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Envelopes of stationary stochastic processes

1. Introduction

Models describing the behaviour of soil under cyclic load take into account both the number of cycles, their magnitude and shape [4], [5], Considering the influence of stochastic loads one is interested not only in the average magnitude of amplitudes. The variability of loads in time and the connected with it the probability of crossing of a certain level or barrier and the probability distributions of local extremes are of a great importance for the proper estimation of structural safety [2].

If the soil is concerned, the strains and the generated by cyclic loads pore water pressures depend to a great extent additonally to the type of soil and the boundary conditions on the magnitude and the number of cyclic loads. In many practical cases, a distinct regularity in cyclic loads is noticed, and both their frequencies and the amplitudes are quite well defined. Such a phenomenon occurs for example in case of machine foundations. In other cases the knowledge of the cyclic load parameters is less well known but there are rather known the characteristics of the load process. These are natural phenomena such as the earthquakes, wind and wave loads. There is always in such cases a high degree of uncertainty concerning the magnitude and variability of loading and therefore the uncertainty of soil stability for long lasting cyclic loading. Calculations in such cases should, if possible, take into account the stochastic character of the phenomenon. The basic factor in such calculation is, additionally to having a suitable model, to know the parameters of the process of loading.

The purpose of this work is to describe and define the envelope for the autoregressive second order process AR(2) and for a linear combination of uncorrelated AR(2) processes, because these processes are well suited for approximation of processes of cyclic behaviour.

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2. Envelopes

An arbitrary stationary process $n(t)$ of an average value equal to zero may be presented in the following form [3]:

$$n(t)=x(t)\cos\omega_0 t+y(t)\sin\omega_0 t \tag{1}$$

where $x(t)$ and $y(t)$ are stationary processes defined below and ω_0 is an arbitrary constant. For construction of $x(t)$ and $y(t)$ process a Hilbert transformation $n'(t)$ of the process $n(t)$ is required:

$$n'(t)=\frac{1}{\pi}\int_{\infty}^{\infty}\frac{n(\tau)}{t-\tau}d\tau. \tag{2}$$

An analytical process is formed:

$$z(t)=n(t)+jn'(t) \tag{3}$$

and two processes $a(t)$ and $b(t)$ defined:

$$a(t)=\mathrm{Re}\left[z(t)\mathrm{e}^{-j\omega_0 t}\right]=n(t)\cos\omega_0 t+n'(t)\sin\omega_0 t \tag{4}$$

$$b(t)=\mathrm{Im}\left[z(t)\mathrm{e}^{-j\omega_0 t}\right]=n'(t)\cos\omega_0 t-n(t)\sin\omega_0 t \tag{5}$$

Solving equations (4) and (5):

$$n(t)=a(t)\cos\omega_0 t-b(t)\sin\omega_0 t \tag{6}$$

$$n'(t)=a(t)\sin\omega_0 t+b(t)\cos\omega_0 t \tag{7}$$

If a substitution $a(t)=x(t)$ and $b(t)=-y(t)$ is made the expression (6) becomes identical to expression (1). It's seen comparing (6) and (7) that the Hilbert transformation changes $\cos\omega_0 t$ into $\sin\omega_0 t$ and a $\sin\omega_0 t$ into $\cos\omega_0 t$ with a negative sign.

This can be also verified by a direct transformation using expression (2). It's evident from (2), (4) and (5) that if $n(t)$ is stationary, the processes $n'(t), a(t), b(t)$ are stationary too. The process given by the expression (6) may be now presented as:

$$n(t)=A(t)\sin(\omega_0 t+\phi(t)) \tag{8}$$

where $A(t)$ is the process of the amplitude:

$$A(t)=\sqrt{a(t)^2+b(t)^2} \tag{9}$$

and $\phi(t)$ is the process of an initial phase:

$$\phi(t)=\arctan\left[-\frac{a(t)}{b(t)}\right]=\arctan\left[\frac{x(t)}{y(t)}\right] \tag{10}$$

It's easy to verify that:

$$a(t)^2+b(t)^2=n(t)^2+n'(t)^2. \tag{11}$$

So the process of the amplitude may be defined as the square root of the sum of squares of the process $n(t)$ and its Hilbert transformation $n'(t)$:

$$A(t)=\sqrt{n(t)^2+n'(t)^2} \tag{12}$$

The Hilbert transformation using formula (2) and calculation of the amplitude is simple if $n(t)$ process is given in form of a function and may be integrated. For the discrete process the integral must be replaced by a sum. Substituting $\tau=p\Delta\tau$, $t=k\Delta\tau$, $n(\tau)=n(p)$, $n'(\tau)=n'(k)$ one receives:

$$n'(k)=\frac{1}{\pi}\sum_{p=-\infty}^{p=\infty}\frac{n(p)}{k-p} \tag{13}$$

For $k\to p$ the discontinuity exists with the values of $+\infty$ and $-\infty$ depending on the sign of $n(p)$ and whether k tends to p for the values higher or lower then p. Eliminating the discontinuity point two sums are received:

$$n'(k)=\frac{1}{\pi}\left[\sum_{p=-\infty}^{p=k-1}\frac{n(p)}{k-p}+\sum_{p=k+1}^{p=\infty}\frac{n(p)}{k-p}\right] \tag{14}$$

The calculation, made by the authors, shows some inconvenience of this type of transformation. It lays in the fact that both the past $p<k$ and the future $p>k$ values of the process must be known. Additionally, the sum (14) is relatively slowly convergent and the numerical calculation, made for the transformation of ideal cosine into sine function, showed that when taking 30 summation points for each sum, it is 60 points together, the error in estimation of a sine was from 23% for $\Delta\tau=2\pi/10$ to 16% for $\Delta\tau=2\pi/30$. For 60 points (120 together) from 22% to 12%, and for 100 points (200 together) from 21% to 5%. By dividing the T period in 100 parts and taking for calculation $2\cdot1000$ points the error decreased to 3% and for $2\cdot2000$ points to 2.5%. Naturally, for the transformation of an ideal cosine function the iteration may be accelerated, but for the stochastic process the acceleration may be difficult depending on the process parameters. Due to that, the Hilbert transformation seems to be not sufficiently effective for the discrete process.

The utilization of the Hilbert transformation for the presentation of the $n(t)$ process according to formula (6) is not unique. If a spectrum $S_{nn}(\alpha)$ of the process $n(t)$ is known for $-\infty<\alpha<\infty$ then the spectrum of the analytical process (3) is given as [3]:

$$S_{zz}=4S_{nn}(\alpha)\,U(\alpha) \tag{15}$$

where $U(\alpha)$ is a function with a unit jump.

Designating by $S_l(\alpha)$ the spectrum of $z(t)e^{-j\omega_0 t}$ process:

$$S_l(\alpha)=S_{zz}(\alpha+\omega_0) \tag{16}$$

Making use of the properties of the Fourier transformation of complex functions, two components of the spectrum $S_{nn}(\alpha)$ in relation to ω_0 may be calculated:

$$S_{aa}(\alpha)=\frac{S_l(\alpha)+S_l(-\alpha)}{2} \tag{17}$$

which is the real part of the spectrum, and:

$$S_{ba}(\alpha)=\frac{S_l(\alpha)-S_l(-\alpha)}{2j} \tag{18}$$

which is an imaginary part.

