A R C H I W U M H Y D R O T E C H N I K I
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Turbulent Flow Around a Cylinder

1. The problem formulation

The paper deals with determination of horizontal, averaged in time, fields of velocity and pressure in a real fluid, forming a steady turbulent stream around a motionless cylinder with vertical axis. The text is of methodical and inspirational character and is a continuation of the paper (Sawicki 1984).

The problem, which has been solved by means of approximated method of series expanding, is not a new idea. In the bibliography one can find many papers, both experimental and theoretical, devoted to description of the phenomenon characteristics. However analytical description of the flow around a cylinder can be obtained only in laminar case or under some simplifications (e.g. plane potential flow (Łojcjańskij 1973)). Equations for turbulent conditions of flow are solved by means of numerical methods (Gryboś 1989) and in consequence it is not possible to apply these solutions in analytical considerations.

It would be very purposeful to have at one's disposal a simple but efficient method of receiving analytical formulas, describing not only a flow around a body, but also other problems of hydrodynamics. It seems that especially promising impression makes the method of series expanding, applied in this paper.

The velocity vector in the case under considerations has two components:

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta \tag{1}$$

while the boundary conditions can be written as follows (Fig. 1)

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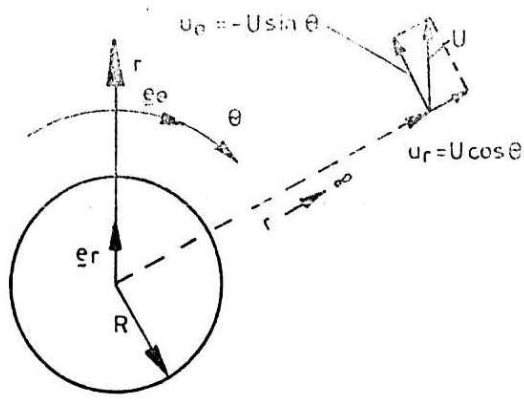

Fig. 1.

$$\left.\begin{array}{ll} r = R & ; u_r = u_\theta = 0 \\ r \to \infty & ; u_r = U\cos\theta \\ & \quad u_\theta = -U\sin\theta \\ & \quad p = p_0 \end{array}\right\} \tag{2}$$

Taking into account the form of the equation of mass conservation we can define the stream function ψ:

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, u_\theta = -\frac{\partial\psi}{\partial r} \tag{3}$$

Eliminating by means of differentiation the pressure from the Navier-Stokes equations (Łojcjańskij 1973) and making use of Eq. 3 one can obtain:

$$\begin{aligned} L(\psi) &= \frac{\partial\psi}{\partial\theta}\ \frac{\partial(\Delta\psi)}{\partial r} - \frac{\partial\psi}{\partial r}\ \frac{\partial(\Delta\psi)}{\partial\theta} - \nu r\Delta\Delta\psi + \\ &- \frac{\partial}{\partial\theta}\left[\frac{1}{r}\ \frac{\partial(r\pi_{rr})}{\partial r} + \frac{1}{r}\ \frac{\partial\pi_{r\theta}}{\partial\theta} - \frac{\pi_{\theta\theta}}{r}\right] + \\ &+ \frac{\partial}{\partial r}\left[r\ \frac{\partial\pi_{r\theta}}{\partial r} + \frac{\partial\pi_{\theta\theta}}{\partial\theta} + 2\pi_{r\theta}\right] = 0 \end{aligned} \tag{4}$$

2. Determination of Reynolds tensor π

The selection of the turbulence model is very important element of the task formulation. The problem under considerations has rather simple geometrical charakteristics, so it is possible to make use of results of comparative study (Piwecki, Sawicki 1986-1987, 1988-1989; Sawicki 1989). In this paper the diffusive model of turbulence has been choosen, when the Reynolds tensor is a linear function of averaged strain rate tensor S:

$$\left.\begin{aligned} S_{rr} &= \frac{1}{r}\ \frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\ \frac{\partial\psi}{\partial\theta} \\ S_{\theta\theta} &= -\frac{1}{r}\ \frac{\partial^2\psi}{\partial r\partial\theta} \\ S_{r\theta} &= \frac{1}{2r^2}\ \frac{\partial^2\psi}{\partial\theta^2} - \frac{1}{2}\ \frac{\partial^2\psi}{\partial r^2} \end{aligned}\right\} \tag{5}$$

