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De Saint–Venant equations for non–Newtonian fluids

1. Introduction

One–dimensional flow of Newtonian fluid in a free–surface system can be described by so–called de Saint–Venant equations (Eqs 4, 5). These equations, the manner of initial and boundary conditions formulation and the solving process are broadly presented in the bibliography and can be regarded as a complete tool of one–dimensional hydraulic problems description.

However, there exists a demand for similar methods in different branches of science and technology. For example, one can enumerate free–surface systems in chemical engineering, or underwater landslides, especially interesting in geotechnics.

These phenomena very often have the same kinematic characteristics like open channel flows, but the problem lies in the different physical properties of flowing medium (e.g. wood pulp in paper industry or liquefied flotation tailings), which shows non–Newtonian features.

The problem bibliography does not contain credible equations which could describe the phenomena mentioned above. Practical problems are usually solved under assumption that the Newtonian model can be accepted in the case under study (Hughes, Brighton 1967), or by means of rheological models, but with some simplifications (e.g. for quasi–steady case (Jeyapalan 1982).

It seems that hydraulics and hydromechanics are obliged to formulate proper relations, which could describe one–dimensional unsteady flow of non–Newtonian medium in free–surface systems. Paper is devoted to this question.

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Notation

B	–	the channel width
C	–	Chézy coefficient
$\underset{\sim}{f}$	–	unit mass force
Fr	–	Froude number
g	–	gravitational constant
h	–	water depth
i_o	–	bottom slope
m	–	Manning coefficient
n	–	material constant
$\underset{\sim}{p}$	–	unit surface force
o_τ	–	bottom shear stress
P	–	stress tensor
P_y	–	yield stress tensor
Q	–	discharge of fluid
r	–	coordinate
R	–	radius
R_H	–	hydraulic radius
S	–	cross–section area
t	–	time
u	–	mean velocity in a cross–section
u^P, u^B	–	mean velocity after power law and Bingham models respectively
w_p	–	wetted perimeter
x	–	coordinate
κ	–	material constant
λ	–	Nikuradse coefficient
μ	–	dynamic viscosity
μ_p	–	plastic viscosity
ν	–	kinematic viscosity
ρ	–	fluid density
σ	–	fluid surface
τ	–	fluid volume
τ_y	–	component of the yield stress tensor

2. De Saint–Venant equations for Newtonian fluids

The starting point is determined by the equation of mass conservation, according to which the mass of fluid contained in the fluid volume τ does not change in time:

$$\frac{d}{dt}\int_{\tau}\rho d\tau = 0 \tag{1}$$

and linear momentum conservation law (which states that the velocity of linear momentum change is equal to the resultant force which acts on the fluid volume τ):

$$\frac{d}{dt}\int_{\tau}\rho\underset{\sim}{u}\,d\tau = \int_{\tau}\rho\underset{\sim}{f}\,d\tau + \oint_{\tau}\underset{\sim}{p}\,d\sigma \tag{2}$$

In consideration of the geometrical specifity of free-surface streams it is very convenient to choose the volume element $d\tau$ as a part of a „slice", cut out in the stream by two succesive cross-sections (Fig. 1):

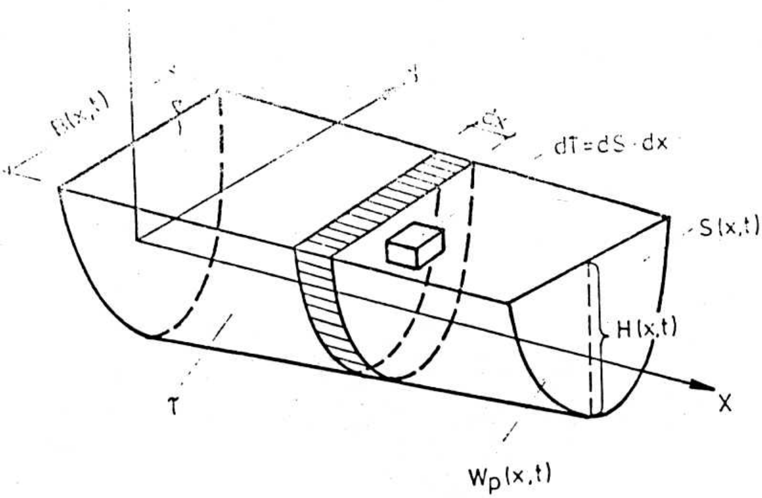

Fig. 1.

