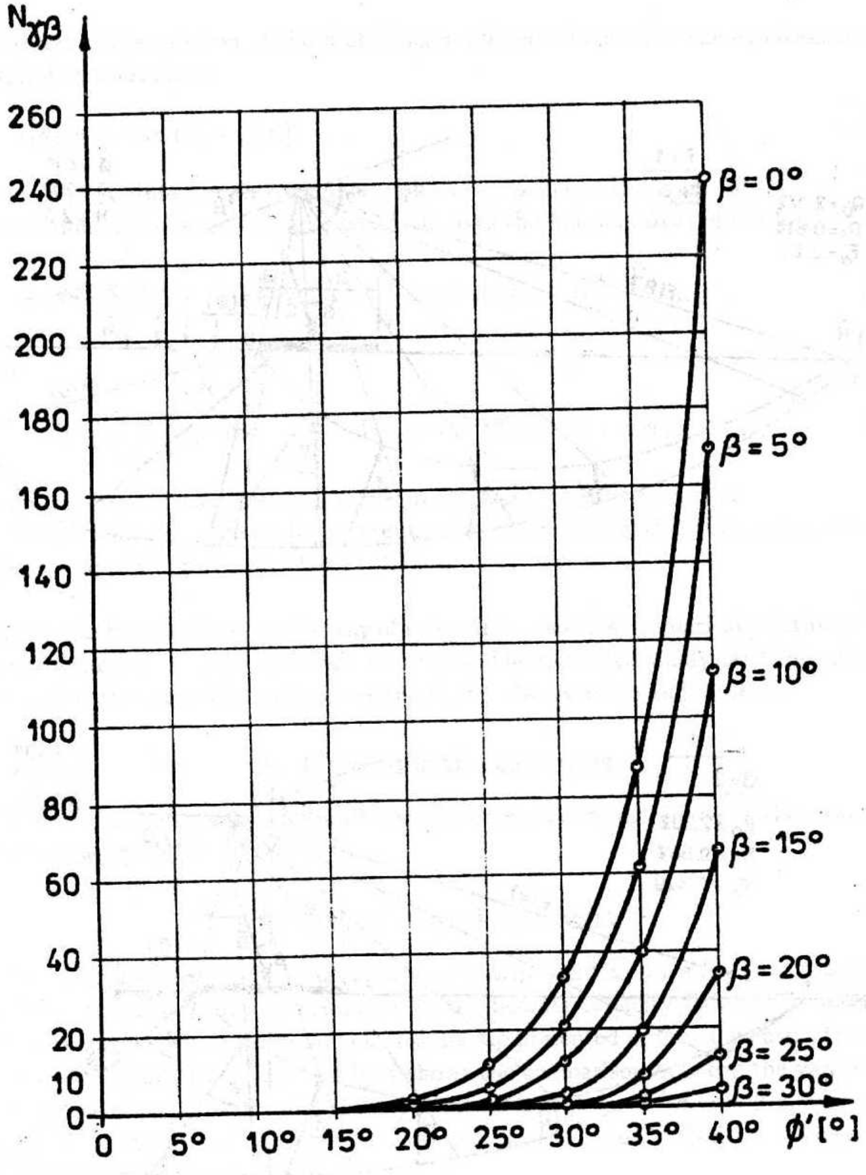

Fig. 3. Graph of the bearing capacity factor $N_{\gamma\beta}$ for various inclination angles of load determined by variational method

