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The numerical presentation of cyclic shear strain accumulation procedure and its modification

1. Introduction

In the last several years the interest in structures subjected to cyclic loading has increased rapidly. It was mainly due to the necessity of a safe design of offshore platforms resting on sea beds and subjected to cyclic loading from waves, wind and currents. The static and cyclic loads cause in such conditions a very complex stress state in the soil under the platform. The example of simplified stress conditions for a few typical elements along a potential failure surface is shown in Fig. 1, (Eide et al. 1984). The average shear stress τ_a is composed of the initial shear stress τ_o in the soil prior to the installation of the platform and an additional shear stress $\Delta\tau$, which is induced by the submerged weight of the structure. The cyclic shear stress τ_{cy} is caused by the cyclic load. In a storm, the wave height and its period vary continuously from one wave to another, so the cyclic shear stress will also vary from cycle to cycle.

The mode of a soil failure during cyclic loading is quite different then in the case of a static shearing. In the undrained conditions the cyclic loading causes an increase in the mean pore water pressure and corresponding reduction of the effective normal stresses and increased shear strains. When the effective stresses approach zero failure by liquefaction occurs.

The shear strain amplitudes are usually used as a failure criterion. In practice a double shear strain amplitude, i.e. the difference between the maximum and the minimum values of strains in a single cycle equal to ±3% or ±15% are regarded as a failure.

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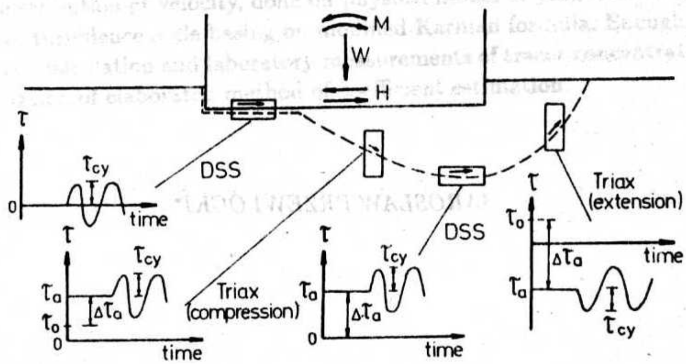

Fig. 1. Stress conditions for typical elements under the platform

The cyclic shear strength for this failure shear strain is defined as the sum of the average τ_α and the cyclic τ_{cy} shear stresses at failure :

$$\tau_f = (\tau_\alpha + \tau_{cy})_f \tag{1}$$

Generally, the cyclic shear strength depends on the stress path (e.g. triaxial or direct simple shearing), stress history and type of soil. For given an OCR test type and a soil, the cyclic shear strength becomes the function of the load history only and can be expressed as a function of the number of cycles and the shear stress amplitude.

A real storm is composed of waves with varying heights and periods, and the soil element beneath a platform is subjected to a cyclic shear stress which varies from one cycle to another. Such working conditions should be simulated in a laboratory. However, for practical reasons the laboratory tests are usually run with a constant cyclic shear stress throughout the test, and the soil strength for a real storm loading must be predicted from these tests. There are some procedures to do that, and one of them, most common, was proposed by Andersen et al. (1978), the so-called cyclic shear strain accumulation method. This method, its proposed numerical presentation and a modification introduced by the author will be described below.

2. Cyclic shear strain accumulation method

The method is based on a cyclic shear strain contour diagram, which is constructed from laboratory undrained cyclic tests run with a constant cyclic shear stress amplitude. An axample of such a diagram is presented in Fig. 2. The procedure postulates that the soil remembers the cyclic shear strain to which it has previously been subjected.

The cyclic shear strains are accumulated independently of the sequence of the applied load. The assumption implies that a cyclic shear strain amplitude increases monotonously without regarding how succesive values of shear stresses vary. It seems to be