As can be seen from the above, knowing the spectrum $S_{nn}(\alpha)$ of $n(t)$ process, given by expression (6) one can calculate the spectrum $S_{aa}(\alpha)$ of the process $a(t)$. This gives its covariance function $R_{aa}(\tau)$ and the covariance function $R_{bb}(\tau)=R_{aa}(\tau)$. For stationarity $R_{ab}(0)$ should be equal to zero, and it means that random variables $a(t)$ and $b(t)$ are uncorrelated. It doesn't mean hovewer that the processes $a(t)$ and $b(t)$ are uncorrelated and therefore $S_{ba}(\alpha)\neq 0$. The general expression for the autocovariance function of the $n(t)$ process is therefore as follows:

$$R_{nn}(\tau)=R_{aa}(\tau)\cos\omega_0\tau-R_{ba}(\tau)\sin\omega_0\tau \tag{19}$$

The amplitude process is a slow varying process if the process $n(t)$ is a narrow band process such that $S_{nn}(\alpha)$ can only take large values around a particular frequency ω. The regularity factor expressed as:

$$a=\frac{\lambda_2}{\sqrt{\lambda_0\lambda_4}} \tag{20}$$

where spectral moments:

$$\lambda_j=2\int_0^\infty \alpha^j S_{nn}(\alpha)\,d\alpha \tag{21}$$

is then close to unity [2].

For slow varying processes both the amplitude and the initial phase are changing slowly so it may be assumed that $\dot{A}(t)\approx 0$ and $\dot{\phi}(t)\approx 0$. The first derivative of $n(t)$ process is then as follows:

$$\dot{n}(t)=A(t)\,\omega_0\cos(\omega_0 t+\phi(t)). \tag{22}$$

There is no problem now in expressing the amplitude as the function of $n(t)$ and $\dot{n}(t)$. From (8) and (22) one receives:

$$A(t)=\left[n^2(t)+\left(\frac{\dot{n}(t)}{\omega_0}\right)^2\right]^{0,5}. \tag{23}$$

Madsen [2] refering to Krenk shows that when the second derivative is taken:

$$\ddot{n}(t)=-A(t)\,\omega_0^2\sin(\omega_0 t+\phi(t)) \tag{24}$$

the expression for the amplitude takes the form:

$$A(t)=\left[\left(\frac{\dot{n}(t)}{\omega_0}\right)^2+\left(\frac{\ddot{n}(t)}{\omega_0}\right)^2\right]^{0,5} \tag{25}$$

This of course requires the assumption that not only the first but also the second derivatives of $a(t)$ and $\phi(t)$ are close to zero and this may not always be true. It is shown in the numerical examples presented further.

Designating the values of the process in three consecutive steps in times $t_1=t-\Delta t$, $t_2=t$ and $t_3=t+\Delta t$ by $n(t_1)=n_1$, $n(t_2)=n_2$ and $n(t_3)=n_3$ and substituting for $\dot{n}(t)$ and $\ddot{n}(t)$their respective first order difference quotients the following formula is received for the expression (23):

$$A(t)=\left[n_2^2+\left(\frac{n_3-n_1}{2\omega_0\,\Delta t}\right)^2\right]^{0.5} \tag{26}$$

and for expression (25):

$$A(t)=\left[\left(\frac{n_3-n_1}{2\omega_0\,\Delta t}\right)^2+\left(\frac{n_3-2n_2+n_1}{\omega_0^2\,\Delta t}\right)^2\right]^{0.5} \tag{27}$$

As it's seen from formulae (8), (9) and (10) the values of the amplitude and the initial phase are processes and therefore change in consecutive steps. If $n(t)$ is a slow varying process, and it may be assumed that in the neighbouring points amplitudes and also initial phases differ little from themselves, the amplitude $A(t)$ can be calculated by finding the parameters of a sine function. It is assumed that it passes through three neighbouring points, the middle of which is $n(t)$. Using notations as before, after simple calculations, one receives:

$$\omega_0\,\Delta t=\arccos\left(\frac{n_1+n_3}{2n_2}\right) \tag{28}$$

$$\omega_0\,t+\phi=\operatorname{arccot}\left(\frac{n_3-n_1}{2n_2\sin\omega_0\,\Delta t}\right) \tag{29}$$

$$A(t)=\left(\frac{n_2}{\sin(\omega_0\,\Delta t+\phi)}\right) \tag{30}$$

One has to realize that if the process isn't a slow varying one the errors when making calculation with formulae (28), (29) and (30) may be high.

3. The envelope of the autoregressive second order AR(2) process

The Gaussian second order autoregressive process is quite often used for describing the stochastic phenomena having cyclic character. KNABE [1] presented the possibility of such approach for loads or displacements of the offshore structure foundations by using sums of uncorrelated AR(2) processes. The knowledge of the parameters of the envelope of such a process may be essential if the computational model requires it and if the real process may be approximated by sum of AR(2) processes. The envelope may of course be calculated in any way described before. Below, the method is presented, worked out by Knabe for finding the accurate value of the envelope based on all values of the process up to the current time.

The AR(2) process may be presented as follows:

$$X(k)=\phi_1\,X(k-1)+\phi_2\,X(k-2)+V(k) \tag{31}$$

where:

$$\phi_1 = 2e^{-\eta\Delt t}\cos\omega\Delta t \tag{32}$$

$$\phi_2 = -e^{-2\eta\Delta t} \tag{33}$$

η – is a damping parameter,

$$\omega = \frac{2\pi}{T} \tag{34}$$

T – average period of a cycle in a process, Δt – the time interval from step k to $k+1$, $V(k)$ – white noise process having normal distributon $N[0, \sigma_v^2]$.

The variance of $V(k)$ is related to the variance σ_x^2 of the $X(k)$ process by expression:

$$\sigma_x^2 = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_v^2}{[1-\phi_2]^2-\phi_1^2} \tag{35}$$

Using the backshift operator B the $X(k)$ process may be written as

$$[1-\phi_1 B-\phi_2 B^2]\,X(k) = V(k) \tag{36}$$

$$\phi(B)\,X(k) = V(k)$$

$$X(k) = \phi^{-1}(B)\,V(k) = \psi(B)\,V(k) \tag{37}$$

and this is a moving average process:

$$X(k) = V(k) + \psi_1 V(k-1) + \psi_2 V(k-2) + \ldots + \psi_p V(k-p) \tag{38}$$

The expression (38) is equivalent to (31) and it will be further used in constructing the envelope.