The problem shows an axial symmetry, so according to the Wang's theory (Sawicki 1989) one can expect that the phenomenon will prove the transversal isotropy. This feature yields the following form of constitutive equation $\pi(S)$:

$$\left.\begin{aligned} \pi_{rr} &= -2/3\rho k + 2\mu_T S_{rr} \\ \pi_{\theta\theta} &= -2/3\rho k + 2\mu_T S_{\theta\theta} \\ \pi_{r\theta} &= 2\mu_T S_{r\theta} \end{aligned}\right\} \tag{6}$$

where: k - kinetic energy of turbulence. The coefficient of turbulent viscosity has been described by means of algebraic formula (Piwecki, Sawicki 1986-1987, 19881989):

$$\mu_T = 0.037\ \ \rho\ \ \mathrm{RU}\ \ \frac{(u_r^2 + u_\theta^2)^3}{U^6} \tag{7}$$

3. Determination of velocity field

The problem has been solved by means of the series expansion (Michlin, Smolicki 1972; Sawicki 1984). It was assumed that the solution of Eq. 4 can be expressed by the following series:

$$\psi \approx \frac{UR}{2}\left[\frac{2r}{R} - 3 + \frac{R^2}{r^2}\right]\sin\theta + \sum_{n=1}^{N} A_n \left(\frac{r}{R} - 1\right)^{n+2} \exp\left(-\frac{r}{R}\right)\sin[(n+1)\theta] \tag{8}$$

which obeys boundary conditions (2). Calculations were performed for $N = 1$ (as it was done in (Sawicki 1984)). The coefficient A_1 has been found by means of the least square method, from the following condition:

$$\frac{\partial J}{\partial A_1} = \frac{\partial}{\partial A_1}\int_D L(\psi)dD = 0 \tag{9}$$

The operator L is given by Eq. 4, and the integration area is a part of the surface (r, θ), for which $r \geq R$.

Substituting (5, 6, 7) into (4) one obtains rather complex expression, so only the integral along 0-axis has bean calculated analytically, whereas the integral along the r-axis was calculated numerically, what yielded the expression:

$$\begin{aligned} E^4 &- E^3\left(2.68 + \frac{65.4}{Re}\right) + E^2\left(2.79 + \frac{131.5}{Re} + \frac{2673.2}{Re^2}\right) + \\ &- E\left(1.2 + \frac{88.08}{Re} + \frac{2149.2}{Re^2}\right) + \left(0.202 + \frac{19.67}{Re} + \frac{720.0}{Re^2}\right) = 0 \end{aligned} \tag{10}$$

where:

$$E = A_1/UR \quad ; \quad Re = 2UR/\nu \tag{11}$$

The solution $E(Re)$ of Eq. 10 has been approximated by the function:

$$E = \frac{A_1}{UR} = 0.67 + \frac{16.35}{Re} \tag{12}$$

what gives the following stream function:

$$\begin{aligned} \psi \;\approx\; & \frac{UR}{2}\left[\frac{2r}{R} - 3 + \frac{R^2}{r^2}\right)\sin\theta + \\ + \;& UR\left[0.67 + \frac{16.35}{Re}\right]\;\left[\frac{r}{R} - 1\right]^3 \exp\left(-\frac{r}{R}\right)\sin 2\theta \end{aligned} \tag{13}$$

The velocity field is described by Eq. 3.

4. Determination of turbulent pressure

The term „turbulent pressure" denotes the sum of averaged presure and isotropic part of turbulent normal stress (Piwecki, Sawicki 1986-1987, 19881989):

$$P_c = p + 2/3\rho k \tag{14}$$

The distribution of p_c is given by the Helmholtz equation, which can be obtained from the Navier-Stokes equation (Sawicki 1984):