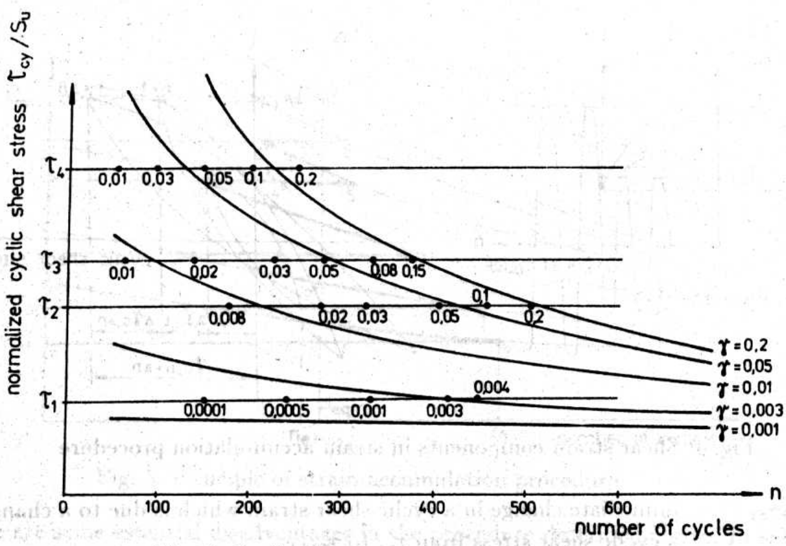

Fig. 2. Cyclic shear strain contour diagram

justified especially for the load history with increasing cyclic shear stress amplitudes. Such a load history is presented in Fig. 3.

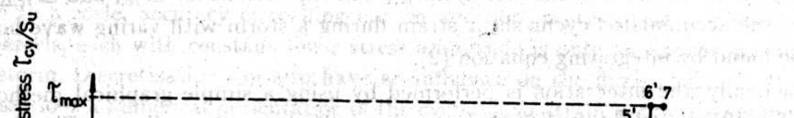

Fig. 3. Load history in form of parcels

The cyclic shear strain after $n + \Delta n$ cycles (Fig. 4) can be expressed as follows (Eide et al. 1984):

$$\gamma_{c,n+\Delta n} = \gamma_{c,n} + \Delta\gamma_{c,\Delta n} + \Delta\gamma_{c,i} \tag{2}$$

where:

$\gamma_{c,n}$ – cyclic shear strain after n cycles for a cyclic shear stress amplitude $\tau_{c,i}$

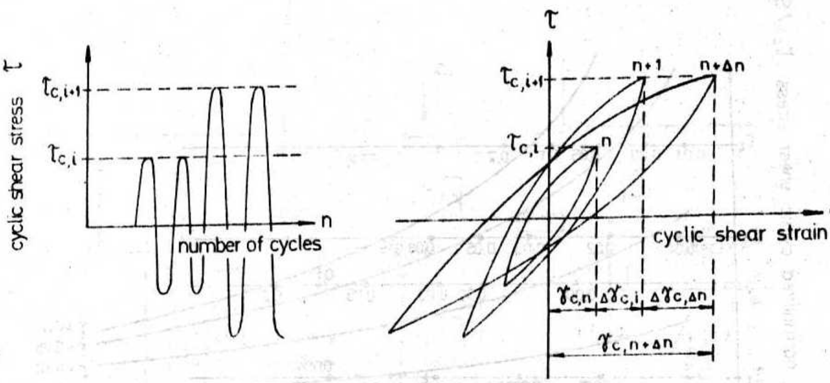

Fig. 4. Shear strain components in strain accumulation procedure

$\Delta\gamma_{c,i}$ – immediate change in a cyclic shear strain which is due to a change in a cyclic shear stress from $\tau_{c,i}$ to $\tau_{c,i+1}$

$\Delta\gamma_{c,\Delta n}$ – increase in a cyclic shear strain due to Δn cycles with a cyclic shear stress $\tau_{c,n+1}$

If the cyclic shear strain in the first cycle $\gamma_{c,1}$ and expressions for $\Delta\gamma_{c,i}$ and $\Delta\gamma_{c,\Delta n}$ are known, the accumulated cyclic shear strain during a storm with varing wave heights may be found by integrating equation (2).

Practically, the integration is performed by using a simple graphical method. It starts from the smallest cyclic shear stress amplitude τ_1. According to Fig. 3 there are n_1 cycles with such an amplitude. Drawing this two values on the plot (Fig. 5a) one obtains a cyclic shear strain γ_1 (point 2). Then increasing the cyclic shear stress level from τ_1 to τ_2, and assuming that the soil remembers the previous shear strain, transition to point 2′ takes place. However, when the cyclic shear stress is increased, the cyclic shear strain will also increase. It is assumed as an approximation that this increase is independent of the cyclic stress history and that it can be determined as the difference in cyclic shear strains for the cyclic shear stresses τ_1 and τ_2, from the curve for $n = 1$ (Fig. 5b). The increase is put on the diagram (Fig.5a) and it corresponds to the distance 2′2″.

The cyclic shear strain in point 2″ can be found from the diagram. It corresponds to the moment when the consecutive loading parcel starts with the cyclic shear stress amplitude τ_2 and n_2 number of cycles. The cyclic shear strain in point 2″ is equivalent to a strain which would appear if the smallest amplitude was τ_2 and a number of cycles in the first parcel was n_{e1}, which is determined from Fig. 5a.