$$d\tau = dS\ dx \tag{3}$$

In this manner we can introduce mean parameters of the flow, averaged in the stream cross-section. Detailed derivation of governing equations can be found in the problem bibliography (e.q. Puzyrewski, Sawicki 1987; Yevjevich Mahmood 1975), so in this place we can present only the final form of de Saint-Venant system (for the channel of constant width B):

$$\frac{\partial h}{\partial t} + \frac{\partial(hu)}{\partial x} = 0 \tag{4}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + g\,\frac{\partial h}{\partial x} = gi_o - p_\tau\,\frac{w_p}{\rho S} \tag{5}$$

The system (4,5) is valid for each continuous liquid. Differentiation of specific medium can be performed by proper choice of function p_τ. The way in which this function was introduced for Newtonian fluids is very important for further considerations, so we must present it in details.

In order to calculate shear stress p_τ it is necessary to determine the flow velocity and pressure field, what is possible only in special, simple cases. Especially useful is the classical Hagen-Poiseuille solution, according to which velocity profile of steady laminar flow in semicircular open channels is of a parabolic character (Fig. 2):

$$u_x(r) = \frac{gi_o}{4\nu}\,(R^2 - r^2) \tag{6}$$

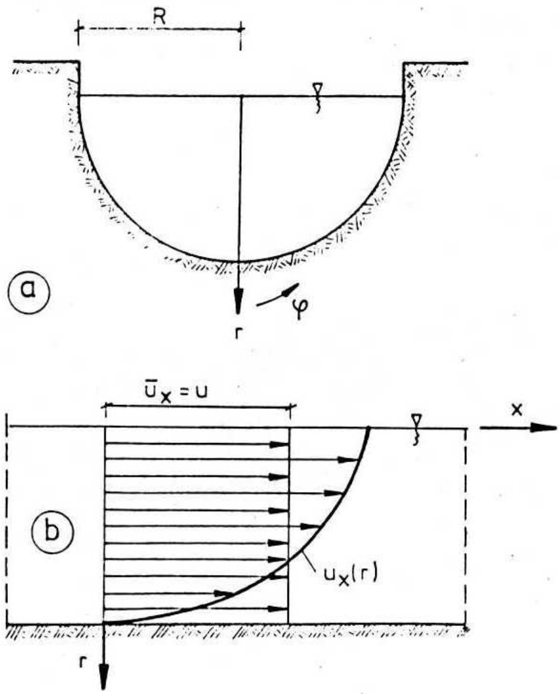

Fig. 2.

The latest equation makes it possible to determine the mean velocity:

$$\overline{u_x} = u(x,t) = \frac{gi_o R^2}{8\nu} \tag{7}$$

The bottom shear stress p_τ can be found be means of Newton hypothesis:

$$p_\tau = -\mu\frac{\partial u_x(r)}{\partial r} = \frac{\rho g i_o R}{2} \tag{8}$$

Replacing the bottom slope i_o in Eq. 8 by the mean velocity $u(x,t)$ (Eq. 7) we have instead of Eq. 8:

$$p_\tau(x,t) = \frac{4\mu u(x,t)}{R} \tag{9}$$

However so regular channels and uniform flow conditions occur very seldom in practice. Usually we have to do with turbulent flows in natural channels of complex shapes. In order to widen the range of Eq. 9 applicability we generalize this relation, replacing the radius R by hydraulic radius R_H:

$$R = 2\ R_H \tag{10}$$

and introducing coefficient λ (after Nikuradse), what makes possible to write the following formula:

$$p_\tau = \lambda\ \frac{\rho u^2}{8} \tag{11}$$

The coefficient λ for free-surface flows can be replaced by classical Chézy coefficient, according to the formula:

