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Surface wave scattering in a basin with stochastically irregular bottom

Scattering by bottom irregularities is one of the effective mechanisms of long surface gravitational waves dissipation (Howe 1971, Elter et al. 1972, Pelinovsky 1970). The shallow-water theory predicts that the coefficient of dissipation increases with decreasing wave length. Since the shallow-water theory fails for short waves it is necessary to study the mechanism of wave dissipation due to scattering by bottom irregularities within a correct theory. In this paper such problem for potential wave motion is discussed. Propagation of small amplitude surface waves in a basin of depth $H(x,y)$ is governed by the boundary-value problem:

$$\Delta\varphi + \partial^2\varphi/\partial z^2 = 0, \qquad -H(x,y) < z < 0, \tag{1}$$

$$\frac{\partial\varphi}{\partial z} = \frac{\omega^2}{g}\varphi, \qquad z = 0 \tag{2}$$

$$\frac{\partial\varphi}{\partial z} = -\nabla H \cdot \nabla\varphi, \qquad z = -H(x,y) \tag{3}$$

where:

φ – velocity potential of wave with frequency ω,
g – acceleration due to gravity,
$H(x,y) = h_0 + h(x,y)$ – depth, h_0 – its mean value,
$\nabla = (\partial/\partial x, \partial/\partial y)$.

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Let $< >$ denote average and $< h(x,y) >= 0$, $\varepsilon^2 =< h^2 > /h_0^2 \ll 1$.

Since $h(x,y)$ is small, let us reduce the boundary condition (3) from the surface $z = -H(x,y)$ to plane $z = -h_0$ by expanding it into a Taylor series about $z = -h_0$ in terms of $h(x,y)$, and retaining only the first-order term in such expansion (by using Eq. (1)):

$$\frac{\partial\varphi}{\partial z} = -\nabla(h\nabla\varphi), \quad z = -h_0 \tag{3a}$$

To solve this boundary-value problem (Eqs 1, 2, 3a) let us apply the modified perturbation method. The same method has been applied for surface and internal water waves propagation (Ermakov 1977). Let us separate the wave field into two distinct parts: mean wave field φ_0, and φ_1 – fluctuation of the field about φ_0. It means that we imagine a sequence of identical observations, where φ_0 represents coherent component in the sense of an average. Then φ_1 represents the fluctuation of the actual field about this mean in any particular realization (thus $< \varphi_1 > = 0$).

The boundary-value problem for φ_0 we can derive by averaging Eqs (1), (2), (3a):

$$\Delta\varphi_0 + \partial^2\varphi/\partial z^2 = 0, \quad -h_0 < z < 0 \tag{4}$$

$$\frac{\partial\varphi_0}{\partial z} = \frac{\omega^2}{g}\varphi_0, \quad z = 0 \tag{5}$$

$$\frac{\partial\varphi_0}{\partial z} = -\nabla < h\nabla\varphi_1 >, \quad z = -h_0 \tag{6}$$

The boundary condition (6) contains the gradient of fluctuation field. Therefore the problem for φ_0 is incomplete without knowing φ_1. To obtain the boundary-value problem for φ_1 we subtract average expressions (4), (5), (6) from complete expressions (Eqs 1, 2, 3a), thus

$$\Delta\varphi_1 + \partial^2\varphi_1/\partial z^2 = 0, \quad -h_0 < z < 0 \tag{7}$$

$$\frac{\partial\varphi_1}{\partial z} = \frac{\omega^2}{g}\varphi_1, \quad z = 0 \tag{8}$$

$$\frac{\partial\varphi_1}{\partial z} = -\nabla(h\nabla\varphi_0) - [\nabla(h\nabla\varphi_1) - < \nabla(h\nabla\varphi_1) >], \quad z = -h_0 \tag{9}$$