The sector 2″3 corresponds to the increment of a cyclic shear strain due to the second parcel of cyclic loading with τ_2 and n_2. In this manner the development of γ_{cy} during an arbitrary load history can be predicted.

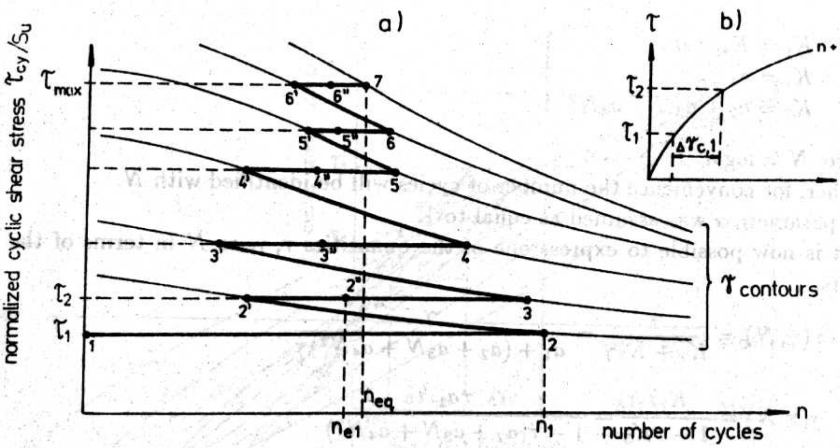

Fig. 5. Principle of strain accumulation procedure

There are some essential disadventages in the procedure described above. In spite of some rather controversial assumptions, the application of the procedure is time consuming and the results obtained by hand-calculation and interpolation are subjected to errors. The first drawback is the construction of the cyclic shear strain contour diagram. It is generally drawn by hand. Thus the final shear strain determined using such a diagram is subjective. Besides, as in each graphical method there are some additional errors due to a scale, accuracy of reading etc. In addition, representing a storm by a set of parcels, each with constant shear stress amplitude is only an approximation of a real storm. Discretization can also have an influence on the final results. In the following sections a numerical presentation of the cyclic shear strain accumulation method and the modified analytical method will be presented. The proposed approaches essentially eliminate the disadventages described above. The computer programme developed provides a means for fast and exact determination of the cyclic shear strain for an arbitrary load history results.

3. Cyclic shear strain contour diagram

The cyclic shear strain contour diagram gives the relation between the number of cycles n with a constant shear stress amplitude τ necessary to reach a shear strain γ (Fig. 2).

A general form of a stress-strain relation can be written according to Ronald and Madsen (1987) as:

$$\tau = \frac{\gamma + K_1\gamma^\alpha}{K_2 + K_3\gamma + K_4\gamma^\alpha} \tag{3}$$

The parameters K_1 to K_4 can be chosen as function of n to obtain a good fit to the data from the laboratory tests (Fig. 2). Based on the analysis of several models, chose the following representation of these parameters:

$$\left.\begin{aligned} K_1 &= K_4 = 0 \\ K_2 &= a_1 \\ K_3 &= a_2 + a_3N + a_4N^2 \end{aligned}\right\} \tag{4}$$

where: $N = \log n$
Further, for convenience the number of cycles will be identified with N.
The parametr α was assumed as equal to 1.

It is now possible to express one of the quantities τ, γ, or N in terms of the two others:

$$\tau(\gamma, N) = \frac{\gamma}{K_2 + K_3\gamma} = \frac{\gamma}{a_1 + (a_2 + a_3N + a_4N^2)\gamma} \tag{5}$$

$$\gamma(\tau, N) = \frac{K_2\tau}{1 - \tau K_3} = \frac{\tau a_1}{1 - \tau(a_2 + a_3N + a_4N^2)} \tag{6}$$

$$N(\tau, \gamma) = \frac{-\tau\gamma a_3 + \sqrt{(\tau\gamma a_3)^2 - 4\tau\gamma a_4(\tau a_1 + \tau\gamma a_2 - \gamma)}}{2\tau\gamma a_4} \tag{7}$$

There are, of course, some limitations concerning the above expressions, connected with the assumed relationship $\tau - \gamma - N$. The following condition should be satisfied in expression (5):

$$a_1 + (a_2 + a_3N + a_4N^2) > 0$$

in expression (6):

$$1 - \tau(a_2 + a_3N + a_4N^2) \;>\; 0 \quad \text{and} \quad a_1 > 0$$

and in (7) :

$$(\tau\gamma a_3)^2 - 4\tau\gamma a_4(\tau a_1 + \tau\gamma a_2 - \gamma) \;>\; 0 \quad \text{and} \quad a_4 > 0$$

Knowing values of the parameters a_1 to a_4 it is possible to express the relationship between the cyclic shear stress amplitude, cyclic shear strain and the number of cycles in the analytical way. The parameters are determined from the laboratory results (Fig. 2), using approximation procedures. Roland and Madsen (1987) used the maximum likelihood method in their work.