$$\lambda = \frac{8g}{C^2} \tag{12}$$

what leads to the relation:

$$p_\tau = \frac{\rho g u^2}{C^2} \tag{13}$$

Non-uniform steady flow (when $\partial/\partial t = 0$ but $\partial/\partial x \neq 0$) is described by the equation of so-called swelling curve (Puzyrewski, Sawicki 1987), which can be obtained from Eqs 4,5 and has the form:

$$Q = Bh(x)\ u(x) = \text{const.} \tag{14}$$

$$\frac{dh(x)}{dx} = \frac{i_o - u^2/c^2 R_H}{1 - Fr^2} \tag{15}$$

where the Froude number Fr is defined as follows:

$$Fr = \frac{u}{\sqrt{gh}} = \frac{Q}{S\sqrt{gh}} \tag{16}$$

In the simplest case of steady uniform flow we have Chézy (or Manning) formula:

$$u = C\sqrt{R_H\ i_o} = \frac{1}{m}\ R_H^{2/3}\ i_o^{1/2} \tag{17}$$

3. Rheological models selection

There is a serious amount of different rheological models. According to some suggestions presented in the problem bibliography (e.g. Hughes Brighton 1967; Schowalter 1978) two families of non-Newtonian fluids can be useful in our considerations, namely:

- power law fluids;
- Bingham plastics;

(it is necessary to underline, that the argumentation presented below can be applied for each other medium).

In the first case the constitutive equation, which links the stress tensor P with shear strain rate tensor D is a nonlinear relation:

$$P = \kappa D^n \tag{18}$$

where the values of material constants (κ and n) depend on the type of considered fluid and dimension of κ contains the value of n (e.g. $[\kappa] = N$ sec^n m^{-2}).

For Bingham plastics in turn the constitutive equation has the form:

$$\left.\begin{array}{lll} P = P_y + \mu_P D & \text{when} & P \geq P_y \\ D = 0 & \text{when} & P < P_y \end{array}\right\} \tag{19}$$

when P_y and μ_P must be determined experimentally for analysed medium.

4. De Saint–Venant equations for power law fluids

According to the way of acting described above, in the first order one has to solve the problem of laminar flow in semicircular open channel (Fig. 2). This solution is known in the bibliography (e.g. Hughes, Brighton 1967) and is described by the function:

$$u_x^P(r) = \frac{n}{n+1} \left(\frac{\rho g i_o}{2\kappa}\right)^{1/n} \left[R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right] \tag{20}$$

The mean velocity for this case is given by the formula:

$$\overline{u_x^P} = u^P = \frac{n}{3n+1} \left(\frac{\rho g i_o}{2}\right)^{1/n} R^{\frac{n+1}{n}} \tag{21}$$

Velocity profile in non-dimensional coordinates:

$$\frac{u_x^P(r/R)}{u_P} = \frac{3n+1}{n+1} \left[1 - (r/R)^{\frac{n+1}{n}}\right] \tag{22}$$

is shown in fig. 3. Shear stress on the channel bottom according to Eqs 20, 21 can be expressed as follows:

$$p_\tau^P = \kappa \left[-\frac{\partial u_x^P(r)}{\partial r}\right]^n \Bigg|_{r=R} = \kappa \left[\frac{3n+1}{n} \frac{u^P}{R}\right]^n \tag{23}$$