3.1. The response function for a unit impulse

As it's visible from expression (38) the impulse applied in point $k-p$ influences the value of the process in point k where $k \geqslant p$. Coefficient ψ_p can be calculated by doing $k-p$ operations according to formula (31) on a unit impulse. So when the unit impulse $V(k)=1$ is applied at a point $k=0$, its influence on the process values at points $k=1, 2, 3, \ldots$ will be according to (31) as follows:

$$X(0) = V(0) = 1 \qquad \Rightarrow \psi_0 = 1$$

$$X(1) = \phi_1 X(0) = \phi_1 V(0) \qquad \Rightarrow \psi_1 = \phi_1 \tag{39}$$

$$X(2) = \phi_1 X(1) + \phi_2 X(0) = \phi_1^2 V(0) + \phi_2 V(0) \;\Rightarrow \psi_2 = \phi_1^2 + \phi_2$$

It's simple to prove that the response of the process $X(k)$ to a unit impulse is in a form of decaying sine function:

$$(k) = e^{-\eta k\Delta t}(A\cos k\omega\Delta t + B\sin k\omega\Delta t) \tag{40}$$

Assuming that the impulse is applied for $k=0$, and that for $k=-1$, $X(-1)=0$ the values of parameters A and B are as follows: $A=V(0)$, $B=V(0)\cot\omega\Delta t$.

The expression for $X(k)$ takes therefore the form:

$$X(k)=V(0)\,d^k(\cos k\alpha+\cot\alpha\sin k\alpha) \tag{41}$$

where:

$$d=e^{-\eta\Delta t} \quad \text{and} \quad \alpha=\omega\Delta t \tag{42}$$

The expression (41) may be written as:

$$X(k)=V(0)\,d^k\frac{\sin[(k+1)\alpha]}{\sin\alpha} \tag{43}$$

So, for the consecutive values of k, the following expressions are received:

$$\begin{aligned} k&=0 \Rightarrow X(0)=V(0)\\ k&=1 \Rightarrow X(1)=V(0)\,d\cos 2\alpha\\ k&=2 \Rightarrow X(2)=V(0)\,d^2[4\cos^2\alpha-1]\\ k&=3 \Rightarrow X(3)=V(0)\,d^3[8\cos^3\alpha-4\cos\alpha]\\ k&=4 \Rightarrow X(4)=V(0)\,d^4[16\cos^4\alpha-12\cos^2\alpha+1] \end{aligned} \tag{44}$$

Identical expressions are received from (39) by substituting for ϕ_1 and ϕ_2 formulae (32) and (33) respectively. Let's now assume, that the impulses $V(0)$, $V(1)$, $V(2)$, …, $V(p)$ are applied consecutively in points $p=0, 1, 2, \ldots, p$. The response at point k to each separate impulse where $0\leqslant p\leqslant k$ is (using expression (41) as follows):

$$\begin{aligned} p&=0 \quad \Rightarrow X_0(k)=V(0)\,d^k[\cos k\alpha+C\sin k\alpha] \quad \text{where } C=\cot\alpha\\ p&=1 \quad \Rightarrow X_1(k)=V(1)\,d^{k-1}[\cos[(k-1)\alpha]+C\sin[(k-1)\alpha]]\\ p&=p \quad \Rightarrow X_p(k)=V(p)\,d^{k-p}[\cos[(k-p)\alpha]+C\sin[(k-p)\alpha]]\\ p&=k \quad \Rightarrow X_k(k)=V(k) \end{aligned}$$

Summing up the responses of all impulses from $p=0$ to $p=k$ one receives:

$$X(k)=\cos k\alpha\Big[\sum_{p=0}^{p=k}V(p)\,d^{k-p}[\cos p\alpha-C\sin p\alpha]\Big]+ \\ +\sin k\alpha\Big[\sum_{p=0}^{p=k}V(p)\,d^{k-p}[\sin p\alpha+C\cos p\alpha]\Big] \tag{45}$$

3.2. Process AR(2) as a sine with an amplitude and initial phase as stochastic processes

Designating in (45) the coefficients:

$$B(k)=\sum_{p=0}^{p=k}V(p)\,d^{k-p}[\cos p\alpha-C\sin p\alpha] \tag{46}$$

$$C(k)=\sum_{p=0}^{p=k}V(p)\,d^{k-p}[\sin p\alpha+C\cos p\alpha] \tag{47}$$

and having in mind that $\alpha=\omega\Delta t$ and $C=\cot\alpha$ the expression (45) takes form of expression (1) where instead of $n(t)$, $x(t)$ and $y(t)$ the processes $X(k)$, $B(k)$, $C(k)$ are substituted respectively and the time $t=k\Delta t$

$$X(k)=B(k)\cos k\omega\Delta t+C(k)\sin k\omega\Delta t \tag{48}$$

This is a form of the process which according to (9) permits to calculate the envelope process and according to (10) the initial phase process. Both $B(k)$ and $C(k)$ are processes as their values change in each consecutive step depending on the value of a random impulse $V(k)$. In the $k+1$ step the $B(k)$ coefficient has the value:

$$B(k+1)=\sum_{p=0}^{p=k+1} V(p)\,d^{k+1-p}[\cos p\alpha-C\sin p\alpha]$$

which one can write as:

$$B(k+1)=d\sum_{p=0}^{p=k} V(p)\,d^{k-p}[\cos p\alpha-C\sin p\alpha]+ \\ +d^0V(k+1)[\cos[(k+1)\alpha]-C\sin[(k+1)\alpha]]$$

The sum in above expression is identical to expression (46) so

$$B(k+1)=dB(k)-V(k+1)\frac{\sin k\alpha}{\sin\alpha} \tag{49}$$

Similarly:

$$C(k+1)=dC(k)+V(k+1)\frac{\cos k\alpha}{\sin\alpha} \tag{50}$$

As can be seen from the above, the processes $B(k)$ and $C(k)$ may be regarded as the first order autoregressive processes. They aren't stationary as the variance of the white noise isn't constant but varies within a period and is equal to

$$\sigma_v^2\frac{\sin\cdot k\alpha}{\sin^2\alpha} \quad \text{for } B(k)$$

and

$$\sigma_v^2\frac{\cos^2 k\alpha}{\sin^2\alpha} \quad \text{for } C(k)$$

As $\alpha=\omega\Delta t=\frac{2\pi}{T}\Delta t=\frac{2\pi}{n}$, where n is a number of divisions in which a period T is divided, the variance of the white noise of the process $B(k)$ is equal to zero for $k=0$ and for $k=$ $=0.5n+pn$; $p=1, 2, 3, \ldots,$. The variance reaches its maximum for $k=0.25n+pn$ and for $k=0$ as then $\sin k\alpha=\pm\sin\pi/2$. For the white noise of the process $C(k)$ the variances have identical values but are shifted by a phase $\pi/2$.