4. The numerical solution

Knowing the analytical expression for the relatioship $\tau - \gamma - N$ and values of the parameters a_1 to a_4, the final cyclic shear strain for a given storm can be determined in a simple way, according to the scheme presented in Fig. 5a. Using the notation in Fig. 6, one can write :

$$\left.\begin{aligned} \gamma_{m1} &= \gamma_{N1} = \gamma(\tau_1, N_1) \\ \gamma_{m2} &= \gamma_{N1} + \gamma_{N2} = \gamma[\tau_2, n(\tau_2, \gamma_{m1} + \Delta\gamma_{1,2}) + n_2] \\ \gamma_{m3} &= \gamma_{N1} + \gamma_{N2} + \gamma_{N3} = \gamma[\tau_3, n(\tau_3, \gamma_{m2} + \Delta\gamma_{2,3}) + n_3] \end{aligned}\right\} \tag{8}$$

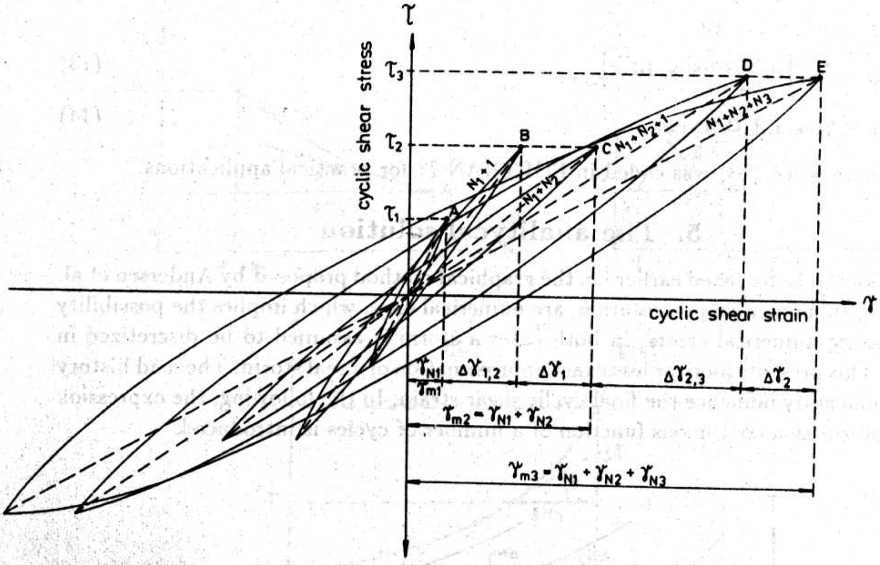

Fig. 6. Basic notation used for numerical procedure

where:

$$\left.\begin{aligned}\Delta\gamma_{1,2} &= \gamma(\tau_2,1)-\gamma(\tau_1,1)=\frac{\tau_2 a_1}{1-\tau_2 a_2}-\frac{\tau_1 a_1}{1-\tau_1 a_2}\\ \Delta\gamma_{2,3} &= \gamma(\tau_3,1)-\gamma(\tau_2,1)=\frac{\tau_3 a_1}{1-\tau_3 a_2}-\frac{\tau_2 a_1}{1-\tau_2 a_2}\end{aligned}\right\} \tag{9}$$

γ_{mi} – denotes the accumulated cyclic shear strain after the i-th parcel (after $n_1 + n_2 + \cdots + n_i$ cycles).

Generally, one can write a recurrent expression for γ_{mk}, the cyclic shear strain due to a storm consisting of k parcels with increasing cyclic shear stress amplitudes from τ_1 to τ_k:

$$\gamma_{mk} = \gamma(\tau_k, N_{ek}) = \gamma[\tau_k, n(\tau_k, \gamma_{m,k-1} + \Delta\gamma_{k-1,k}) + n_k] \tag{10}$$

where:

$$\Delta\gamma_{k-1} = \gamma(\tau_k, 1) - \gamma(\tau_{k-1}, 1) \tag{11}$$

Using expression (6), the total accumulated cyclic shear strain for a given load history is:

$$\gamma_{mk} = \frac{a_1\tau_k}{1-\tau_k(a_2 + a_3 N_{ek} + a_4 N_{ek}^2)} \tag{12}$$

where:

$$N_{ek} = \log\left[10^{N(\tau_k,\gamma_{zk})} + 10^{N_k}\right] \qquad (13)$$

$$\gamma_{zk} = \gamma_{m,k-1} + \Delta\gamma_{k-1,k} \qquad (14)$$

The expression (13) was coded in FORTRAN 77 for practical applications.