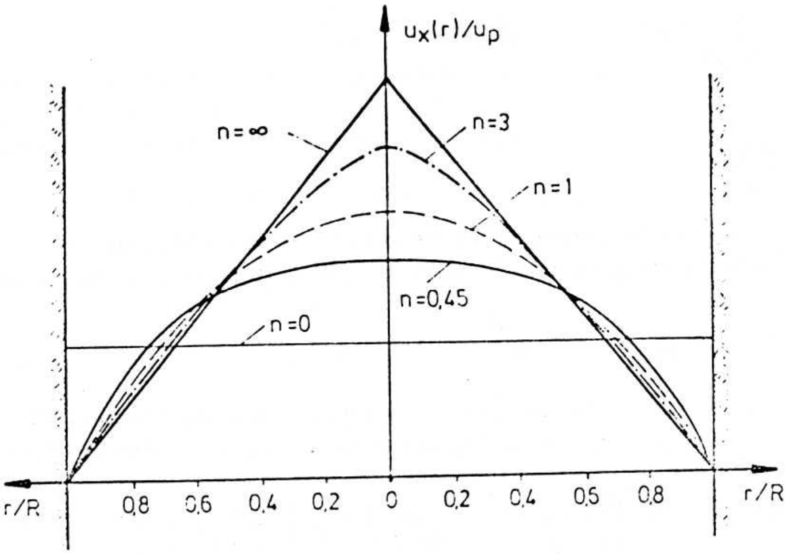

Fig. 3.

For optional cross-section of the stream we can replace the radius R by R_H (Eq. 10) and we obtain the following form of de Saint-Venant system for power law fluids:

$$\frac{\partial h}{\partial t} + \frac{\partial (hu^P)}{\partial x} = 0 \tag{24}$$

$$\frac{du^P}{dt} + g\,\frac{\partial h}{\partial x} = g\;i_o - \frac{\kappa}{\rho R_H}\left(\frac{3n+1}{n}\,\frac{u^P}{R_H}\right)^n \tag{25}$$

These equations refer to the laminar flow, but considering that the flow of liquefied sediments (and similar mediums) very often has creeping character it seems that just this case is especially interesting in practice.

For steady non-uniform case we have from Eqs 24, 25:

$$\frac{dh(x)}{dx} = \frac{i_o - \frac{\kappa}{\rho g R_H}\left[\frac{3n+1}{2n}\,\frac{u^P}{R_H}\right]^n}{1 - Fr^2} \tag{26}$$

while for steady uniform case we have the following analogon od Chézy formula:

$$u^P = \frac{2nR_H}{3n+1}\left(\frac{\rho g i_o R_H}{\kappa}\right)^{1/n} \tag{27}$$

5. De Saint–Venant equations for Bingham plastics

Velocity field of Bingham plastic laminar flow in semicircular open channel is given by the function (e.g. Hughes, Brighton 1967):

$$u_x^B(r) = \frac{\rho g i_o}{4\mu_P}\,(R^2 - r^2) - \frac{\tau_y}{\mu_P}\,(R - r) \quad (\text{for } r_P < r < R) \tag{28}$$

$$u_x^B = \frac{\tau_y^2}{\rho g i_o \mu_P} \, (R/r_P - 1)^2 \quad (\text{for } 0 < r < r_P) \tag{29}$$

where:

$$r_P = \frac{2\tau_y}{\rho g i_o} \tag{30}$$

and $\tau_y = P_y^{xr}$ is a (x, r) – component of the yield stress tensor P_y (Eq. 19).

According to the Bingham plastics definition the bottom shear stress is equal to:

$$p_\tau^B = \tau_y - \mu_P \left. \frac{\partial u}{\partial r} \right|_{r=R} = \frac{\rho g i_o R}{2} \tag{31}$$

Similarly to the previously discussed case it is convenient to express the bottom slope i_o by mean velocity u_B, which according to Eqs 28, 29 can be written as follows:

$$\overline{u_x^B} = n^B(x,t) = \frac{R^2 \rho g i_o}{8\mu_p} \left[1 - \frac{4}{3} \left(\frac{2\tau_y}{\rho g i_o R} \right) + \frac{1}{3} \left(\frac{2\tau_y}{\rho g i_o R} \right)^4 \right] \tag{32}$$