The sum of these variances is constant and independent of t and is equal to $\sigma_v^2/\sin^2\alpha$. The covariance matrix has a form

$$\mathbf{K}=\frac{\sigma_v^2}{\sin^2\alpha}\begin{bmatrix}\sin^2 k\alpha, & -0.5\sin 2k\alpha\\ -0.5\sin 2k\alpha & \cos^2 k\alpha\end{bmatrix} \tag{51}$$

The correlation matrix is therefore as follows:

$$\mathbf{C}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{52}$$

and so there is a full negative correlation between the random impulses of $B(k)$ and $C(k)$ processes. Squaring the $B(k+1)$ given by (49) and taking the expected value one receives:

$$E[B^2(k+1)]=d^2E[B^2(k)]-2d\frac{\sin k\alpha}{\sin\alpha}E[B(k)V(k+1)]+\frac{\sin^2 k\alpha}{\sin^2\alpha}E[V^2(k+1)]$$

As $B(k)$ and $V(k+1)$ are uncorrelated and $E[B(k)]=0$ the relation, between the variances $\gamma^B(k)$ and $\gamma^B(k+1)$ of the $B(k)$ process is as follows:

$$\gamma^B(k+1)=d^2\gamma^B(k)+\frac{\sin^2 k\alpha}{\sin^2\alpha}\sigma_v^2 \tag{53}$$

For a given value of $\alpha=\omega\Delta t$ the above variance isn't constant but depends on $\sin k\alpha$ and the expression (53) may be treated as the first order autoregressive process of a form

$$X(k+1)=\phi X(k)+V(k)$$

Here the coefficient $\phi=d^2$ whereas in the $B(k)$ process (expression (49)) $\phi=d$. The impulse $V(k)$ is here deterministic, depending on the $\sin k\alpha$ value. Similarly to (53) the relation for variances of the $C(k)$ process is as follows:

$$\gamma^C(k+1)=d^2\gamma^C(k)+\sigma_v^2\frac{\cos^2 k\alpha}{\sin^2\alpha} \tag{54}$$

The sum of variances of the $B(k)$ and $C(k)$ processes is then:

$$\gamma^B(k+1)+\gamma^C(k+1)=d^2(\gamma^B(k)+\gamma^C(k))+\frac{\sigma_v^2}{\sin^2\alpha}. \tag{55}$$

As the $n(t)$ process is stationary, one may expect that its envelope process will also be stationary and so the expected value of $B^2(k)+C^2(k)$ be constant. As this expected value is equal to the sum of variances (55), it will be independent of k if:

$$\gamma^B(k)+\gamma^C(k)=\frac{\sigma_v^2}{(1-d^2)\sin^2\alpha} \tag{56}$$

As it's seen from the above the $B(k)$ and $C(k)$ processes may be regarded as AR(1) processes of constant expected values equal to zero but of varying variances depending on the cyclic variability of random impulses. The random impulses are fully correlated and in a such way that the sum of their variations is constant.

If for $k=0$ the value of $B(k)$ is equal to $B(O)$, the values for the consecutive steps may be expressed as:

$$B(k)=d^kB(0)-\sum_{i=1}^{k}V(i)\,d^{k-i}\frac{\sin i\alpha}{\sin\alpha} \tag{57}$$

and similarly for $C(k)$:

$$C(k)=d^k C(0)+\sum_{i=1}^{k} V(i)\,d^{k-i}\frac{\cos i\alpha}{\sin\alpha} \tag{58}$$

3.3. Envelope of the AR(2) process

Because the $X(k)$ process has been presented by formula (45) as a linear combination of cosine and sine functions, with the coefficients in form of $B(k)$ and $C(k)$ processes (formulae (46) and (47)), the sum of squares of these coefficients may be treated as a square of an amplitude of the process given by (45). Let's form a vector process with processes $B(k)$ and $C(k)$ as components:

$$\mathbf{D}(k+1)=\begin{bmatrix} B(k+1) \\ C(k+1) \end{bmatrix}=d\begin{bmatrix} B(k) \\ C(k) \end{bmatrix}+\begin{bmatrix} -\sin k\alpha/\sin\alpha \\ \cos k\alpha/\sin\alpha \end{bmatrix}V(k+1) \tag{59}$$

$$\mathbf{D}(k+1)=d\mathbf{D}(k)+\mathbf{E}(k)\,V(k+1) \tag{60}$$

where

$$\mathbf{E}^T(k)=\left[-\frac{\sin k\alpha}{\sin\alpha},\ \frac{\cos k\alpha}{\sin\alpha}\right]$$

The process of the squared amplitude takes form:

$$A^2(k+1)=\mathbf{D}^T(k+1)\,\mathbf{D}(k+1) \tag{61}$$

and after substituting (60):

$$A^2(k+1)=d^2A^2(k)+2d\mathbf{D}^T(k)\mathbf{E}(k)\,V(k+1)+V^2(k+1)\sin^{-2}\alpha \tag{62}$$

As the random impulse:

$$V(k+1)=\sigma_v\,U(k+1) \tag{63}$$

where $U(k+1)$ is a random impulse with the normal distribution $N[0, 1]$:

$$A^2(k+1)=d^2A^2(k)+[2d\sigma_v\,\mathbf{D}^T(k)\mathbf{E}(k)]\,U(k+1)+U^2(k+1)\,\frac{\sigma_v^2}{\sin^2\alpha}. \tag{64}$$

This may be written as:

$$A^2(k+1)=d^2A^2(k)+W(k+1) \tag{65}$$

where:

$$W(k+1)=[2d\sigma_v\,\mathbf{D}^T(k)\mathbf{E}(k)]\,U(k+1)+U^2(k+1)\,\frac{\sigma_v^2}{\sin^2\alpha} \tag{66}$$

As it's seen the process of the squared amplitude of the AR(2) process is the first order autoregressive process (65) in which the random impulse $W(k+1)$ is a sum of two correlated impulses. One of the impulses has a normal distribution with zero mean value but varying variation and the second impulse has a constant variation and the chi-square distribution with one freedom degree. The varying variance in (66) depends on the product of multiplication of two vectors $\mathbf{D}^T(k)$ and $\mathbf{E}(k)$. This product may be presented as:

$$\mathbf{D}^T(k)\,\mathbf{E}(k) = -B(k)\frac{\sin k\alpha}{\sin\alpha} + C(k)\frac{\cos k\alpha}{\sin\alpha}$$

Substituting for $B(k)$ and $C(k)$ the expressions (57) and (58) respectively:

$$\mathbf{D}^T(k)\,\mathbf{E}(k) = d^k\left[\frac{C(0)\cos k\alpha - B(0)\sin k\alpha}{\sin\alpha}\right] + \sum_{i=1}^{k} V(i)\,d^{k-i}\frac{\sin[(k+i)\alpha]}{\sin^2\alpha} \tag{67}$$

For high values of k, $d^k \to 0$ and the first component may be disregarded. Substituting $V(i) = \alpha_v\, U(i)$ one receives:

$$\mathbf{D}^T(k)\,\mathbf{E}(k) = \sigma_v \sum_{i=1}^{k} U(i)\,d^{k-i}\frac{\sin[(k+i)\alpha]}{\sin^2\alpha} \tag{68}$$

So the random impulse in the squared amplitude process:

$$W(k+1) = 2\sigma_v^2\left[\sum_{i=1}^{k} U(i)\,d^{k-i}\frac{\sin[(k+i)\alpha]}{\sin^2\alpha}\right]U(k+1) + \frac{\sigma_v^2\,U^2(k+1)}{\sin^2\alpha} \tag{69}$$

Though the expression (69) explains more clearly than (66) the character of the random impulse $W(k+1)$, it is nevertheless less convenient in numerical calculations, because the varying coefficient, by which $U(k+1)$ is multiplied, must, in each step, be calculated as a sum of weighted impluses $U(i)$. It's much more convenient to use formula (66) where, for calculating $\mathbf{D}(k+1)$ in the consecutive $k+1$ step, the recursive formula (60) may be used. $\mathbf{E}(k)$ is a deterministic vector and d, α_v and α are constant values.