$$\begin{aligned} \Delta p \;=\; & \rho\frac{\partial^3\psi}{\partial\theta^3}\left[\frac{2}{r^2}\;\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\;\frac{\partial\psi}{\partial r}\right] - \rho\frac{\partial^2\psi}{\partial r\partial\theta}\left[\frac{2}{r^2}\;\frac{\partial^2\psi}{\partial r\partial\theta} + \right. \\ + \;& \left.\frac{1}{r^2}\;\frac{\partial\psi}{\partial r}\right] - \rho\frac{2}{r^4}\left(\frac{\partial\psi}{\partial\theta}\right)^2 - \frac{1}{r}\;\frac{\partial^2(r\pi_{rr})}{\partial r^2} - \frac{2}{r}\;\frac{\partial^2\pi_{r\theta}}{\partial r\partial\theta} + \\ + \;& \frac{1}{r}\;\frac{\partial\pi_{\theta\theta}}{\partial r} - \frac{1}{r^2}\;\frac{\partial^2\pi_{\theta\theta}}{\theta\theta^2} - \frac{2}{r^2}\;\frac{\partial\pi_{r\theta}}{\partial\theta} \end{aligned} \tag{15}$$

Assuming that the function p_c can be described by the series:

$$p_c = p_o + \sum_{n=1}^{N} M_n\left(\frac{r}{R}\right)^{n-2}\exp\left(-\frac{r}{R}\right)\cos[(n+1)\theta] \tag{16}$$

the following value of the coefficient M_1 has been obtained ($N = 1$, the least square method):

$$M_1 = -\rho U^2\left[1.59 + \frac{30.79}{Re} + \frac{5324.9}{Re^2}\right] \tag{17}$$

The turbulent pressure is described by the formula:

$$p_c = p_o - \rho U^2\left[1.59 + \frac{30.79}{Re} + \frac{5324.9}{Re^2}\right]\;\left(\frac{r}{R}\right)^3\exp\left(-\frac{r}{R}\right)\cos 2\theta \tag{18}$$

5. Determination of drag coefficient

The force acting on the cylinder (related to the unit of its length) is given by the following integral (Łojcjańskij 1973):

$$W = R \int_0^{2\pi} \left[-p_c \cos\theta - \mu \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \sin\theta \right] \Bigg|_{r=R} d\theta \tag{19}$$

which after calculations gives:

$$W = \rho R U^2 \left(0.92 + \frac{27.22}{Re} + \frac{3077.8}{Re^2} \right) \tag{20}$$

The drag coefficient is defined as a ratio of the drag force and hydrodynamic thrust:

$$C_D = W/\rho R U^2 \tag{21}$$

Substituting (20) in (21) we have:

$$C_D = 0.92 + \frac{27.22}{Re} + \frac{3077.8}{Re^2} \tag{22}$$

6. Discussion

Results obtained in this paper have been presented graphically. The velocity distribution is shown in Fig. 2, as a diagram $u_B = f(y/D)$ for $x = 8D$, where u_B is a normalized velocity:

$$u_B = \frac{U - u_x(x,y)}{U - u_x(x, y=0)} \tag{23}$$

The dashed line in this figure shows experimental data, taken from the paper (Kovásznay 1948/1949). Experiments have been carried out for $Re = 56$. Comparison of both curves enables us to state, that calculated velocity distribution shows mixed conformity with experimental results. This conformity is acceptable in the region of the $0x$-axis, but is getting worse when the variable y/D is increasing. It can be seen that calculated spread of the wake behind the cylinder is greater than observed one.

As a second parameter the drag coefficient has been analysed. The full line in Fig. 3 shows the relation $C_D(Re)$ according to Eq. 22, whereas experimental data after (Walden, Stasiak 1971) shows the dashed line. The consistence of both curves for $Re > 100$ is apparently better than in the previous case, what confirms the statement presented above, that the calculated velocity profile is better related to the real one near the wall. A big disaccord of both curves in Fig. 3 for low Reynolds numbers ($Re <$ 100) is self-evident, as the model under study does not refer to the laminar flow.

Reassuming, comparison of calculated and measured results does not speak very well for the model presented in this paper. However one has to remember that this model is simplified (transversal isotropy has been assumed) and the equations were solved by means of approximated method. Taking into account that the paper is of methodological character, the qualitative conformity of obtained and experimental results leads to the conclusion, that it is purposeful to continue efforts in this direction.