When the term which contains the yield stress is relatively small, i.e. when:

$$\frac{2\tau_y}{\rho g i_o R} \ll 1 \tag{33}$$

one can linearize Eq. 32, what gives a simplified relation:

$$i_o = \frac{8\mu_p u^B}{R^2 \rho g} + \frac{8\tau_y}{3\rho g R} \tag{34}$$

Substituting the latter equation in Eq. 24 and taking into account Eq. 10 we have:

$$p_\tau^B = \frac{2\mu_p u^B}{R_H} - \frac{4\tau_y}{3} \tag{35}$$

Finally we can write de Saint–Venant equations in the following form:

$$\frac{\partial h}{\partial t} + \frac{\partial (h u^B)}{\partial x} = 0 \tag{36}$$

$$\frac{du^B}{dt} + g \, \frac{\partial h}{\partial x} = g i_o - \left(\frac{2\mu_p}{\rho R_H^2} \, u^B - \frac{4\tau_y}{3\rho g R_H} \right) \tag{37}$$

For steady non-uniform flow we have an evident relation:

$$\frac{dh(x)}{dx} = \frac{i_o - 2\mu_p Q/(S\rho g R_H^2) - 4\tau_y/(3\rho g R_H)}{1 - Fr^2} \tag{38}$$

whereas for steady, uniform case:

$$u^B = \frac{\rho g i_o R_H^2}{2\mu_p} - \frac{2\tau_y R_H}{3\mu_p} \tag{39}$$

6. Examples

In order to present the influence of the constitutive equation choice on calculations results, three models described above, i.e. the system (4, 5, 13), (24, 25) and (36, 37) have been solved for the following initial conditions:

- Newtonian fluid (Eq. 17, $m = 0.023$, $\rho = 1000$ kg/m^3)

$$t = 0 \quad - \quad h(x,0) = 1.0 \text{ m}$$
$$u(x,0) = 0.43 \text{ m/s}$$

- power law fluid (Eq. 27, $n = 0.45$, $\kappa = 1.6[N\sec^{0.45}\text{m}^{-2}]$, $\rho = 1600$ kg/m^3)

$$t = 0 \quad - \quad h^P(x,0) = 1.0 \text{ m}$$
$$u^P(x,0) = 0.36 \text{ m/s}$$

- Bingham plastic (Eq. 39, $\mu_p = 0.15$ [kg/m sec], $\tau_y = 1.05$ N/m^2, $\rho = 1600$ kg/m^3)

$$t = 0 \quad - \quad h^B(x,0) = 1.0 \text{ m}$$
$$u^B(x,0) = 0.57 \text{ m/s}$$

and with boundary conditions:

$$x = 0 - h(0,t) = h^P(0,t) = h^B(0,t) = f(t) \quad \text{- given function (Fig. 4)}$$

$$x = L = 1000 \text{ m} - h(L,t) = h^P(L,t) = h^B(L,t) = 1.0 \text{ m}$$

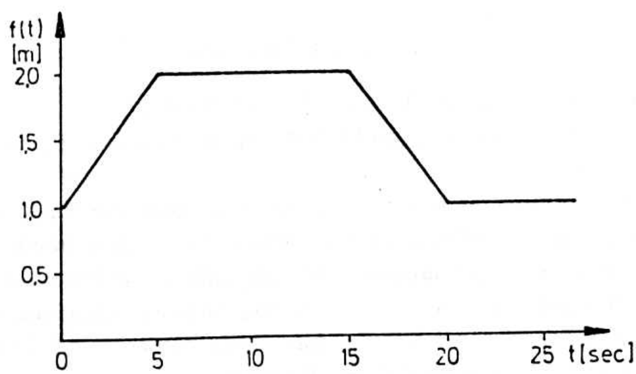

Fig. 4.

So formulated problem has been solved numerically (finite differences method, explicit Lax scheme (Potter 1973), for linear channel (when $R_h = h$). Obtained results are presented graphically in Fig. 5, as free-surface profiles after $\Delta t_1 = 25$ sec, $\Delta t_2 = 50$ sec, $\Delta t_3 = 75$ sec (for $i_o = 0.0001$).