The expected value of the squared amplitude is received from (62)

$$E[A^2(k+1)] = \overline{A}^2(k+1) = d^2\overline{A}^2(k) + \frac{\sigma_v^2}{\sin^2\alpha} \tag{70}$$

For stationarity, the expected value is constant and so:

$$\overline{A}^2(k) = \frac{\sigma_v^2}{(1-d^2)\sin^2\alpha} \tag{71}$$

This expression is identical with (56) for the sum of variations of $B(k)$ and $C(k)$ processes. The amplitude process is received by taking the square root of (65):

$$A(k+1) = [d^2A^2(k) + W(k+1)]^{0,5} \tag{72}$$

Finding the expected value and covariance function of this process seems to be difficult and wasn't done here. The average value of the squared process was only calculated and this is as follows:

$$\overline{A}^2(k) = \sigma_x^2\frac{1+2d^2+d^4-4d^2\cos^2\alpha}{(1+d^2)\sin^2\alpha} \tag{73}$$

It was received by substituting α_v^2 from (35) to (70) and utilizing expressions (32), (33) and (42). For $\Delta t \to 0$, $d \to 1$ and $\alpha \to 0$ so the expression (73) is indeterminate. Evaluating the limit:

$$\overline{A}^2(k)=2\sigma_x^2\frac{\eta^2+\omega^2}{\omega^2} \tag{74}$$

Knowing the average of the squared amplitude from (74) one can calculate the square of average amplitude if a variance of the process is known:

$$(\overline{A}(k))^2=\overline{A}^2(k)-\mathrm{Var}[A(k)]. \tag{75}$$

Summarizing one can say that though the autoregressive AR(2) process is relatively simple one, the exact autocorrelation function of its envelope is difficult to receive. It would be possible to get an estimation of it using a generally known approximating formulae, but this wasn't done in this paper. Nevertheless, it's easy to generate the envelope process calculating $B(k)$ and $C(k)$ using formulae (46) and (47) or recursively (59) and then the amplitude:

$$A(k)=[B^2(k)+C^2(k)]^{0,5}. \tag{76}$$

It may be also calculated using formula (65) in which the random impulse $W(k+1)$ is given by (66).

3.4. Transformation in the state space

WILDE [6] presented the procedure for receiving the curves tangent to the autoregressive second order process for the model of the AR(2) process in the state space and having therefore some properties of the envelope

$$\mathbf{X}(n+1)=\phi\mathbf{X}(n)+g\mathbf{V}(n+1) \tag{77}$$

where:

$$\mathbf{X}^T(n)=[X(n-1),X(n)].$$

For the observation, described by the relation:

$$\mathbf{Z}(n)=h\mathbf{X}(n)+\sigma_v\mathbf{U}(n) \tag{78}$$

a linear transformation is performed:

$$\mathbf{Y}(n)=\gamma\mathbf{X}(n) \tag{79}$$

where γ is a transformation matrix such that:

$$E[\mathbf{Y}(n)\mathbf{Y}^T(n)]=\gamma k_{xx}(0)\gamma^T=k(0)\mathbf{I} \tag{80}$$

and $k(0)$ is a variance of $X(n)$ process.

The following expressions for $\mathbf{Y}(n+1)$, $\mathbf{Y}(n)$ and $\mathbf{Z}(n)$ are received:

$$\begin{aligned}\mathbf{Y}(n+1)&=\phi_Y\mathbf{Y}(n)+g_Y\mathbf{V}(n+1)\\ \mathbf{Z}(n)&=h_Y\mathbf{Y}(n)+\sigma_v\mathbf{U}(n)\end{aligned} \tag{81}$$

For $\eta \to 0$, g_y becomes a zero matrix and ϕ_y the rotation matrix. The express ion for $\mathbf{Z}(n)$ takes then the form:

$$\mathbf{Z}(n) = Y_1(n) \sin[(\omega n - \omega/2)\Delta t] + Y_2(n) \cos[(\omega n - \omega/2)\Delta t] \tag{82}$$

where $Y_1(n)$ and $Y_2(n)$ are the elements of matrix $\mathbf{Y}(n)$. The amplitude is calculated as:

$$A(n) = \sqrt{\mathbf{Y}^T(n)\,\mathbf{Y}(n)} \tag{83}$$

For $\eta \neq 0$ the expressions for $Y_1(n)$ and $Y_2(n)$ take form:

$$Y_1(n) = \frac{\sqrt{2}}{2}\left[\frac{\mathrm{ch}\,\eta\Delta t}{\mathrm{ch}\,\eta\Delta t - \cos\omega\Delta t}\right]^{0.5} [X(n-1) - X(n)] \tag{84}$$

$$Y_2(n) = \frac{\sqrt{2}}{2}\left[\frac{\mathrm{ch}\,\eta\Delta t}{\mathrm{ch}\,\eta\Delta t + \cos\omega\Delta t}\right]^{0.5} [X(n-1) + X(n)] \tag{85}$$

If Δt tends to zero then;

$$Y_1(n) = \frac{-1}{\sqrt{\omega^2 + \eta^2}} \frac{X(n) - X(n-1)}{\Delta t} \tag{86}$$

and for $\eta = 0$ the expression (86) is a derivative of $a\cos(\omega t + \phi)$. The expression (85) is for $\eta = 0$

$$Y_2(n) = \frac{X(n-1) + X(n)}{2} \tag{87}$$

and it's the average value from the interval $n, n-1$.

In the vicinity of extremes $Y_1(n)$ tends, for small Δt, to zero as $X(n)$ is close to $X(n-1)$, so the expression (83) gives directly the $Y_2(n)$ value and forms a sort of an envelope. The values of the envelope calculated by the above method differ relatively little from values calculated by the method proposed by the authors (formulae (49), (50) and (76)). The results of numerical calculations are presented later.

3.5. Envelope as a square root of a sum of two independent (RA2) processes

Assume, that a spectrum $S_{nn}(\alpha)$ or a correlation function $R_{nn}(\tau)$ of a stationary $n(t)$ process is given. As a special case, the $n(t)$ process may be an AR(2) one. The $n(t)$ process may be presented in a form given by formula (1). Let's assume that processes $x(t)$ and $y(t)$ are uncorrelated which suggests that the imaginary component of a spectrum (see (18)) in relation to ω_0 is equal to zero. Let's also assume that with a sufficient dose of accuracy the processes $x(t)$ and $y(t)$ may be approximated by AR(2) ones. For such assumptions the correlation function of the $n(t)$ process has the following form:

$$R_{nn}(\tau) = R_{xx}(\tau)\cos\omega_0\tau \tag{88}$$

and a spectrum of this process:

$$S_{nn}(\alpha) = \frac{S_{xx}(\alpha - \omega_0) + S_{xx}(\alpha + \omega_0)}{2}. \tag{89}$$

For the case when $x(t)$ is an AR(2) process, the correlation function is given by the following formula:

$$R_{xx}(\tau)=e^{-\eta_x\tau}\left[\cos\omega_x\tau+\frac{\eta_x}{\omega_x}\sin\omega_x\tau\right] \tag{90}$$

Therefore

$$R_{nn}(\tau)=e^{-\eta_x\tau}\left[\cos\omega_x\tau+\frac{\eta_x}{\omega_x}\sin\omega_x\tau\right]\cos\omega_0\tau \tag{91}$$