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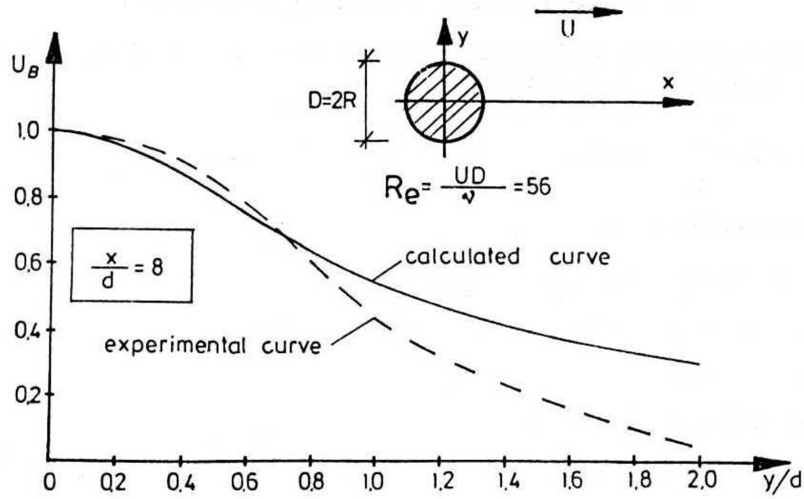
U
U_B
y
x
D=2R
$R_e = \frac{UD}{\nu} = 56$
1,0
0,8
0,6
0,4
0,2
$\frac{x}{d} = 8$
calculated curve
experimental curve
0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
y/d

Fig. 2.

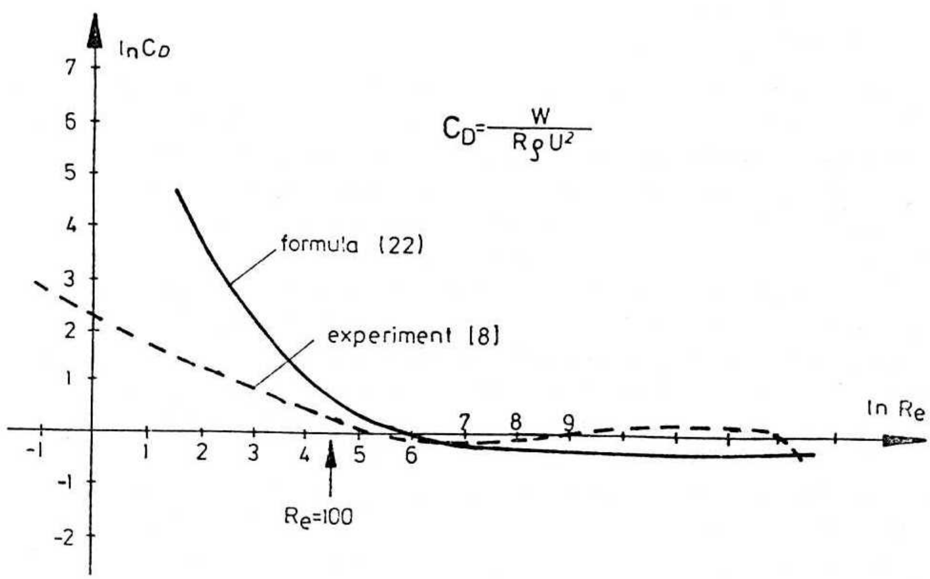
$\ln C_D$
$C_D = \frac{W}{R \rho U^2}$
formula (22)
experiment [8]
ln Re
$R_e = 100$
7
6
5
4
3
2
1
-1
-2
1
2
3
4
5
6
7
8
9

Fig. 3.

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Summary

The paper is devoted to the problem of mathematical modelling of velocity and pressure fields for steady turbulent flow around a cylinder. The governing equations have been formulated and solved by means of the mean-square method. Eddy viscosity has been described by algebraic formula (7). The results are compared with experimental data.

Streszczenie

Praca poświęcona jest przybliżonemu określeniu pola prędkości i ciśnień dla stacjonarnego turbulentnego opływu cylindra. Sformułowano równania zachowania i rozwiązano je metodą najmniejszych kwadratów. Współczynnik lepkości burzliwej określono wzorem algebraicznym (7). W oparciu o otrzymane rezultaty wyznaczono współczynnik oporu cylindra. Wyniki obliczeń porównano z materiałem eksperymentalnym i przeanalizowano.