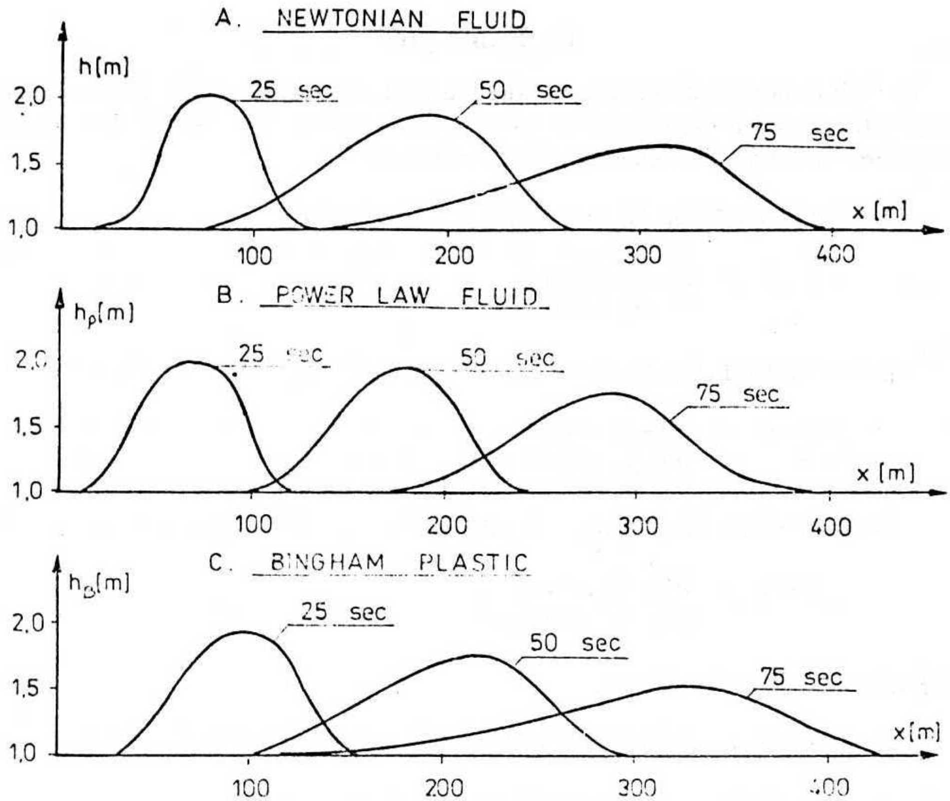

Fig. 5.

7. Conclusions

There is no sense to justify the usefulness of rheological models of fluids, but it is purposeful to formulate a system of equations, which could describe the motion of non-Newtonian fluid.

The essence of this paper lies in adaptation of mass and linear momentum conservation equations for one-dimensional unsteady flow with a free-surface, which are known as the Saint-Venant equations. This adaptation has been carried out for two types of non-Newtonian fluids, viz. for power law fluids and Bingham plastics. General equations of unsteady, non-uniform flow have been derived, and then – equations of the swelling curve and analogon of Chézy formula.

Relations obtained in the paper have been presented on an example of a simple wave propagation. The analysis of calculated results leads to the conclusion that the proper choice of adequate rheological model of considered medium is a very important question. Although according to the theory of characteristics the velocity of wave propagation (u_f) in an open channel depends mainly on the stream depth, but the effective velocity and especially the process of energy dissipation are closely related to the character of considered medium, what can be seen in Fig. 5.

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Summary

The paper is devoted to the system well-known in open-channel hydraulics, of the Saint-Venant equations, adapted for non-Newtonian fluids. These equations have been derived for two different kinds of rheological models, viz. power law fluids and Bingham plastics. An example has been enclosed, which presents the influence of the model choice on calculated results.

Streszczenie

Praca poświęcona jest dobrze znanemu w hydraulice koryt otwartych układowi równań de Saint-Venanta, ale odniesionemu do przepływu cieczy nienewtonowskich. Równania wyprowadzono dla dwóch modeli reologicznych – dla płynu potęgowego i dla płynu Binghama. Omówiono postać ogólną równań, a także ważne przypadki szczególne – przepływ stacjonarny, niejednostajny oraz stacjonarny, jednostajny. Załączono przykładowe rozwiązania.