The spectrum of $x(t)$ process for $\alpha\geqslant 0$:

$$S_{xx}(\alpha)=\frac{4\sigma_x^2\eta_x(\eta_x^2+\omega_x^2)}{\pi[\eta_x^2+(\omega_x+\alpha)^2][\eta_x^2+(\omega_x-\alpha)^2]} \tag{92}$$

According to (89) if $S_{xx}(\alpha)$ is given by (92) then:

$$S_{nn}(\alpha)=\frac{4}{\pi}\sigma_x^2\eta_x(\eta_x^2+\omega_x^2)\left[\frac{1}{[\eta_x^2+(\omega_x-\alpha-\omega_0)^2][\eta_x^2+(\omega_x-\alpha+\omega_0)^2]}+\right.$$
$$\left.+\frac{1}{[\eta_x^2+(\omega_x-\alpha+\omega_0)^2][\eta_x^2+(\omega_x-\alpha-\omega_0)^2]}\right] \tag{93}$$

If a real spectrum of $n(t)$ process $S_{nn}^r(\alpha)$ is now given, the parameters η_x, α_x, and ω_x of the $x(t)$ process can be calculated for the condition that:

$$\int_0^\infty (S_{nn}(\alpha)-S_{nn}^r(\alpha))^2 d\alpha=\text{minimum} \tag{94}$$

or

$$\int_0^\infty |S_{nn}(\alpha)-S_{nn}^r(\alpha)|d\alpha=\text{minimum} \tag{95}$$

In numerical calculations the integral is replaced by the sum. Instead of a spectrum a correlation function could be used. Designating by $R_{nn}^r(\tau)$ the correlation function of a given process, the integral of a square of the difference of this function and that defined by formula (91) should be minimized. Practically, ω_0 should be assumed to be equal to the abscissa corresponding to the maximum of the spectrum of the $n(t)$ process and equal to approximately $2\pi/T$ where T is an average period. For AR(2) process the peak of the spectrum is for $\alpha=(\omega_0^2-\eta^2)^{0.5}$ and for small η, α is approximately equal to ω_0.

The above assumptions may be further simplified by assuming that the $x(t)$ process is an AR(2) one with $T=\infty$, that means $\omega_x=0$. The expression (90) in this case simplifies to:

$$R_{xx}(\tau)=e^{-\eta_x\tau}(1+\eta_x\tau) \tag{96}$$

and the spectrum formula (93) for the $n(t)$ process takes the form:

$$S_{nn}(\alpha)=\frac{8\sigma_x^2\eta_x^3}{\pi}\left[\frac{1}{[\eta_x^2+(\alpha+\omega_0)^2][\eta_x^2+(\alpha-\omega_0)^2]}\right]. \tag{97}$$

In this case only the parameter, η_x will be sought for. When η_x is known the $x(t)$ and $y(t)$ processes may be generated and the amplitude $A(t)$ calculated

$$A(t)=\sqrt{x^2(t)+y^2(t)} \tag{98}$$

It's shown in presented further numerical example how the η_x value changes depending on η and ω parameters of the $n(t)$ process which in numerical example was assumed to be an AR(2) one. Generally, but not always, the approximation received by using formulae (96) and (97) is sufficiently accurate.

4. Numerical examples of the envelope calculations

To illustrate the character of the envelopes, calculated by different formulae, some realizations of an AR(2) process were generated with the minicomputer IBM PC-XT. The same average period $T=10$ s was used for all generations. Two damping parameters were adopted; $\eta=0.01$ for more regular processes, differing relatively little from sine curve, and $\eta=0.1$ for processes with higher irregularities. Two time intervals were chosen $\Delta t=0.5$ s, so that the average period was divided into 20 parts (Fig. 1) and $\Delta t=1.5$ s producing 6.7 divisions per an average cycle. The calculations were performed using formulae: (26) and (27) with the assumption that the envelope process is a slow varying one, (76) calculating $B(k)$ and $C(k)$ recursively with formulae (49) and (50) and (83) by transformation of an AR(2) process in the state space.

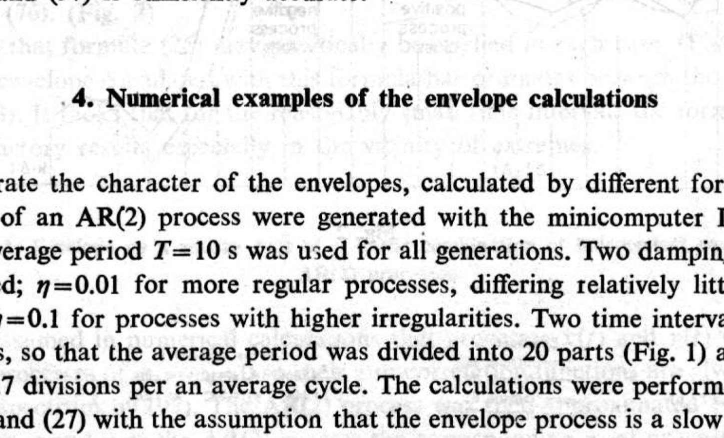

Fig. 1

Conclusions:

– The assumptions that both the first and the second derivatives of the amplitude and the initial shift may be assumed to be zero are for AR(2) process with $\eta=0.01$ not admissible. The envelope calculated with formula (27) isn't tangent to the process but, in several

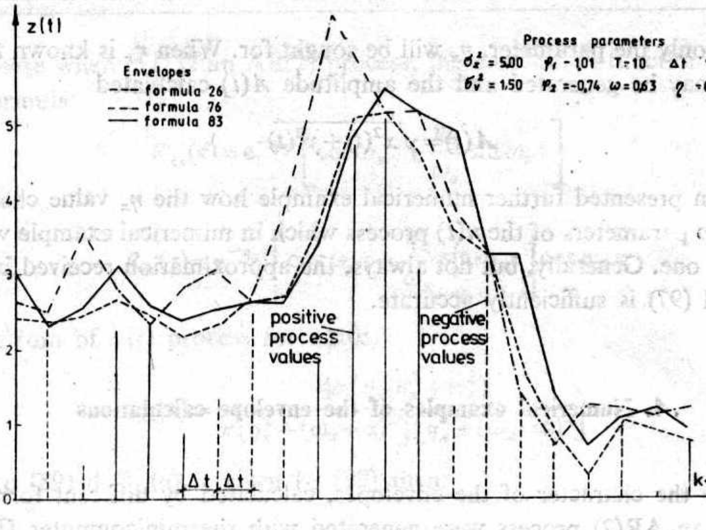

Fig. 2

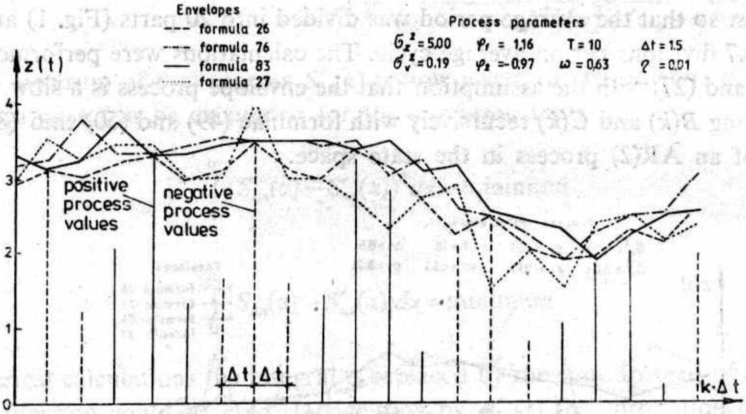

Fig. 3

Table 1

Values of η_x of the $x(t)$ AR(2) process in formula (1) depending on ω and η of approximated AR(2 process $n(t)$