5. The analitycal solution

The methods discussed earlier i.e. the graphical method proposed by Andersen et al. (1978)and its numerical presentation, are numerical ones, which implies the possibility of appearing numerical errors. In both cases a storm is assumed to be discretized in parcels. This presents more or less exact approximation of a real storm. The load history may significantly influence the final cyclic shear strain. In the following, the expression for the storm as a continuous function of a number of cycles is introduced.

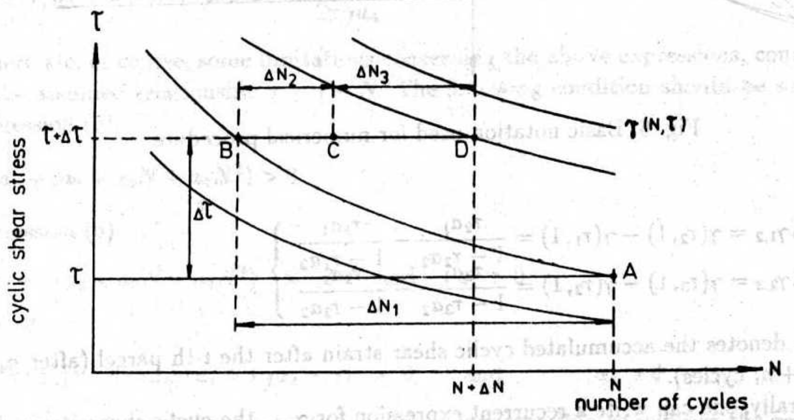

Fig. 7. Equivalent number of cycles components in strain accumulation procedure

According to Fig. 7 the cyclic shear strain for the cyclic shear stress amplitude τ, after N cycles corresponds to point A. For an increased amplitude of $\Delta\tau$ the shear strain goes to point D and can be expressed as :

$$\gamma_D = \gamma(\tau + \Delta\tau, N + \Delta N_1 + \Delta N_2 + \Delta N_3) \qquad (15)$$

It can be computed using formulae (16) if ΔN_1,ΔN_2, ΔN_3 are know. The sum $\Delta N = \Delta N_1 + \Delta N_2 + \Delta N_3$ corresponds to a cyclic shear stress amplitude increment $\Delta\tau$. The analytical solution could be obtained by a transition from the difference equation into the differential one. To do this, it is necessary to derive expression for $\Delta N_1, \Delta N_2$ and ΔN_3 values (Fig. 7). The first ΔN_1 increment is due to the difference of the number of cycles corresponding to stress change $\Delta\tau$ for the same cyclic shear strain $\gamma_A = \gamma_B$ (Fig. 8).

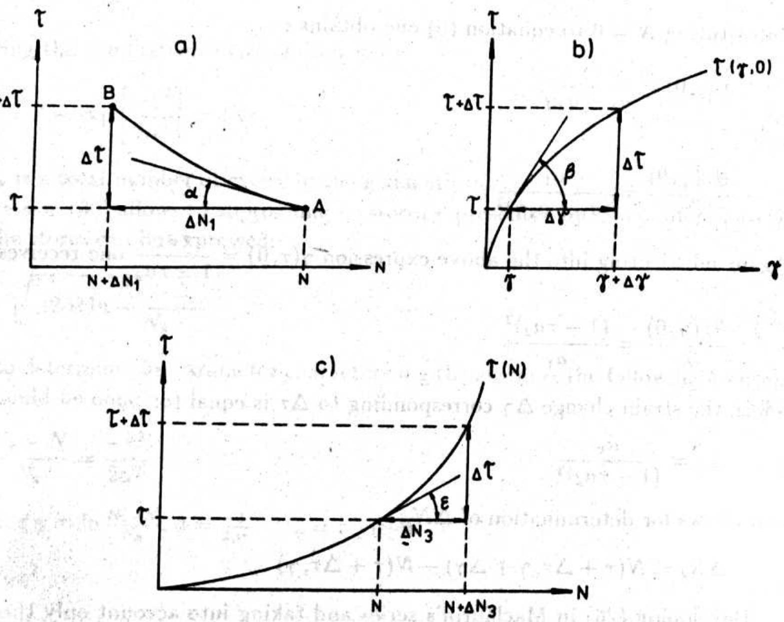

Fig. 8. Schemes of increments of equivalent numbers of cycles

For $\Delta\tau$ approaching zero one can write :

$$\tan\alpha = \frac{\Delta\tau}{\Delta N_1} = \frac{\partial\tau(N,\gamma)}{\partial N} \tag{16}$$