The fluctuation field φ_1 is determined as soon as $h(x,y)$ and φ_0 are specified. Supposing they are known, the boundary-value problem (Eqs 7, 8, 9) may be solved with any accuracy by the method of successive approximations. Using small parameter ε let us neglect the square bracket in Eq. 9. This bracket is of the order of ε^2 (it is the well known Bourret approximation (Bourret 1962)). Now we have two linear boundary-value problems connected by the boundary condition on the plane $z = -h_0$. Let us present it in the form:

$$\varphi_{1,0}(z,\vec{r}) = \frac{1}{4\pi^2}\int\int d^2\vec{r}e^{i\vec{\xi}\cdot\vec{r}}\varphi_{\vec{\xi}_{1,0}}(z) \tag{10}$$

Applying two-dimensional Fourier transformation in the plane (x,y) in Eq. (7) and the boundary conditions (8, 9), we obtain:

$$\frac{d^2}{dz^2}\varphi_{\vec{\xi}_1} - \xi^2\varphi_{\vec{\xi}_1} = 0, \quad -h_0 < z < 0 \tag{11}$$

$$\frac{d}{dz}\varphi_{\vec{\xi}_1} = \frac{\omega^2}{g}\varphi_{\vec{\xi}_1}, \qquad z = 0, \tag{12}$$

$$\frac{d}{dz}\varphi_{\vec{\xi}_1} = \Lambda(\vec{\xi}), \tag{13}$$

where $\Lambda(\vec{\xi}) = -\iint d^2\vec{r}\, e^{-i\vec{\xi}\cdot\vec{r}}\nabla(h\nabla\varphi_0) = \iint d^2\vec{\chi}(\vec{\chi}\cdot\vec{\xi})H(\vec{\chi}-\vec{\xi})\varphi_{\vec{\chi}_0}(-h_0)$ and $H(\vec{\xi}) = \iint d^2\vec{r}\, e^{-i\vec{\xi}\cdot\vec{r}}h(\vec{r})$.

Solution of equation (11) with conditions (12) and (13) yields

$$\varphi_{\vec{\xi}1}(z) = \frac{\Lambda(\vec{\xi})}{\xi}\,\frac{g\xi\cosh(\xi z) + \omega^2\sinh(\xi z)}{\omega^2\cosh(\xi h_0) - g\xi\sinh(\xi h_0)} \tag{14}$$

Applying Fourier transformation to Eqs (4, 5, 6) and using Eq. 14 for $\varphi_{\vec{\xi}_1}$ we obtain equation and the boundary conditions for $\varphi_{\vec{\xi}_0}$:

$$\frac{d^2}{dz^2}\varphi_{\vec{\xi}_0} - \xi^2\varphi_{\vec{\xi}_0} = 0 \qquad -h_0 < z < 0, \tag{15}$$

$$\frac{d}{dz}\varphi_{\vec{\xi}_0} = \frac{\omega^2}{g}\varphi_{\vec{\xi}_0}, \qquad z = 0 \tag{16}$$

$$\frac{d}{dz}\varphi_{\vec{\xi}_0} = \varphi_{\vec{\xi}_0}\iint d^2\vec{\chi}\frac{(\vec{\xi}\cdot\vec{\chi})}{\chi}\phi(\vec{\chi}-\vec{\xi})\frac{g\chi - \omega^2\tanh(\chi h_0)}{\omega^2 - g\chi\tanh(\chi h_0)}, \quad z = -h_0 \tag{17}$$

Here $\phi(\vec{k}) =< H^2(\vec{k}) >$ represents a spectrum of bottom irregularities. General solution of (15) takes the form:

$$\varphi_{\vec{\xi}_0} = A\, e^{\xi z} + B\, e^{-\xi z}$$

Introducing this to the boundary conditions (16) and (17), we connect A with B and obtain dispersion equation for the mean field

$$\frac{\omega^2 - gk\tanh(kh_0)}{gk - \omega^2\tanh(kh_0)} = I \equiv \iint d^2\vec{\xi}\frac{(\vec{k}\cdot\vec{\xi})}{k\xi}\phi(\vec{k}-\vec{\xi})\frac{g\xi - \omega^2\tanh(\xi h_0)}{\omega^2 - g\xi\tanh(\xi h_0)} \tag{18}$$