ω/η	0.010	0.030	0.050	0.070	0.100	0.2500	0.500
0.3	0.021	0.062	0.101	0.138	0.190	0.435	0.535
0.6	0.020	0.063	0.105	0.147	0.205	0.493	0.790
1.0	0.020	0.061	0.105	0.142	0.208	0.490	0.950
1.5	0.020	0.060	0.110	0.150	0.215	0.515	0.970
2.0	0.020	0.060	0.100	0.154	0.215	0.515	0.980

places, takes the values smaller than the process itself. (Fig. 1, Fig. 3) This is also due to the high errors made in numerical calculation of the first and second derivatives.

– For a small time interval ($\Delta t=0.5$ s) and a small damping coefficient ($\eta=0.01$), the envelopes calculated by formulae (26), (76) and (83) differ little from themselves for the whole run of the realization (Fig. 1).

– For a higher time interval ($\Delta t=1.5$ s) and higher damping coefficient ($\eta=0.1$), the differences of the envelopes values increase. For the extreme points, the differences are small. It looks as if the envelope calculated with formula (83) was one time interval a head of formula (76). (Fig. 2)

– It seems that formula (26) may practically be applied in each case. (Figs. 1, 2, 3) Generally the envelope calculated with this formula has ordinates between those of formulae (76) and (83). It looks that for the reasonably small time intervals the formula (26) gives quite satisfactory results especially in the vicinity of extremes.

4.2. Envelope as a square root of a linear combination of independent squared AR(2) processes

It was assumed in numerical calculations that processes $x(t)$ and $y(t)$ in formula (1) are AR(2) processes of $\omega_x=\omega_y=0$ so their autocorrelation functions are given by formula (96) and a spectrum by (97). The AR(2) process was then approximated by formula (1), For different ω and η of the AR(2) process the corresponding η_x of process (1) were estimated. This was done by using formula (95) where $S_{nn}(\alpha)$ is the spectrum of AR(2) process as given by formula (93) without subscript x, and $S^r_{nn}(\alpha)$ is a spectrum (97) of process (1). The results of some calculations are presented below in a table 1.

The results in the table may not be very accurate and slightly different values would be received if the criterion (94) was adopted. Nevertheless it's clearly seen that if $\omega \gg \eta$ and η is small so that the distinct peak in the spectrum is distinguished, then η_x is approximately twice as big as η. The spectra of AR(2) processes are in such cases well approximated by spectra of processes given by formula (1). As an example three of them are presented in Fig. 4. A better approximation would be received if the value of ω_0 is slightly reduced since the peak of AR(2) process, as mentioned before, doesn't occur for ω_0 but for $(\omega_0^2-\eta^2)^{0.5}$. This is a matter of discussion because the assumption of smaller ω_0 will increase the average period $T=2\pi/\omega_0$ in formula (1) and it will therefore not be the same as in the AR(2) process being approximated. There is no problem in making an alternative calculation. In this case not one parameter η_x but two parameters η_x and ω_0 in formula (97) must be found using criterion (94) or (95). The realization of the AR(2) process for $\eta=0.01$ and $\omega=0.63$, approximated by formula (1) is presented in Fig. 5. The parameters for the $x(t)$ and $y(t)$ uncorrelated processes are $\eta_x=0.02$ and $\omega_x=0$. As it's seen the envelope drawn as a continuous line is smooth and practically tangent to the process in the extremes. Additionally an envelope, calculated with the formula (26), is also presented. It's drawn as a dashed line and it's ragged as the numerical calculation of the first derivative is burden with numerical errors and the assumptions made in deriving the formula are never fully valid.

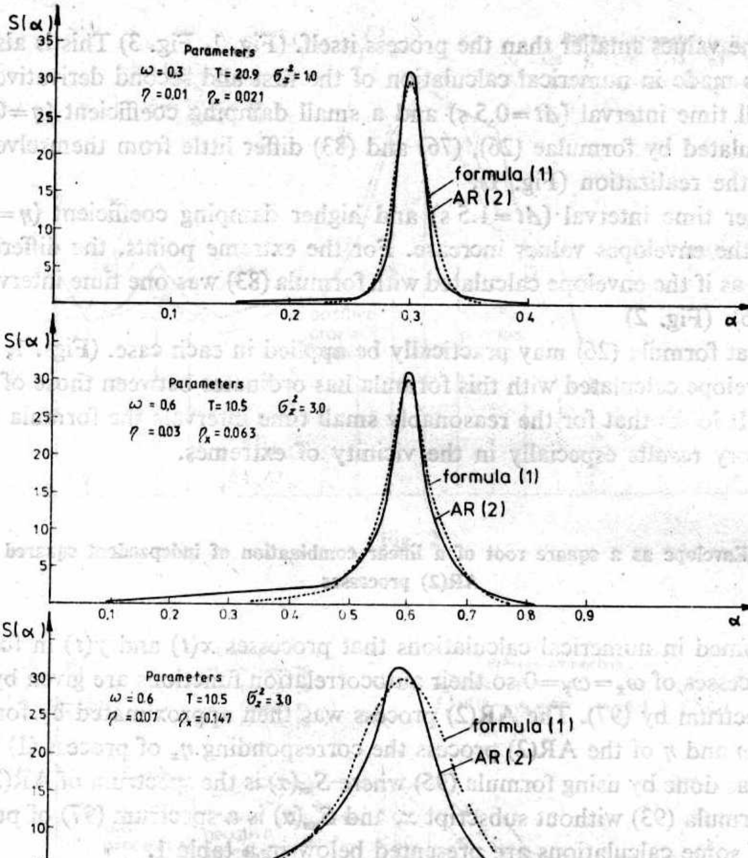
S(α)
Parameters
ω = 0.3 T = 20.9 σ_z² = 1.0
η = 0.01 ρ_x = 0.021
formula (1)
AR (2)
S(α)
Parameters
ω = 0.6 T = 10.5 σ_z² = 3.0
η = 0.03 ρ_x = 0.063
formula (1)
AR (2)
S(α)
Parameters
ω = 0.6 T = 10.5 σ_z² = 3.0
η = 0.07 ρ_x = 0.147
formula (1)
AR (2)
α