Deriving $\tau(N,\gamma)$ expressed by formulae (5) with respect to the number of cycles N one obtains:

$$\frac{\partial\tau(N,\gamma)}{\partial N} + \frac{-(a_3+2a_4N)\gamma^2}{[a_1+(a_2+a_3N+a_4N^2)\gamma]^2} \tag{17}$$

then:

$$\Delta N_1 = \frac{-[a_1+(a_2+a_3N+a_4N^2)\gamma]^2}{(a_3+2a_4N)\gamma^2}\Delta\tau \tag{18}$$

Substituting into the above expression the formulae (6) for γ and after some simple transformations :

$$\Delta N_1 = \frac{-1}{\tau^2(a_3+2a_4N)}\Delta\tau \tag{19}$$

The second increment $-\Delta N_2$ results from the cyclic shear strain change of $\Delta\gamma$ for one cycle $n=1$ or $N=0$ (Fig. 8b). As before one can write:

$$\tan\beta = \frac{\Delta\tau}{\Delta\gamma} = \frac{\partial\tau(\gamma,0)}{\partial\gamma} \tag{20}$$

Substituting $N = 0$ to equation (5) one obtains :

$$\tau(\gamma, 0) = \frac{\gamma}{a_1 + a_2\gamma} \tag{21}$$

so:

$$\frac{\partial\tau(\gamma, 0)}{\partial\gamma} = \frac{a_1}{(a_1 + a_2\gamma)^2} \tag{22}$$

Again substituting into the above expression $\gamma(\tau, 0) = \frac{\tau a_1}{1 - \tau a_2}$ one receives:

$$\frac{\partial\tau(\gamma, 0)}{\partial\gamma} = \frac{(1 - \tau a_2)^2}{a_1} \tag{23}$$

Thus, the strain change $\Delta\gamma$ corresponding to $\Delta\tau$ is equal to:

$$\Delta\gamma = \frac{a_1}{(1 - \tau a_2)^2} \tag{24}$$

and allows for determination of ΔN_2:

$$\Delta N_2 = N(\tau + \Delta\tau, \gamma + \Delta\gamma) - N(\tau + \Delta\tau, \gamma) \tag{25}$$

Developing (26) in Maclaurin's series and taking into account only the first order derivatives one gets:

$$\Delta N_2 = \frac{\partial N(\tau, \gamma)}{\partial\gamma}\Delta\gamma \tag{26}$$

Derivating $N(\tau, \gamma)$ expressed by (7) with respect to γ one obtains:

$$\frac{\partial N(\tau, \gamma)}{\partial N} = \frac{[\tau(a_2 + a_3N + a_4N^2) - 1]^2}{a_1\tau^2\sqrt{a_3^2 + 4a_4(a_3N + a_4N^2)}} \tag{27}$$

and using (21), the expression for ΔN_2 is:

$$\Delta N_2 = \frac{[\tau(a_2 + a_3N + a_4N^2) - 1]^2}{\tau^2(1 - \tau a_2)^2\sqrt{a_3^2 + 4a_4(a_3N + a_4N^2)}} \tag{28}$$

In order to determine ΔN_3 a storm should be described as a continuous function of a number of cycles. Assumining that the shear stress amplitudes probability density function follows Rayleigh's distribution:

$$f(\tau) = \frac{\tau}{\alpha^2}\exp\left[\frac{-\tau^2}{2\alpha^2}\right] \tag{29}$$

the distribution function has a form:

$$F(\tau) = 1 - \exp\left[\frac{-\tau^2}{2\alpha^2}\right] \tag{30}$$

Considering the cumulative curve one can write:

$$\frac{N}{N_k} = 1 - \exp\left[\frac{-\tau^2}{2\alpha^2}\right] = F(\tau) \tag{31}$$

where N_k is a total number of cycles in the given storm.