By solving this relation, the propagation speed and dissipation coefficient of mean wave can be calculated. Due to small fluctuation of depth about the mean value let us represent the wave number k as follows: $k = k + \delta k + i\gamma$, where k satisfies dispersion equation for basin of constant depth h_0 ($\omega^2 = gk\tanh(kh_0)$). By expanding left-hand-side of Eq. 18 in terms of $\delta k + i\gamma$ we obtain:

$$\delta k + i\gamma = \frac{-k}{2kh_0 + \sinh 2kh_0)} I \tag{19}$$

The integral I is complex; the real part of it is equal to the Cauchy principal value and δk provides the variation of velocity propagation and shape of nonsinusoidal wave (this is essential for long-distance propagation i.e. for tsunami (Holloway et al. 1962)). The imaginary part is defined by residue of the integral at the pole $\xi = k$ and represents the damping of the mean field due to scattering of random depth inhomogenities:

$$\gamma = \frac{\pi}{2} \frac{4(2kh_0)^2 k^3}{(2kH_0 + \sinh(2kh_0))^2} \int_0^{2\pi} d\Theta \cos^2\Theta \, \phi\left(2k \sin\frac{\Theta}{2}, \frac{3\pi + \Theta}{2}\right) \tag{20}$$

Expression (20) differs from corresponding expression obtained for long wave in (Pelinovsky 1970) by the factor:

$$\eta = \frac{4(2kh_0)^2}{(2kh_0 + \sinh(2kh_0))^2}$$

which decreases with increasing kh_0 (graph of η is shown in Fig. 1)

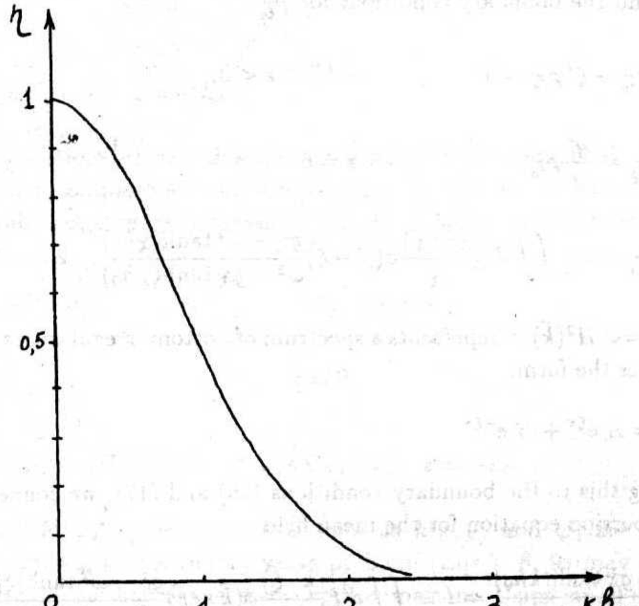

Fig. 1. Function $\eta = f(kh_0)$

Let us consider an example of wave propagation in the ocean. For the north-west part of Pacific $\phi(\vec{k})$ is known to be isotropic in the plane and may be approximated by the formula (Bell 1975).

$$\phi(k) = \begin{cases} 10^5\ \mathrm{m}^2 \cdot \mathrm{km}, & k < 0.1\ \mathrm{km}^{-1} \\ 10^{2.5} k^{-2.5} \mathrm{m}^2 \cdot \mathrm{km}, & k \geq 0.1\ \mathrm{km}^{-1} \end{cases}$$

Using Eq. (20) we obtain the dependence of the dissipation coefficient γ on the wave number (see Fig. 2). It is seen that for small k (i.e. long wavelength) γ is an increasing