Fig. 4

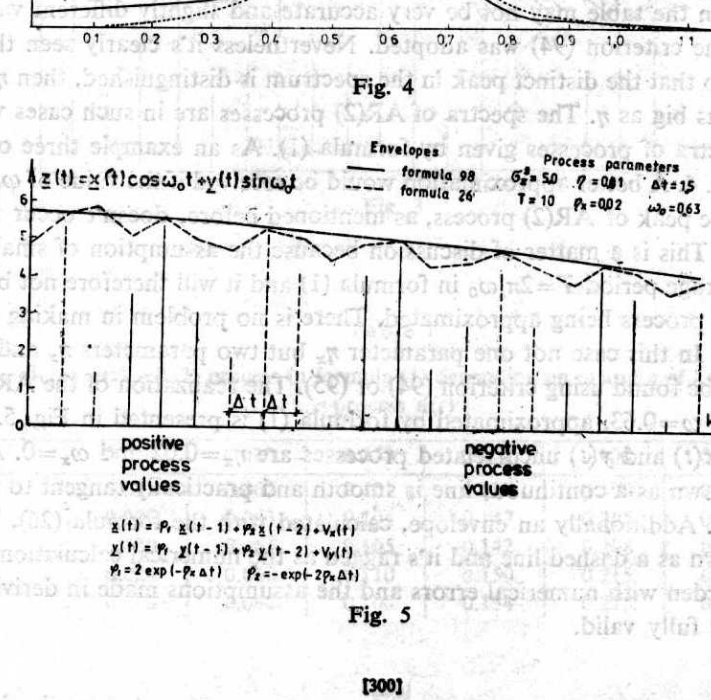
z(t) = x(t) cos ω₀t + y(t) sin ω₀t
Envelopes
formula 98
formula 26'
Process parameters
σ_z² = 5.0 η = 0.01 Δt = 1.5
T = 10 ρ_x = 0.02 ω₀ = 0.63
Δt Δt
k·Δt
positive process values
negative process values
x(t) = φ₁ x(t − 1) + φ₂ x(t − 2) + V_x(t)
y(t) = φ₁ y(t − 1) + φ₂ y(t − 2) + V_y(t)
φ₁ = 2 exp(−ρ_x Δt) φ₂ = − exp(−2ρ_x Δt)

Fig. 5

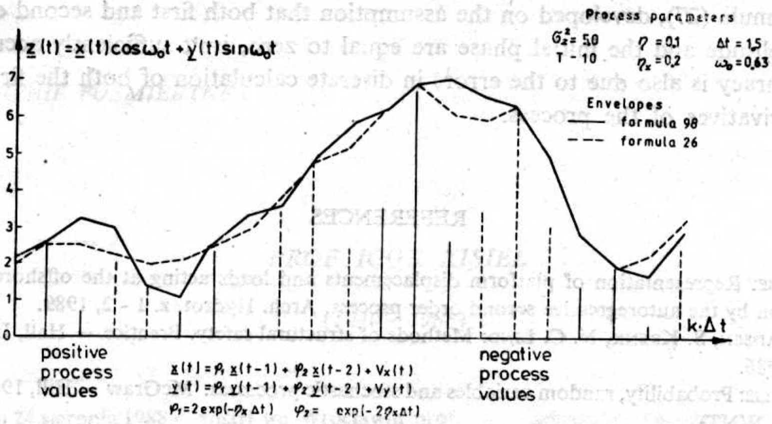

Fig. 6

Similar realization but for the higher $\eta = 0.1$ is presented in Fig. 6. There is a considerably higher variability of the process itself and therefore of its envelope. The envelope marked with a continuous line, calculated on the basis of the transformation of the AR(2) process into the form presented by formula (1), is smooth and tangent to the extremes. The dashed line presents the envelope computed with formula (26). In some parts it coincides with the continuous line but in some points it deviates considerably.

5. Final conclusions

If the stationary process is given in form of a function which could be subjected to Hilbert transformation, the envelope of this process is given by the formula (12). If the whole or sufficiently long realization of the process is known, the discrete Hilbert transformation may be performed using formula (14) and the amplitude calculated using formula (12). This is a time consuming calculation as each value of an amplitude requires one Hilbert transformation.

If a spectrum of a stationary process is given then the spectrum of the $a(t) = x(t)$ process in formula (1) can be calculated (17). Then, assuming the character of this process its parameters should be estimated using expressions (94) and (95). An example for AR(2) process is shown in paragraph 3.5.

For the realization of an AR(2) process, the envelope may be calculated in each consecutive step using formula (76) with parameters found recursively with formulae (49) and (50). Alternatively the formula (72) for the envelope may be used with formulae (66) and (60) for computation of the random impulses. When the realization of an arbitrary, stationary process is given the envelope may be estimated using formula (26) with the assumption that the first derivatives of the amplitude and the initial shift are equal to zero. The resultant envelope is a bit ragged and not smooth contrary to that calculated as described in the paragraph 3.5. (Fig. 5).

The formula (27), developed on the assumption that both first and second derivatives of the amplitude and the initial phase are equal to zero, isn't sufficiently accurate. The high inaccuracy is also due to the errors in discrete calculation of both the first and the second derivatives of the process.

REFERENCES

1. W. Knabe: Representation of platform displacements and loads acting at the offshore plastform foundation by the autoregressive second order process. Arch. Hydrot. z. 1 - 2, 1989.
2. H. O. Madsen, S. Krenk, N. C. Lind: Methods of structural safety. Prentice – Hall, International Series, 1986.
3. A. Papoulis: Probability, random variables and stochastic processes. McGraw – Hill, 1965 (Warszawa 1972, WNT).
4. A. Sawicki, L. W. Morland: Pore-pressure generation in a saturated sand layer subjected to a cyclic horizontal acceleration of its base. J. Mech. Phis. Solids Vol. 33, No. 6, pp. 545-559, 1985 Pergamon Press Ltd.
5. A. Sawicki: Mechanics of sand subjected to cyclic sheating (in Polish) – Zeszyty Naukowe Politechniki Gdańskiej nr 400, 1987, pp. 81 - 160.
6. P. Wilde, A. Kozakiewicz: Analysis of beating by means of the Kalman filter, (in Polish). Arch. Inżynierii Lądowej, tom XXXII, z. 1, 1986.

Obwiednie stacjonarnych procesów stochastycznych

Streszczenie

W pracy opisano dokładne oraz przybliżone metody określania obwiedni procesów stochastycznych. Przedstawiono wykorzystanie transformacji Hilberta oraz określenia składowych synfazowych i kwadraturowych widma względem średniej częstotliwości procesu, pozwalających oszacować przybliżone wartości obwiedni. Wyprowadzono i określono wyrażenia na proces opisujący obwiednię procesu autoregresji rzędu drugiego AR(2). Opisano możliwość oszacowania obwiedni dla tego procesu w przestrzeni stanu, wykonano wiele obliczeń numerycznych ilustrujących sposoby określania obwiedni i różnice wynikające z różnych założeń i przybliżeń.

Огибающие стационарных стохастических процессов

Содержание

В статье описаны точные и приближенные методы определения огибающих стохастических процессов. Представлено применение преобразования Гильберта и определения синфазных и квадратурных составляющих спектра относительно средней частоты процесса, позволяющих оценить приближенные значения огибающей. Выведены и приведены выражения для процесса, описывающего огибающую процесса авторегрессии второго порядка АР (2). Описана возможность оценки огибающей для этого процесса в пространстве состояния, проведен ряд численных расчетов, иллюстрирующих способы определения огибающей и различия, вытекающие из разных предположенний и, приближений.