The expression (30) allows to approximate a storm represented by succesion of parcels. Finally the storm can be expressed:

$$\tau = \sqrt{-2\alpha^2 \ln \frac{N_k - N}{N_k}} \tag{32}$$

In order to determine the parameter characterizing the storm α the following transformation should be done :

$$\ln \frac{N_k - N}{N_k} = \frac{-\tau^2}{2\alpha^2} \tag{33}$$

Substituting $y = \ln \frac{N_k - N}{N_k}$, $a = \frac{-1}{2\alpha^2}$ $\quad x = \tau$ yields:

$$y = ax^2 \tag{34}$$

The parameter a can be found using the least squares method:

$$a = \frac{\Sigma x_i^4}{\Sigma x_i^2 y_i} \tag{35}$$

then :

$$\alpha = \sqrt{\frac{\Sigma x_i^2 y_i}{2\Sigma x_i^4}} \tag{36}$$

Now the increment ΔN_3 can be found, according to Fig. 8c. Similarly as for ΔN_1 and ΔN_2 one can write :

$$\tan \varepsilon = \frac{\Delta \tau}{\Delta N_3} = \frac{d\tau(N)}{dN} \tag{37}$$

$$\frac{d\tau(N)}{dN} = \frac{\alpha}{(N_k - N)\sqrt{-2 \ln \frac{N_K - N}{N_k}}} \tag{38}$$

then:

$$\Delta N_3 = \frac{N_k - N}{\alpha} \sqrt{-2 \ln \frac{N_k - N}{N_k}} \tag{39}$$

or

$$\Delta N_3 = \frac{N_k \tau}{\alpha^2} \exp\left[\frac{-\tau}{2\alpha^2}\right] \tag{40}$$

Having determined all increment components ΔN_1, ΔN_2, ΔN_3, their sum $\Delta N = \Delta N_1 + \Delta N_2 + \Delta N_3$ can be found using (20), (29) and (41):

$$\Delta N = \left[\frac{-1}{\tau^2(a_3 + 2a_4N)} + \frac{[\tau(a_2 + a_3N + a_4N^2) - 1]^2}{\tau^2(1 - \tau a_2)^2\sqrt{a_3^2 + 4a_4(a_3N + a_4N^2)}} + \right.$$
$$\left. + \frac{N_k\tau}{\alpha^2}\exp[-\tau^2/2\alpha^2]\right]\Delta\tau \qquad (41)$$

For the infinitesimal increments $\Delta\tau$ and ΔN the first-order nonlinear differential equation is obtained :

$$\frac{dN}{d\tau} = \frac{-1}{\tau(a_3 + 2a_4N)} + \frac{[\tau(a_2 + a_3N + a_4N^2) - 1]^2}{\tau^2(1 - \tau a_2)^2\sqrt{a_3^2 + 4a_4(a_3N + a_4N^2)}} +$$
$$+ \frac{N_k\tau}{\alpha^2}\exp[-\tau^2/2\alpha^2] \qquad (42)$$

In the above equation the expression (40) for ΔN_3 is substituted instead of (39). It is so because the increment ΔN_3 is being added independently of the actual N. For the given initial condition the equation (42) can be easily solved using numerical methods. The author used the fourth order Runge–Kutta method for solving the example presented below.

6. Numerical example

In order to illustrate the proposed modifications, the numerical example for a storm given in Table 1 is presented. The data are taken from (Anderson et al. 1978).

Table 1

Load composition for example calculation

Cyclic shear stress amplitude τ	Number of cycles n
0.441	10000
0.552	1900
0.576	650
0.630	250
0.693	66
0.738	22
0.801	8
0.864	3
0.900	1