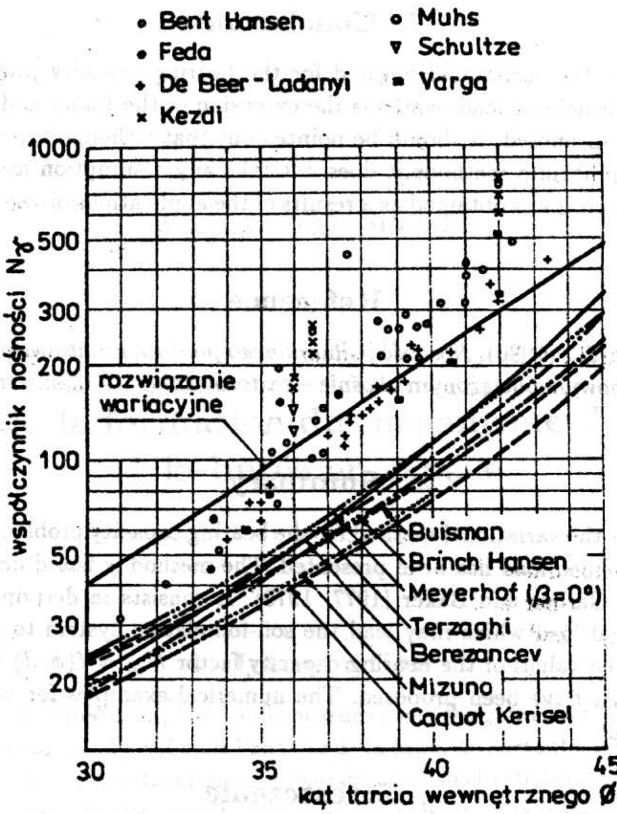

Fig. 4. Comparative graphs of the factor $N_{\gamma\beta}$ for the case $\beta = 0$

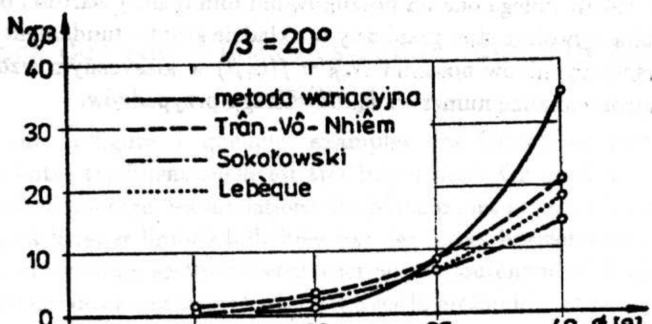

Fig. 5. Comparative graphs of the factor $N_{\gamma\beta}$ for the case $\beta \neq 0$

7. Conclusions

In the paper the variational method for the bearing capacity problem of a strip foundation with inclined load, which is the extension of the Baker and Garber formulation, has been proposed. It should be pointed out that although the method belongs to the limit equilibrium methods it does not take any assumption as to the slip line shape. The log spiral was obtained as a results of the applications of variational calculus theorems, only.

Reference

Lewandowska J., (1990), *Nośność jednorodnego podłoża gruntowego pod fundamentem pasmowym obciążonym ukośnie – zastosowanie rachunku wariacyjnego*, Thesis.

Summary

In the paper the variational method for the bearing capacity problem of a strip foundation with inclined load has been presented. The method is based on the variational formulation of Garber and Baker (1977, 1979). It consists in deriving the minimum value of external load which may lead the soil-foundation system to the limit equilibrium state. The values of the bearing capacity factor $N_{\gamma\beta} = f(\phi, \beta)$ for the Terzaghi classical formula have been proposed. The numerical examples for various data sets were carried out.

Streszczenie

Analiza nośności gruntów niespoistych za pomocą metody wariacyjnej

Przedstawiono w artykule metodę wariacyjną nośności fundamentów pasmowych obciążonych siłą ukośną. Metodę oparto na sformułowaniu wariacyjnym Garbera i Bakera (1977, 1979). Polega ona na poszukiwaniu minimalnej wartości obciążenia zewnętrznego, która wywołuje stan graniczny w układzie grunt – fundament. Zaproponowano wartości współczynników nośności $N_{\gamma\beta} = f(\phi, \beta)$ w klasycznym wzorze Terzaghiego. Przeprowadzono analizę numeryczną dla szeregu przypadków.