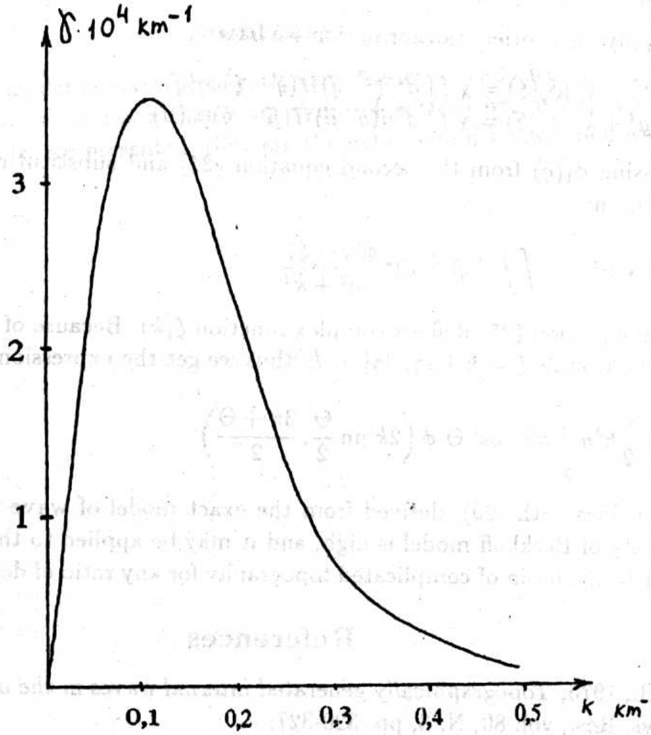

Fig. 2.

function of k; it reaches the greatest value for $k \sim 0.1\ \mathrm{km}^{-1}$ and then falls rapidly. For wavelength 20 – 200 km, damping length is of the order of the ocean size; thus the dissipation is effective.

Let us return to the same problem, but instead of solving the boundary-value problem, we apply the equation proposed by Berkhoff (1976). The approximated solution of (1), (2), (3a) is considered in the form:

$$\varphi(\vec{r}, z) = f(z)\phi(\vec{r}), \quad f(z) = \frac{\cosh k(z + h_0)}{\cosh kh_0} \tag{21}$$

where ω and k are connected by equation $\omega^2 = gk \tanh kh_0$.

We first substitute (21) into (1), multiply it by $f(z)$ and integrate over the interval $[-h_0, 0]$. Then by using the boundary conditions (2), (3a) we obtain:

$$\Delta\phi + k^2\phi = \chi\nabla(h\nabla\phi) \tag{22}$$

where $\chi^2 = \eta$.
Similarly as it was done above, let us expand ϕ in the form: $\phi = \phi_0 + \phi_1$ ($<\phi_1>= 0$).
Using Bourret approximation we have:

$$\begin{aligned} \Delta\phi_0 + k^2\phi_0 &= \chi < \nabla(h\nabla\phi_1) >, \\ \Delta\phi_1 + k^2\phi_1 &= \chi\nabla(h\nabla\phi_0). \end{aligned} \tag{23}$$

And by applying Fourier transformation we have:

$$\begin{aligned} (-\xi^2 + k^2)\phi_0(\vec{\xi}) &= \chi \int\int d^2\vec{q}(\vec{\xi}\cdot\vec{q})H(\vec{q}-\vec{\xi})\phi_1(\vec{q}) \\ (-g^2 + k^2)\phi_1(\vec{q}) &= \chi \int\int d^2\vec{\mu}(\vec{q}\cdot\vec{\mu})H(\vec{\mu}-\vec{q})\phi_0(\vec{\mu}) \end{aligned} \tag{24}$$

By expressing $\phi_1(\vec{q})$ from the second equation (24) and substituting it into the first one, we obtain:

$$-\xi^2 + k^2 = \eta \int\int d^2\vec{q}(\vec{\xi}\cdot\vec{q})^2 \frac{\phi(\vec{q}-\vec{\xi})}{-q^2+k^2} \tag{25}$$

Dispersion equation (25) defines complex function $\xi(k)$. Because of small fluctuations of depth we assume $\xi = k + i\gamma$, $|\gamma| \ll k$; thus we get the expression:

$$\gamma = \frac{\pi}{2}k^3\eta \int_0^{2\pi} d\Theta \cos^2\Theta \, \phi\left(2k\sin\frac{\Theta}{2}, \frac{3\pi+\Theta}{2}\right) \tag{26}$$

which coincides with (20), derived from the exact model of wave propagation. Thus the accuracy of Berkhoff model is high, and it may be applied to the problem of wave scattering in the basin of complicated topography for any ratio of depth to wavelength.

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Summary

In the paper, a model of wave propagation over stochastically irregular bottom is considered under the assumption that a spectrum of bottom irregularities is known. Two various solutions are presented; they are the extention of known long wave solutions.