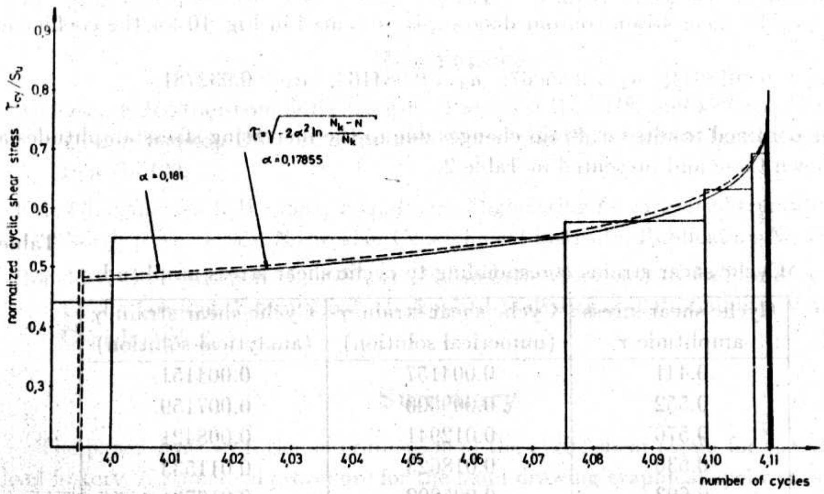

Fig. 9. Analytical function of load history

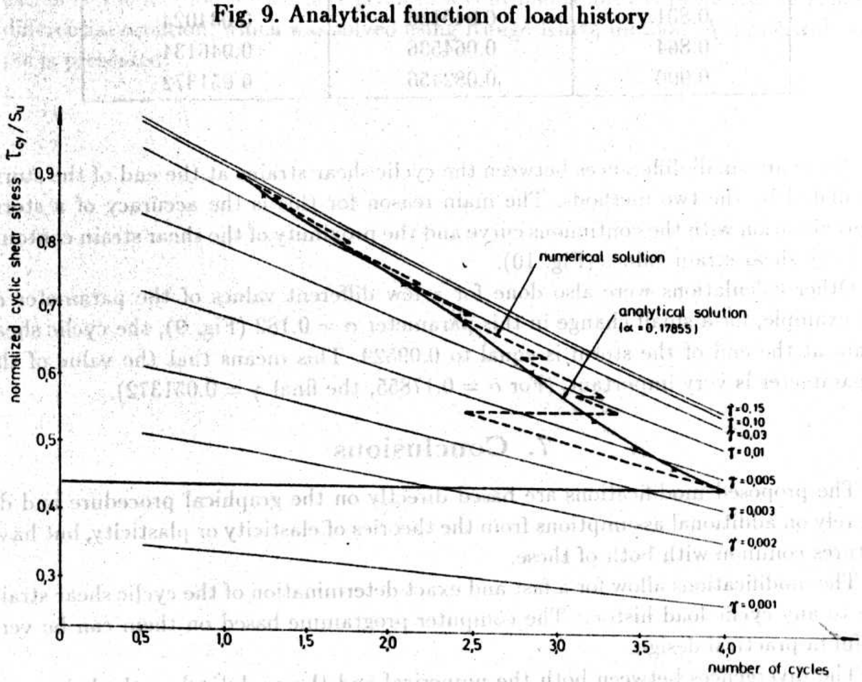

Fig. 10. Comparison of cyclic strains obtained by numerical and analytical solution

Figure 9 presents a typical 6-hour load history divided into 9 parcels. The approximation curve from eqs (33), with the parameter $\alpha = 0.17855$ is also shown.

The cyclic shear strain contour diagram is presented in Fig. 10 for the coefficients:

$$a_1 = 0.0018944; \quad a_2 = 0.95067; \quad a_3 = 0.084163; \quad a_4 = 0.032781.$$

The obtained results i.e. strain changes due to the increasing stress amplitudes are also shown there and presented in Table 2.

Table 2

Cyclic shear strains coresponding to cyclic shear stress amplitudes

Cyclic shear stress amplitude τ	Cyclic shear strain γ (numerical solution)	Cyclic shear strain γ (analytical solution)
0.441	0.004157	0.004151
0.552	0.009500	0.007159
0.576	0.012941	0.008421
0.630	0.018623	0.011533
0.693	0.025892	0.018731
0.738	0.033405	0.025792
0.801	0.045136	0.034024
0.864	0.064536	0.046134
0.900	0.082456	0.051372

There are small differences between the cyclic shear strains at the end of the storm calculated by the two methods. The main reason for this is the accuracy of a storm approximation with the continuous curve and the proximity of the shear strain contours for large shear strain values (Fig. 10).

Other calculations were also done for a few different values of the parameter α. For example, for a small change in this parameter $\alpha = 0.182$ (Fig. 9), the cyclic shear strain at the end of the storm is equal to 0.09529. This means that the value of the α-parameter is very important. (For $\alpha = 0.17855$, the final $\gamma = 0.051372$).

7. Conclusions

The proposed modifications are based directly on the graphical procedure and do not rely on additional assumptions from the theories of elasticity or plasticity, but have features common with both of these.

The modifications allow for a fast and exact determination of the cyclic shear strain due to any cyclic load history. The computer programme based on them can be very useful in practical design.

The divergences between both the numerical and the analytical methods increase for increasing values of cyclic shear strains. The difference depend on the shape and position of the shear strain contours.

With the modified analytical procedure, a determinant parameter for the accumulated shear strain due to a storm is the parameter characterizing the storm, α.

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Summary

The paper deals with the determination of the cyclic shear strain for the arbitrary load history. A numerical procedure for the hand-drawing graphical cyclic shear strain accumulation method proposed by Andersen et al. (1978) is introduced. The modification of the method for a storm given in a continuous form is proposed. It yielded a differential equation, which was solved using Runge-Kutta method. A numerical example is presented.